ONLINE APPENDIX – All Proofs

A. Proofs of Propositions 1-4

Proof of Proposition 1: At time t, an agent with valuation x_t chooses an asset i and a position q in the asset to solve

$$\max_{i\in\{1,2\}} \max_{q\in\{0,1\}} \left[q(\delta + x_t) - |q|\, y - qrp_i\right], \tag{A1}$$

i.e., maximize the flow utility minus the time value of the position's cost. In equilibrium, assets trade at the same price because otherwise no agent would demand a long position in the more expensive asset. Denoting by p the common price, no agent would demand a long position in any asset if $rp > (\delta + \overline{x} - y)$. Conversely, if $rp < (\delta + \overline{x} - y)$, then high-valuation agents would demand long positions, which generates excess demand from Assumption 2. Therefore, $rp = (\delta + \overline{x} - y)$. Under this price, high-valuation agents are indifferent between a long and no position, and all other agents hold no position. ∎

Proof of Proposition 2: In equilibrium, either high-valuation agents accept to buy asset i, or they refuse to do so and the asset is owned only by average-valuation agents. To nest the two cases, we define the variable λ_i by $\lambda_i \equiv \lambda$ if high-valuation agents accept to buy asset i and $\lambda_i \equiv 0$ otherwise. The utilities $V_{\overline{b}}$, $V_{\overline{n}i}$, and $V_{\overline{s}i}$ of being type $\overline{b}$, $\overline{n}i$, and $\overline{s}i$, respectively, are determined by the flow-value equations

$$rV_{\overline{b}} = -\overline{\kappa} V_{\overline{b}} + \sum_{i=1}^{2} \lambda_i \mu_{\overline{s}i} \left(V_{\overline{n}i} - p_i - V_{\overline{b}}\right), \tag{A2}$$

$$rV_{\overline{n}i} = \delta + \overline{x} - y + \overline{\kappa}\left(V_{\overline{s}i} - V_{\overline{n}i}\right), \tag{A3}$$

$$rV_{\overline{s}i} = \delta - y + \lambda_i \mu_{\overline{b}} \left(p_i - V_{\overline{s}i}\right). \tag{A4}$$

For example, (A2) equates the flow value $rV_{\overline{b}}$ of being type $\overline{b}$ to the flow benefits accruing to $\overline{b}$ and the utility derived from the possibility of $\overline{b}$ transiting to other types. The flow benefits are zero because $\overline{b}$ does not own an asset. The transitions are (i) revert to average valuation at rate $\overline{\kappa}$ and exit the market (utility zero and net utility $-V_{\overline{b}}$), and (ii) meet a seller of asset $i \in \{1, 2\}$ at rate $\lambda_i \mu_{si}$, buy at price p_i, and become a non-searcher $\overline{n}i$ (utility $V_{\overline{n}i}$ and net utility $V_{\overline{n}i} - p_i - V_{\overline{b}}$).

The price of asset i is such that the buyer receives a fraction ϕ of the surplus $\hat{\Sigma}_i$. The buyer's net utility from the transaction is $V_{\overline{n}i} - p_i - V_{\overline{b}}$ and the seller's is $p_i - V_{\overline{s}i}$. Therefore, the price satisfies

$$V_{\overline{n}i} - p_i - V_{\overline{b}} = \phi\hat{\Sigma}_i = \phi(V_{\overline{n}i} - V_{\overline{b}} - V_{\overline{s}i}) \Rightarrow p_i = \phi V_{\overline{s}i} + (1-\phi)\left(V_{\overline{n}i} - V_{\overline{b}}\right). \tag{A5}$$

Equilibrium imposes that

$$\lambda_i = \lambda \Leftrightarrow \hat{\Sigma}_i \geq 0, \tag{A6}$$

i.e., high-valuation agents accept to buy asset i if this transaction generates a positive surplus $\hat{\Sigma}_i$.

Subtracting (A2) and (A4) from (A3), and replacing p_i by (A5), we find

$$(r+\overline{\kappa})\hat{\Sigma}_i = \overline{x} - \phi\sum_{j=1}^{2}\lambda_j\mu_{\overline{s}j}\hat{\Sigma}_j - (1-\phi)\lambda_i\mu_{\overline{b}}\hat{\Sigma}_i. \tag{A7}$$

If $\lambda_1 = \lambda_2 = 0$, (A7) implies that $\hat{\Sigma}_i = \overline{x}/(r+\overline{\kappa}) > 0$, a contradiction. If $\lambda_1 = \lambda$ and $\lambda_2 = 0$, (A7) implies that $\hat{\Sigma}_2 > \hat{\Sigma}_1 > 0$, again a contradiction. Therefore, the only possibility is that $\lambda_1 = \lambda_2 = \lambda$, i.e., high-valuation agents accept to buy both assets. For $\lambda_1 = \lambda_2 = \lambda$, the variables $(V_{\overline{n}i}, V_{\underline{n}i}, p_i, \hat{\Sigma}_i)$ are independent of i, and thus the Law of One Price holds. ∎

Proof of Proposition 3: The lending fee is zero by the argument preceding the proposition's statement. Agents' optimization problem is (A1) with the only difference that $q \in \{-1, 0, 1\}$. Same arguments as in Proposition 1 imply that assets trade at the same price p, such that $rp \leq (\delta + \overline{x} - y)$. If $rp < (\delta + \overline{x} - y)$, then high-valuation agents would demand long positions, and average-valuation agents would not demand short positions from Assumption 1. This implies excess demand from Assumption 2, and thus $rp = (\delta + \overline{x} - y)$. Under this price, high-valuation agents are indifferent between a long and no position. Moreover, Assumption 1 implies that low-valuation agents hold short positions and average-valuation agents hold no position. ∎

Proof of Proposition 4: If in equilibrium low-valuation agents refuse to borrow asset i, the asset carries no lending fee, and its owners are high-valuation agents who sell when they switch to average valuation. If instead low-valuation agents accept to borrow asset i, some owners can be average-valuation. Indeed, because the asset carries a positive lending fee, its owners might prefer not to terminate a repo contract when they switch to average valuation, but wait until the borrower wishes to terminate. To nest the two cases, we define the variable ν_i by $\nu_i \equiv \nu$ if low-valuation agents accept to borrow asset i and $\nu_i \equiv 0$ otherwise. We denote by $V_{\overline{\ell}i}$ the utility of a high-valuation agent seeking to lend asset i, $V_{\overline{n}i}$ the utility of a high-valuation agent who is in a repo contract lending asset i, V_{ni} the utility of an average-valuation agent who is in the same repo contract and waits for the borrower to terminate, $V_{\underline{bo}}$ the utility of a low-valuation agent seeking to borrow an asset, and $V_{\underline{n}i}$ the utility of a low-valuation agent who is in a repo contract borrowing asset i. These utilities satisfy the flow-value equations

$$
\begin{aligned}
rV_{\overline{\ell}i} &= \delta + \overline{x} - y + \overline{\kappa}\left(p_i - V_{\overline{\ell}i}\right) + \nu_i \mu_{\underline{bo}}\left(V_{\overline{n}i} - V_{\overline{\ell}i}\right), && \text{(A8)} \\
rV_{ni} &= \delta - y + w_i + \underline{\kappa}\left(p_i - V_{ni}\right), && \text{(A9)} \\
rV_{\underline{bo}} &= -\underline{\kappa} V_{\underline{bo}} + \sum_{i=1}^{2} \nu_i \mu_{\overline{\ell}i}\left(V_{\underline{n}i} + p_i - V_{\underline{bo}}\right). && \text{(A10)}
\end{aligned}
$$

The remaining two equations depend on whether an owner terminates a repo contract immediately upon switching to average valuation, or whether he waits for the borrower to terminate.

We first treat the case of immediate termination, which is characterized by the condition $p_i \geq V_{ni}$. The two remaining flow-value equations are

$$
\begin{aligned}
rV_{\overline{n}i} &= \delta + \overline{x} - y + w_i + \overline{\kappa}\left(p_i - V_{\overline{n}i}\right) + \underline{\kappa}\left(V_{\overline{\ell}i} - V_{\overline{n}i}\right), && \text{(A11)} \\
rV_{\underline{n}i} &= -\delta + \underline{x} - y - w_i + \overline{\kappa}\left(V_{\underline{bo}} - p_i - V_{\underline{n}i}\right) + \underline{\kappa}\left(-p_i - V_{\underline{n}i}\right). && \text{(A12)}
\end{aligned}
$$

To determine the price p_i, note that if $p_i > V_{\overline{\ell}i}$, then high-valuation agents would not demand long positions, and neither would other agents with lower valuations. Conversely, if $p_i < V_{\overline{\ell}i}$, then high-valuation agents would demand long positions. Since the measure of short-sellers does not exceed that of low-valuation agents (and is, in fact, strictly smaller because of the search friction), Assumption 2 implies excess demand for asset i. Therefore, $p_i = V_{\overline{\ell}i}$. The lending fee w_i is such that the lender receives a fraction $\theta \in [0, 1]$ of the surplus Σ_i in a repo transaction. Since a repo transaction turns the lender $\overline{\ell}i$ into type $\overline{n}i$, the lender's surplus is $V_{\overline{n}i} - V_{\overline{\ell}i}$. The borrower's surplus is $p_i + V_{\underline{n}i} - V_{\underline{bo}}$ because the borrower $\underline{bo}$ sells the asset and becomes type $\underline{n}i$. Therefore, the lending fee is implicitly defined by

$$
V_{\overline{n}i} - V_{\overline{\ell}i} = \theta \Sigma_i = \theta(V_{\overline{n}i} - V_{\overline{\ell}i} + p_i + V_{\underline{n}i} - V_{\underline{bo}}). \tag{A13}
$$

Finally, equilibrium imposes (3), i.e., low-valuation agents accept to borrow asset i if this transaction generates a positive surplus Σ_i.

Since $p_i = V_{\overline{\ell}i}$, the surplus is $\Sigma_i = V_{\overline{n}i} + V_{\underline{n}i} - V_{\underline{bo}}$. Subtracting (A10) from the sum of (A11) and (A12), and noting that (A13) implies $p_i + V_{\underline{n}i} - V_{\underline{bo}} = (1-\theta)\Sigma_i$, we find:

$$
(r + \overline{\kappa} + \underline{\kappa})\Sigma_i = \overline{x} + \underline{x} - 2y - (1-\theta)\sum_{j=1}^{2} \nu_j \mu_{\overline{\ell}j} \Sigma_j. \tag{A14}
$$

Eq. (A14) implies $\Sigma_1 = \Sigma_2 \equiv \Sigma$ and thus $\nu_1 = \nu_2$. If $\nu_1 = \nu_2 = 0$, then $\Sigma = (\overline{x}+\underline{x}-2y)/(r+\overline{\kappa}+\underline{\kappa})$, which is positive by Assumption 1, a contradiction. Therefore, $\nu_1 = \nu_2 = \nu$, i.e., low-valuation agents accept to borrow both assets. For $\nu_1 = \nu_2 = \nu$, the variables $(V_{\overline{\ell}i}, V_{\overline{n}i}, V_{\underline{n}i}, p_i, w_i)$ are independent of i, and thus the Law of One Price holds.

We next treat the case $p_i < V_{ni}$. Then, (A11) and (A12) are replaced by

$$\begin{aligned} rV_{\overline{n}i} &= \delta + \overline{x} - y + w_i + \overline{\kappa}\left(V_{ni} - V_{\overline{n}i}\right) + \underline{\kappa}\left(V_{\overline{\ell}i} - V_{\overline{n}i}\right), && \text{(A15)} \\ rV_{\underline{n}i} &= -\delta + \underline{x} - y - w_i + \underline{\kappa}\left(-p_i - V_{\underline{n}i}\right). && \text{(A16)} \end{aligned}$$

The counterpart of (A14) is

$$(r+\underline{\kappa})\Sigma_i = \overline{x} + \underline{x} - 2y - (1-\theta)\sum_{j=1}^{2}\nu_j\mu_{\overline{\ell}j}\Sigma_j + \overline{\kappa}(V_{ni} - V_{\overline{n}i}). \tag{A17}$$

Subtracting (A9) from (A15), we find

$$V_{\overline{n}i} - V_{ni} = \frac{\overline{x}}{r+\overline{\kappa}+\underline{\kappa}}, \tag{A18}$$

and can rewrite (A17) as

$$(r+\underline{\kappa})\Sigma_i = \overline{x} + \underline{x} - 2y - (1-\theta)\sum_{j=1}^{2}\nu_j\mu_{\overline{\ell}j}\Sigma_j - \frac{\overline{\kappa}\overline{x}}{r+\overline{\kappa}+\underline{\kappa}}. \tag{A19}$$

Suppose that $\Sigma_1, \Sigma_2 \leq 0$. Then, a borrower and a lender of asset i are better off agreeing on a repo contract with a fee $w_i \approx 0$. Indeed, since $rp_i = \delta + \overline{x} - y$ from (A8), we have $\delta - y + w_i - rp_i \approx -\overline{x} < 0$ and thus $V_{ni} < p_i$. Therefore, the surplus Σ_i under this contract is given by (A14) and is positive. The lender is better off because of the fee, and if the fee is small the borrower is better off because $\Sigma_i > 0$. Therefore, $\Sigma_1, \Sigma_2 \leq 0$ cannot be part of an equilibrium.

Suppose that $\Sigma_1 > 0$ and $\Sigma_2 \leq 0$. Then, a borrower and a lender of asset 2 are better off agreeing on a repo contract with a fee $w_2 \approx 0$. Indeed, the surplus Σ_2 under this contract is given by (A14). If Σ_1 is given by (A14), then $\Sigma_2 = \Sigma_1 > 0$. If Σ_1 is given by (A17), then $\Sigma_2 = \overline{x}/(r+\overline{\kappa}+\underline{\kappa}) > 0$. Therefore, $\Sigma_1 > 0$ and $\Sigma_2 \leq 0$ cannot be part of an equilibrium, and the only possible outcome is $\Sigma_1, \Sigma_2 > 0$ and $\nu_1 = \nu_2 = \nu$.

Since $\nu_1 = \nu_2 = \nu$, the Law of One Price holds if $p_i \geq V_{ni}$ for both assets or $p_i < V_{ni}$ for both assets. Consider an equilibrium in which $p_1 \geq V_{n1}$ and $p_2 < V_{n2}$. Then (A8) and (A13) imply that

$$rp_i = \delta + \overline{x} - y + \nu\mu_{\underline{bo}}\theta\Sigma_i, \tag{A20}$$

(A8), (A11) and (A13) imply that

$$w_1 = (r + \overline{\kappa} + \underline{\kappa} + \nu\mu_{\underline{bo}})\theta\Sigma_1, \tag{A21}$$

(A8), (A13), (A15) and (A18) imply that

$$w_2 = (r + \underline{\kappa} + \nu\mu_{\underline{bo}})\theta\Sigma_2 + \frac{\overline{\kappa x}}{r + \overline{\kappa} + \underline{\kappa}}, \tag{A22}$$

(A20), (A21) and $\delta - y + w_1 - rp_1 \leq 0$ imply that

$$(r + \overline{\kappa} + \underline{\kappa})\theta\Sigma_1 - \overline{x} \leq 0, \tag{A23}$$

and (A20), (A22) and $\delta - y + w_2 - rp_2 > 0$ imply that

$$(r + \overline{\kappa} + \underline{\kappa})\theta\Sigma_2 - \overline{x} > 0. \tag{A24}$$

Eqs. (A23) and (A24) imply that $\Sigma_2 > \Sigma_1$. But then, a borrower and a lender of asset 1 can be made better off agreeing to a contract with a fee $\tilde{w}_1 > w_1$ such that $\delta - y + \tilde{w}_1 - rp_1$ is slightly positive. Using (A9), this implies that $\tilde{V}_{n1} > p_1$, so that the lender finds it optimal not to terminate when he reverts to an average valuation. Hence, this contract generates surplus Σ_2. Because $\delta - y + \tilde{w}_1 - rp_1$ is slightly positive, we also have that $\tilde{V}_{n1} \approx p_1$, meaning that a lender is nearly indifferent between terminating or not. This means that the change in the lender's utility is

$$\Delta V_{\overline{n}i} \approx \frac{\tilde{w}_1 - w_1}{r + \overline{\kappa} + \underline{\kappa}} > 0,$$

the PV of the lending fee difference assuming that the lender follows the same termination strategy than with w_1. The change in the borrower's utility is $\Sigma_2 - \Sigma_1 - \Delta V_{\overline{n}i}$. Factoring out $1/(r + \overline{\kappa} + \underline{\kappa})$, we can write this as

$$\begin{aligned}
& (r + \overline{\kappa} + \underline{\kappa})(\Sigma_2 - \Sigma_1) - (\tilde{w}_1 - w_1) \\
\approx\ & (r + \overline{\kappa} + \underline{\kappa})(\Sigma_2 - \Sigma_1) - \left[rp_1 - \delta + y - (r + \overline{\kappa} + \underline{\kappa} + \nu\mu_{\underline{bo}})\theta\Sigma_1\right] \\
=\ & (r + \overline{\kappa} + \underline{\kappa})(\Sigma_2 - \Sigma_1) - \left[\delta + \overline{x} - y + \nu\mu_{\underline{bo}}\theta\Sigma_1 - \delta + y - (r + \overline{\kappa} + \underline{\kappa} + \nu\mu_{\underline{bo}})\theta\Sigma_1\right] \\
=\ & (r + \overline{\kappa} + \underline{\kappa})(\Sigma_2 - \Sigma_1) - [\overline{x} - (r + \overline{\kappa} + \underline{\kappa})\theta\Sigma_1] \\
=\ & (1 - \theta)(r + \overline{\kappa} + \underline{\kappa})(\Sigma_2 - \Sigma_1) + [(r + \overline{\kappa} + \underline{\kappa})\theta\Sigma_2 - \overline{x}] > 0.
\end{aligned}$$

Therefore, the conjectured equilibrium is not possible. ∎

B. Population Measures

The measures μ_{bi} and μ_{si} of buyers and sellers of asset i are

$$\mu_{bi} = \mu_{\overline{b}} + \mu_{\underline{b}i} \tag{B1}$$

$$\mu_{si} = \mu_{\overline{s}i} + \mu_{\underline{s}i}. \tag{B2}$$

Since assets are held by either lenders or sellers, market clearing implies that

$$\mu_{\overline{\ell}i} + \mu_{si} = S. \tag{B3}$$

Moreover, since there is equal measure of high- and low-valuation agents involved in repo contracts,

$$\mu_{\overline{n}i} \equiv \mu_{\overline{n}\underline{s}i} + \mu_{\overline{n}\underline{n}i} + \mu_{\overline{n}\underline{b}i} = \mu_{\underline{s}i} + \mu_{\underline{n}i} + \mu_{\underline{b}i} \tag{B4}$$

To write the inflow-outflow equations, we condense types $(\overline{n}\underline{s}i, \overline{n}\underline{n}i, \overline{n}\underline{b}i)$ into a type $\overline{n}i$, and denote that type's measure by $\mu_{\overline{n}i}$ as in (B4) above. We also denote by f_i the inflow from type $\overline{n}i$ to type $\overline{\ell}i$. The inflow-outflow equations are

$$\textit{Buyers } \overline{b} \qquad \overline{F} = \overline{\kappa}\mu_{\overline{b}} + \sum_{i=1}^{2} \lambda \mu_{si}\mu_{\overline{b}} \tag{B5}$$

$$\textit{Lenders } \overline{\ell}i \qquad \lambda\mu_{\overline{b}}\mu_{si} + f_i = \overline{\kappa}\mu_{\overline{\ell}i} + \nu_i \mu_{\underline{bo}}\mu_{\overline{\ell}i} \tag{B6}$$

$$\textit{Non-searchers } \overline{n}i \qquad \nu_i\mu_{\underline{bo}}\mu_{\overline{\ell}i} = f_i + \overline{\kappa}\mu_{\overline{n}i} \tag{B7}$$

$$\textit{Sellers } \overline{s}i \qquad \overline{\kappa}\mu_{\overline{\ell}i} + \overline{\kappa}\mu_{\underline{s}i} = \lambda\mu_{bi}\mu_{\overline{s}i} \tag{B8}$$

$$\textit{Borrowers } \underline{bo} \qquad \underline{F} + \sum_{i=1}^{2} \overline{\kappa}(\mu_{\underline{s}i} + \mu_{\underline{n}i}) = \underline{\kappa}\mu_{\underline{bo}} + \sum_{i=1}^{2} \nu_i \mu_{\underline{bo}}\mu_{\overline{\ell}i} \tag{B9}$$

$$\textit{Sellers } \underline{s}i \qquad \nu_i\mu_{\underline{bo}}\mu_{\overline{\ell}i} = \overline{\kappa}\mu_{\underline{s}i} + \underline{\kappa}\mu_{\underline{s}i} + \lambda\mu_{bi}\mu_{\underline{s}i} \tag{B10}$$

$$\textit{Non-searchers } \underline{n}i \qquad \lambda\mu_{bi}\mu_{\underline{s}i} = \overline{\kappa}\mu_{\underline{n}i} + \underline{\kappa}\mu_{\underline{n}i} \tag{B11}$$

$$\textit{Buyers } \underline{b}i \qquad \underline{\kappa}\mu_{\underline{n}i} = \overline{\kappa}\mu_{\underline{b}i} + \lambda\mu_{\underline{b}i}\mu_{si}, \tag{B12}$$

For example, (B5) equates the inflow into type $\overline{b}$, which is $\overline{F}$ because of the new entrants, to the outflow, which is the sum of (i) $\overline{\kappa}\mu_{\overline{b}}$ because some buyers revert to average valuation and exit the market, and (ii) $\sum_{i=1}^{2} \lambda\mu_{si}\mu_{\overline{b}}$ because some buyers meet with sellers.

We determine population measures by the system of (B1)-(B5) and (B8)-(B12). The total number of equations is 18 (because some are for each asset), and the 18 unknowns are the measures of the 14 types $\overline{b}$, $\underline{bo}$, $\{\overline{\ell}i, \overline{n}i, \overline{s}i, \underline{s}i, \underline{n}i, \underline{b}i\}_{i\in\{1,2\}}$ and $\{\mu_{bi}, \mu_{si}\}_{i\in\{1,2\}}$. A solution to the system satisfies (B6) and (B7), which is why we do not include them into the system. Indeed, adding (B10)-(B12), and using (B4), we find

$$\nu_i \mu_{\underline{bo}} \mu_{\overline{\ell}i} = \overline{\kappa}\mu_{\underline{s}i} + \underline{\kappa}\mu_{\overline{n}i} + \lambda\mu_{\underline{b}i}\mu_{si}.$$

Therefore, (B7) holds with $f_i = \underline{\kappa}\mu_{\underline{s}i} + \lambda\mu_{\underline{b}i}\mu_{si}$. For this value of f_i, (B6) becomes $\lambda\mu_{bi}\mu_{si} + \underline{\kappa}\mu_{\underline{s}i} = \overline{\kappa}\mu_{\overline{\ell}i} + \nu_i\mu_{\underline{bo}}\mu_{\overline{\ell}i}$, and is redundant because it can be derived by adding (B8) and (B10).

To solve the system, we reduce it to a simpler one in the six unknowns $\mu_{\underline{bo}}$, $\mu_{\overline{b}}$, and $\{\mu_{bi}, \mu_{si}\}_{i\in\{1,2\}}$. Adding (B10) and (B11), we find

$$\mu_{\underline{s}i} + \mu_{\underline{n}i} = \frac{\nu_i \mu_{\underline{bo}} \mu_{\overline{\ell}i}}{\overline{\kappa} + \underline{\kappa}}. \tag{B13}$$

Plugging into (B9), and using (B3), we find

$$\underline{F} = \underline{\kappa}\mu_{\underline{bo}} + \frac{\underline{\kappa}}{\overline{\kappa} + \underline{\kappa}} \sum_{i=1}^{2} \nu_i \mu_{\underline{bo}} (S - \mu_{si}). \tag{B14}$$

Eqs. (B10) and (B3) imply that

$$\mu_{\underline{s}i} = \frac{\nu_i \mu_{\underline{bo}} (S - \mu_{si})}{\overline{\kappa} + \underline{\kappa} + \lambda\mu_{bi}}. \tag{B15}$$

Eq. (B11) implies that

$$\mu_{\underline{n}i} = \frac{\lambda \mu_{\underline{s}i} \mu_{bi}}{\overline{\kappa} + \underline{\kappa}} \tag{B16}$$

and (B12) implies that

$$\mu_{\underline{b}i} = \frac{\underline{\kappa}\mu_{\underline{n}i}}{\overline{\kappa} + \lambda\mu_{si}}. \tag{B17}$$

Combining these equations to compute $\mu_{\underline{b}i}$, and using (B1), we find

$$\mu_{bi} = \mu_{\overline{b}} + \frac{\underline{\kappa}\lambda\mu_{bi}\nu_i\mu_{\underline{bo}}(S - \mu_{si})}{(\overline{\kappa} + \underline{\kappa})(\overline{\kappa} + \underline{\kappa} + \lambda\mu_{bi})(\overline{\kappa} + \lambda\mu_{si})}. \tag{B18}$$

Noting that $\mu_{\overline{\ell}i} + \mu_{\underline{s}i} = S - \mu_{\overline{s}i}$, we can use (B8) to compute $\mu_{\overline{s}i}$:

$$\mu_{\overline{s}i} = \frac{\overline{\kappa}S}{\overline{\kappa} + \lambda\mu_{bi}}. \tag{B19}$$

Adding (B15) and (B19), and using (B2), we find

$$\mu_{si} = \frac{\overline{\kappa}S}{\overline{\kappa}+\lambda\mu_{bi}} + \frac{\nu_i \mu_{\underline{bo}}(S-\mu_{si})}{\underline{\kappa}+\overline{\kappa}+\lambda\mu_{bi}}. \tag{B20}$$

The new system consists of (B5), (B14), (B18), and (B20). These are six equations (because some are for each asset), and the six unknowns are $\mu_{\underline{bo}}$, $\mu_{\overline{b}}$, and $\{\mu_{bi},\mu_{si}\}_{i\in\{1,2\}}$. Once this system is solved, the other measures can be computed as follows: $\mu_{\underline{s}i}$ from (B15), $\mu_{\underline{n}i}$ from (B16), $\mu_{\underline{b}i}$ from (B17), $\mu_{\overline{s}i}$ from (B19), $\mu_{\overline{\ell}i}$ from (B3), and $\mu_{\overline{n}i}$ from (B4).

To cover the case where search frictions are small, we make the change of variables $\varepsilon \equiv 1/\lambda$, $n \equiv \nu/\lambda$, $\alpha_i \equiv \nu_i/\nu$, $\gamma_{si} \equiv \lambda\mu_{si}$, and $\gamma_{\underline{bo}} \equiv \nu\mu_{\underline{bo}}$. Under the new variables, (B5), (B14), (B18), and (B20) become

$$\overline{F} = \overline{\kappa}\mu_{\overline{b}} + \sum_{i=1}^{2}\mu_{\overline{b}}\gamma_{si}, \tag{B21}$$

$$\underline{F} = \frac{\varepsilon\underline{\kappa}\gamma_{\underline{bo}}}{n} + \frac{\underline{\kappa}}{\overline{\kappa}+\underline{\kappa}}\sum_{i=1}^{2}\alpha_i\gamma_{\underline{bo}}(S-\varepsilon\gamma_{si}), \tag{B22}$$

$$\mu_{bi} = \mu_{\overline{b}} + \frac{\underline{\kappa}\mu_{bi}\alpha_i\gamma_{\underline{bo}}(S-\varepsilon\gamma_{si})}{(\overline{\kappa}+\underline{\kappa})\left[\varepsilon(\overline{\kappa}+\underline{\kappa})+\mu_{bi}\right](\overline{\kappa}+\gamma_{si})}, \tag{B23}$$

$$\gamma_{si} = \frac{\overline{\kappa}S}{\varepsilon\overline{\kappa}+\mu_{bi}} + \frac{\alpha_i\gamma_{\underline{bo}}(S-\varepsilon\gamma_{si})}{\varepsilon(\underline{\kappa}+\overline{\kappa})+\mu_{bi}}, \tag{B24}$$

respectively.

B.1. Existence and Uniqueness

We next show that the system of (B21)-(B24) has a unique symmetric solution when $\alpha_1=\alpha_2=1$ (the "symmetric" case), and a unique solution when $\alpha_1=1$ and $\alpha_2=0$ (the "asymmetric" case). Using (B23) to eliminate $\gamma_{\underline{bo}}$ in (B24), we find

$$\gamma_{si} = \frac{\overline{\kappa}S}{\varepsilon\overline{\kappa}+\mu_{bi}} + (\mu_{bi}-\mu_{\overline{b}})\frac{(\overline{\kappa}+\underline{\kappa})(\overline{\kappa}+\gamma_{si})}{\underline{\kappa}\mu_{bi}}.$$

Multiplying by μ_{bi}, and setting $i=1$, we find

$$\gamma_{s1}\mu_{\overline{b}} = \frac{\overline{\kappa}S\mu_{b1}}{\varepsilon\overline{\kappa}+\mu_{b1}} + (\mu_{b1}-\mu_{\overline{b}})\frac{\overline{\kappa}}{\underline{\kappa}}(\overline{\kappa}+\underline{\kappa}+\gamma_{s1}). \tag{B25}$$

In the rest of the proof, we use (B21), (B22), (B23) for $i \in \{1,2\}$, and (B24) for $i=2$, to determine $\mu_{\overline{b}}$ and μ_{b1} as functions of $\gamma_{s1} \in (0, S/\varepsilon)$. We then plug these functions into (B25), and show that the resulting equation in the single unknown γ_{s1} has a unique solution.

We first solve for $\mu_{\overline{b}}$. In the asymmetric case, (B23) implies that $\mu_{b2} = \mu_{\overline{b}}$, (B24) implies that $\gamma_{s2} = \overline{\kappa} S/(\varepsilon\overline{\kappa} + \mu_{\overline{b}})$, and (B21) implies that

$$\overline{F} = \overline{\kappa}\mu_{\overline{b}} + \mu_{\overline{b}}\left(\gamma_{s1} + \frac{\overline{\kappa}S}{\varepsilon\overline{\kappa} + \mu_{\overline{b}}}\right). \tag{B26}$$

The RHS of (B26) is (strictly) increasing in $\mu_{\overline{b}} \in (0,\infty)$, is equal to zero for $\mu_{\overline{b}} = 0$, and goes to ∞ for $\mu_{\overline{b}} \to \infty$. Therefore, (B26) has a unique solution $\mu_{\overline{b}} \in (0,\infty)$. This solution is decreasing in γ_{s1} because the RHS is increasing in γ_{s1}. In the symmetric case, (B21) implies that $\mu_{\overline{b}} = \overline{F}/(\overline{\kappa} + 2\gamma_{s1})$. This solution is again decreasing in γ_{s1}.

We next solve for μ_{b1}. Eq. (B22) implies that

$$\gamma_{\underline{bo}} = \frac{\underline{F}}{\frac{\varepsilon\underline{\kappa}}{n} + \frac{\underline{\kappa}}{\overline{\kappa}+\underline{\kappa}}\sum_{i=1}^{2}\alpha_i(S - \varepsilon\gamma_{si})} = \frac{\underline{F}}{\frac{\varepsilon\underline{\kappa}}{n} + \frac{\underline{\kappa}}{\overline{\kappa}+\underline{\kappa}}(1+\alpha_2)(S - \varepsilon\gamma_{s1})},$$

where the second step follows because in the symmetric case $\gamma_{s2} = \gamma_{s1}$ and in the asymmetric case $\alpha_2 = 0$. Plugging into (B23), setting $i=1$, and dividing by μ_{b1}, we find

$$1 = \frac{\mu_{\overline{b}}}{\mu_{b1}} + \frac{(S - \varepsilon\gamma_{s1})n\underline{F}}{\left[\varepsilon(\overline{\kappa} + \underline{\kappa}) + \mu_{b1}\right](\overline{\kappa} + \gamma_{s1})\left[\varepsilon(\overline{\kappa} + \underline{\kappa}) + n(1+\alpha_2)(S - \varepsilon\gamma_{s1})\right]}. \tag{B27}$$

The RHS of (B27) is decreasing in $\mu_{b1} \in (0,\infty)$, goes to ∞ for $\mu_{b1} \to 0$, and goes to zero for $\mu_{b1} \to \infty$. Therefore, (B26) has a unique solution $\mu_{b1} \in (0,\infty)$. This solution is decreasing in γ_{s1} because the RHS is decreasing in γ_{s1} and increasing in $\mu_{\overline{b}}$ (which is decreasing in γ_{s1}).

We next substitute $\mu_{\overline{b}}$ and μ_{b1} into (B25), and treat it as an equation in the single unknown γ_{s1}. To show uniqueness, we will show that the LHS is increasing in γ_{s1} and the RHS is decreasing. In the symmetric case, the LHS is equal to

$$\gamma_{s1}\mu_{\overline{b}} = \frac{\gamma_{s1}\overline{F}}{\overline{\kappa} + 2\gamma_{s1}},$$

and is increasing. In the asymmetric case, (B26) implies that the LHS is equal to

$$\gamma_{s1}\mu_{\overline{b}} = \overline{F} - \overline{\kappa}\mu_{\overline{b}} - \frac{\overline{\kappa}S\mu_{\overline{b}}}{\varepsilon\overline{\kappa} + \mu_{\overline{b}}},$$

and is increasing because $\mu_{\overline{b}}$ is decreasing in γ_{s1}. The first term in the RHS is increasing in μ_{b1}, and thus decreasing in γ_{s1}. To show that the second term is also decreasing, we multiply (B27) by $\mu_{b1}(\overline{\kappa}+\underline{\kappa}+\gamma_{s1})$:

$$(\mu_{b1}-\mu_{\overline{b}})(\overline{\kappa}+\underline{\kappa}+\gamma_{s1}) = \frac{\mu_{b1}(\overline{\kappa}+\underline{\kappa}+\gamma_{s1})(S-\varepsilon\gamma_{s1})n\underline{F}}{\left[\varepsilon(\overline{\kappa}+\underline{\kappa})+\mu_{b1}\right](\overline{\kappa}+\gamma_{s1})\left[\varepsilon(\overline{\kappa}+\underline{\kappa})+n(1+\alpha_2)(S-\varepsilon\gamma_{s1})\right]}.$$

The RHS of this equation is decreasing in γ_{s1} because it is decreasing in γ_{s1} and increasing in μ_{b1} (which is decreasing in γ_{s1}). Therefore, the second term in the RHS of (B25) is decreasing in γ_{s1}.

To show existence, we note that for $\gamma_{s1}=0$, the LHS of (B25) is equal to zero, while the RHS is positive. Moreover, for $\gamma_{s1}=S/\varepsilon$, the LHS is equal to $S\mu_{\overline{b}}/\varepsilon$, while the RHS is equal to

$$\frac{\overline{\kappa}S\mu_{\overline{b}}}{\varepsilon\overline{\kappa}+\mu_{\overline{b}}} < \frac{S\mu_{\overline{b}}}{\varepsilon}$$

because $\mu_{b1}=\mu_{\overline{b}}$. Therefore, there exists a solution $\gamma_{s1}\in(0,S/\varepsilon)$.

B.2. Small Search Frictions

The case of small search frictions corresponds to small ε. Thus, the solution in this case is close to that for $\varepsilon=0$ provided that continuity holds. Our proof so far covers the case $\varepsilon=0$, except for existence. We next show that Assumption 2 ensures existence for $\varepsilon=0$. We also compute the solution in closed form and show continuity.

To emphasize that $\varepsilon=0$ is a limit case, we use m and g instead of μ and γ. Eqs. (B21)-(B24) become

$$\overline{F} = \overline{\kappa}m_{\overline{b}}+\sum_{i=1}^{2}m_{\overline{b}}g_{si}, \tag{B28}$$

$$\underline{F} = \frac{\underline{\kappa}}{\overline{\kappa}+\underline{\kappa}}\sum_{i=1}^{2}\alpha_i g_{\underline{bo}}S, \tag{B29}$$

$$m_{bi} = m_{\overline{b}}+\frac{\underline{\kappa}\alpha_i g_{\underline{bo}}S}{(\overline{\kappa}+\underline{\kappa})(\overline{\kappa}+g_{si})}, \tag{B30}$$

$$g_{si} = \frac{\overline{\kappa}S}{m_{bi}}+\frac{\alpha_i g_{\underline{bo}}S}{m_{bi}}. \tag{B31}$$

We first solve the system of (B28)-(B31) in the symmetric case ($\alpha_1 = \alpha_2 = 1$), suppressing the asset subscript because of symmetry. Eq. (B29) implies that

$$g_{\underline{bo}} = \frac{(\overline{\kappa} + \underline{\kappa})\underline{F}}{2\underline{\kappa}S}, \tag{B32}$$

(B31) implies that

$$g_s = \frac{\overline{\kappa}S + \frac{\overline{\kappa}+\underline{\kappa}}{2\underline{\kappa}}\underline{F}}{m_b}, \tag{B33}$$

and (B28) implies that

$$m_{\overline{b}} = \frac{\overline{F}}{\overline{\kappa} + 2g_s}. \tag{B34}$$

Substituting $g_{\underline{bo}}$, g_s, and $m_{\overline{b}}$ from (B32)-(B34) into (B30), we find that m_b solves the equation

$$1 = \frac{\overline{F}}{\overline{\kappa}m_b + 2\overline{\kappa}S + \frac{\overline{\kappa}+\underline{\kappa}}{\underline{\kappa}}\underline{F}} + \frac{\underline{F}}{2\overline{\kappa}m_b + 2\overline{\kappa}S + \frac{\overline{\kappa}+\underline{\kappa}}{\underline{\kappa}}\underline{F}}. \tag{B35}$$

This equation has a positive solution because of Assumption 2.

We next consider the asymmetric case ($\alpha_1 = 1$, $\alpha_2 = 0$), and use $\hat{m}$ and $\hat{g}$ instead of m and g. Eq. (B30) implies that $\hat{m}_{b2} = \hat{m}_{\overline{b}}$, (B31) implies that

$$\hat{g}_{s2} = \frac{\overline{\kappa}S}{\hat{m}_{\overline{b}}}, \tag{B36}$$

(B29) implies that

$$\hat{g}_{\underline{bo}} = \frac{(\overline{\kappa} + \underline{\kappa})\underline{F}}{\underline{\kappa}S}, \tag{B37}$$

(B31) implies that

$$\hat{g}_{s1} = \frac{\overline{\kappa}S + \frac{\overline{\kappa}+\underline{\kappa}}{\underline{\kappa}}\underline{F}}{\hat{m}_{b1}}, \tag{B38}$$

and (B28) implies that

$$\hat{m}_{\overline{b}} = \frac{\overline{F} - \overline{\kappa}S}{\overline{\kappa} + \hat{g}_{s1}}. \tag{B39}$$

Substituting $\hat{g}_{\underline{bo}}$, $\hat{g}_{s1}$, and $\hat{m}_{\overline{b}}$ from (B37)-(B39) into (B30), we find

$$\hat{m}_{b1} = \frac{\overline{F}}{\overline{\kappa}} - 2S - \frac{\underline{F}}{\underline{\kappa}}, \tag{B40}$$

which is positive because of Assumption 2.

To show continuity at $\varepsilon = 0$, we write (B25) as

$$\gamma_{s1}\mu_{\overline{b}} - \frac{\overline{\kappa} S \mu_{b1}}{\varepsilon\overline{\kappa} + \mu_{b1}} - (\mu_{b1} - \mu_{\overline{b}})\frac{\overline{\kappa}}{\underline{\kappa}}(\overline{\kappa} + \underline{\kappa} + \gamma_{s1}) = 0,$$

and denote by $R(\gamma_{s1}, \varepsilon)$ the RHS (treating $\mu_{\overline{b}}$ and μ_{b1} as functions of $(\gamma_{s1}, \varepsilon)$). Because $\mu_{\overline{b}}, \mu_{b1} > 0$ for $(\gamma_{s1}, \varepsilon) = (g_{s1}, 0)$ (symmetric case) and $(\gamma_{s1}, \varepsilon) = (\hat{g}_{s1}, 0)$ (asymmetric case), the functions $\mu_{\overline{b}}$ and μ_{b1} are continuously differentiable around that point, and so is the function $R(\gamma_{s1}, \varepsilon)$. Moreover, our uniqueness proof shows that the derivative of $R(\gamma_{s1}, \varepsilon)$ w.r.t. γ_{s1} is positive. Therefore, the Implicit Function Theorem ensures that for small ε, (B25) has a continuous solution $\gamma_{s1}(\varepsilon)$. Because of uniqueness, this solution coincides with the one that we have identified.

C. Utilities and Prices

The flow-value equations are

$$\begin{aligned}
rV_{\overline{b}} &= -\overline{\kappa} V_{\overline{b}} + \sum_{i=1}^{2} \lambda \mu_{si}(V_{\overline{\ell} i} - p_i - V_{\overline{b}}) && \text{(C1)}\\
rV_{\overline{\ell} i} &= \delta + \overline{x} - y + \overline{\kappa}(V_{\overline{s} i} - V_{\overline{\ell} i}) + \nu_i \mu_{\underline{bo}}(V_{\overline{n}\underline{s} i} - V_{\overline{\ell} i}) && \text{(C2)}\\
rV_{\overline{n}\underline{s} i} &= \delta + \overline{x} - y + w_i + \overline{\kappa}(V_{\overline{s} i} - V_{\overline{n}\underline{s} i}) + \underline{\kappa}(V_{\overline{\ell} i} - V_{\overline{n}\underline{s} i}) + \lambda \mu_{bi}(V_{\overline{n}\underline{n} i} - V_{\overline{n}\underline{s} i}) && \text{(C3)}\\
rV_{\overline{n}\underline{n} i} &= \delta + \overline{x} - y + w_i + \overline{\kappa}(C_i - V_{\overline{n}\underline{n} i}) + \underline{\kappa}(V_{\overline{n}\underline{b} i} - V_{\overline{n}\underline{n} i}) && \text{(C4)}\\
rV_{\overline{n}\underline{b} i} &= \delta + \overline{x} - y + w_i + \overline{\kappa}(C_i - V_{\overline{n}\underline{b} i}) + \lambda \mu_{si}(V_{\overline{\ell} i} - V_{\overline{n}\underline{b} i}) && \text{(C5)}\\
rV_{\overline{s} i} &= \delta - y + \lambda \mu_{bi}(p_i - V_{\overline{s} i}) && \text{(C6)}\\
rV_{\underline{bo}} &= -\underline{\kappa} V_{\underline{bo}} + \sum_{i=1}^{2} \nu_i \mu_{\overline{\ell} i}(V_{\underline{s} i} - V_{\underline{bo}}) && \text{(C7)}\\
rV_{\underline{s} i} &= -w_i + \overline{\kappa}(V_{\underline{bo}} - V_{\underline{s} i}) - \underline{\kappa} V_{\underline{s} i} + \lambda \mu_{bi}(V_{\underline{n} i} + p_i - V_{\underline{s} i}) && \text{(C8)}\\
rV_{\underline{n} i} &= -\delta + \underline{x} - y - w_i + \overline{\kappa}(V_{\underline{bo}} - C_i - V_{\underline{n} i}) + \underline{\kappa}(V_{\underline{b} i} - V_{\underline{n} i}) && \text{(C9)}\\
rV_{\underline{b} i} &= -\delta - y - w_i + \overline{\kappa}(-C_i - V_{\underline{b} i}) + \lambda \mu_{si}(-p_i - V_{\underline{b} i}), && \text{(C10)}
\end{aligned}$$

where C_i denotes the cash collateral seized by the lender when the borrower cannot deliver instantly.

The lending fee w_i is such that the lender receives a fraction $\theta \in [0, 1]$ of the surplus Σ_i in a repo transaction. Since a repo transaction turns the lender $\overline{\ell} i$ into type $\overline{n}\underline{s} i$, the lender's surplus is $V_{\overline{n}\underline{s}i} - V_{\overline{\ell}i}$. The borrower's surplus is $V_{\underline{s}i} - V_{\underline{bo}}$ because the borrower $\underline{bo}$ becomes a seller $\underline{s}i$. Therefore, the lending fee is implicitly defined by

$$V_{\overline{n}\underline{s}i} - V_{\overline{\ell}i} = \theta\Sigma_i = \theta(V_{\overline{n}\underline{s}i} - V_{\overline{\ell}i} + V_{\underline{s}i} - V_{\underline{bo}}). \tag{C11}$$

The price is determined by (2). The reservation value of type $\overline{b}$ is $\Delta_{\overline{b}} = V_{\overline{\ell}i} - V_{\overline{b}}$ because after buying the asset, $\overline{b}$ becomes a lender $\overline{\ell}i$. The reservation value of type $\overline{s}i$ is $\Delta_{\overline{s}i} = V_{\overline{s}i}$ because after selling the asset, $\overline{s}i$ exits the market with utility zero. Substituting in (2), we find

$$p_i = \phi V_{\overline{s}i} + (1-\phi)\left(V_{\overline{\ell}i} - V_{\overline{b}}\right). \tag{C12}$$

Using (C1)-(C12), we compute below the lending fee w_i and the price p_i as a function of the short-selling surplus Σ_i. We then derive a linear system for Σ_1 and Σ_2.

C.1. Lending Fee

Subtracting (C2) from (C3), we find

$$(r + \overline{\kappa} + \underline{\kappa} + \nu_i\mu_{\underline{bo}})(V_{\overline{n}\underline{s}i} - V_{\overline{\ell}i}) = w_i + \lambda\mu_{bi}(V_{\overline{n}\underline{n}i} - V_{\overline{n}\underline{s}i}), \tag{C13}$$

subtracting (C3) from (C4), we find

$$(r + \overline{\kappa} + \underline{\kappa} + \lambda\mu_{bi})(V_{\overline{n}\underline{n}i} - V_{\overline{n}\underline{s}i}) = \overline{\kappa}(C_i - V_{\overline{s}i}) + \underline{\kappa}(V_{\overline{n}\underline{b}i} - V_{\overline{\ell}i}), \tag{C14}$$

and subtracting (C4) from (C5), we find

$$(r + \overline{\kappa} + \underline{\kappa})(V_{\overline{n}\underline{b}i} - V_{\overline{n}\underline{n}i}) = \lambda\mu_{si}(V_{\overline{\ell}i} - V_{\overline{n}\underline{b}i}). \tag{C15}$$

Eqs. (C14) and (C15) imply that

$$V_{\overline{n}\underline{b}i} - V_{\overline{n}\underline{s}i} = \frac{\overline{\kappa}}{r + \overline{\kappa} + \underline{\kappa} + \lambda\mu_{bi}}(C_i - V_{\overline{s}i}) + \frac{\underline{\kappa}(r + \overline{\kappa} + \underline{\kappa}) - \lambda\mu_{si}(r + \overline{\kappa} + \underline{\kappa} + \lambda\mu_{bi})}{(r + \overline{\kappa} + \underline{\kappa})(r + \overline{\kappa} + \underline{\kappa} + \lambda\mu_{bi})}(V_{\overline{n}\underline{b}i} - V_{\overline{\ell}i}).$$

Adding $V_{\overline{n}\underline{s}i} - V_{\overline{\ell}i}$ on both sides and solving for $V_{\overline{n}\underline{b}i} - V_{\overline{\ell}i}$, we find

$$V_{\overline{n}\underline{b}i} - V_{\overline{\ell}i} = \frac{(r + \overline{\kappa} + \underline{\kappa})(r + \overline{\kappa} + \underline{\kappa} + \lambda\mu_{bi})}{(r + \overline{\kappa} + \underline{\kappa})(r + \overline{\kappa} + \lambda\mu_{bi}) + \lambda\mu_{si}(r + \overline{\kappa} + \underline{\kappa} + \lambda\mu_{bi})}\left[\frac{\overline{\kappa}(C_i - V_{\overline{s}i})}{r + \overline{\kappa} + \underline{\kappa} + \lambda\mu_{bi}} + V_{\overline{n}\underline{s}i} - V_{\overline{\ell}i}\right].$$

Substituting $V_{\overline{n}\underline{b}i} - V_{\overline{\ell}i}$ from this equation into (C14), we find

$$V_{\overline{n}\underline{n}i} - V_{\overline{n}\underline{s}i} = \frac{\overline{\kappa}(r + \overline{\kappa} + \underline{\kappa} + \lambda\mu_{si})(C_i - V_{\overline{s}i}) + \underline{\kappa}(r + \overline{\kappa} + \underline{\kappa})(V_{\overline{n}\underline{s}i} - V_{\overline{\ell}i})}{(r + \overline{\kappa} + \underline{\kappa})(r + \overline{\kappa} + \lambda\mu_{bi}) + \lambda\mu_{si}(r + \overline{\kappa} + \underline{\kappa} + \lambda\mu_{bi})}.$$

Substituting $V_{\overline{n}\underline{n}i} - V_{\overline{n}\underline{s}i}$ from this equation into (C13), and using (C11), we can determine the lending fee as a function of the short-selling surplus:

$$\begin{aligned} & \left[r + \overline{\kappa} + \underline{\kappa}\frac{(r+\overline{\kappa})(r+\overline{\kappa}+\underline{\kappa}+\lambda\mu_{si}) + \lambda\mu_{si}(\underline{\kappa}+\lambda\mu_{bi})}{(r+\overline{\kappa}+\lambda\mu_{bi})(r+\overline{\kappa}+\underline{\kappa}) + \lambda\mu_{si}(r+\overline{\kappa}+\underline{\kappa}+\lambda\mu_{bi})} + \nu_i\mu_{\underline{bo}}\right]\theta\Sigma_i \\ = \quad & w_i + \frac{\overline{\kappa}\lambda\mu_{bi}(r+\overline{\kappa}+\underline{\kappa}+\lambda\mu_{si})}{(r+\overline{\kappa}+\underline{\kappa})(r+\overline{\kappa}+\lambda\mu_{bi}) + \lambda\mu_{si}(r+\overline{\kappa}+\underline{\kappa}+\lambda\mu_{bi})}(C_i - V_{\overline{s}i}). \end{aligned} \tag{C16}$$

C.2. Price

Eq. (C6) implies that

$$V_{\overline{s}i} - p_i = \frac{\delta - y - rp_i}{r + \lambda\mu_{bi}}. \tag{C17}$$

Subtracting rp_i from both sides of (C2), and using (C11) and (C17), we find

$$V_{\overline{\ell}i} - p_i = \frac{1}{r+\overline{\kappa}}\left[\delta + \overline{x} - y - rp_i + \nu_i\mu_{\underline{bo}}\theta\Sigma_i + \overline{\kappa}\frac{\delta - y - rp_i}{r+\lambda\mu_{bi}}\right]. \tag{C18}$$

Substituting (C17) and (C18) into (C12), we find

$$\delta - y - rp_i + \frac{(1-\phi)(r+\lambda\mu_{bi})}{r+\overline{\kappa}+(1-\phi)\lambda\mu_{bi}}\left[\overline{x} + \nu_i\mu_{\underline{bo}}\theta\Sigma_i - (r+\overline{\kappa})V_{\overline{b}}\right] = 0. \tag{C19}$$

Substituting $d - y - rp_i$ from (C19) into (C18), we find

$$V_{\overline{\ell}i} - p_i = \frac{\phi(\overline{x} + \nu_i\mu_{\underline{bo}}\theta\Sigma_i) + (1-\phi)(r+\overline{\kappa}+\lambda\mu_{bi})V_{\overline{b}}}{r+\overline{\kappa}+(1-\phi)\lambda\mu_{bi}}.$$

Substituting $V_{\overline{\ell}i} - p_i$ from this equation into (C1) and solving for $V_{\overline{b}}$, we find

$$V_{\overline{b}} = \frac{\phi\sum_{j=1}^2 \frac{\lambda\mu_{sj}}{r+\overline{\kappa}+(1-\phi)\lambda\mu_{bj}}(\overline{x} + \nu_j\mu_{\underline{bo}}\theta\Sigma_j)}{(r+\overline{\kappa})\left[1 + \phi\sum_{j=1}^2 \frac{\lambda\mu_{sj}}{r+\overline{\kappa}+(1-\phi)\lambda\mu_{bj}}\right]}.$$

Substituting $V_{\overline{b}}$ from this equation into (C19), we can determine the price as a function of the short-selling surplus:

$$p_i = \frac{\delta - y}{r} + \frac{(1-\phi)(r+\lambda\mu_{bi})}{r\left[r+\overline{\kappa}+(1-\phi)\lambda\mu_{bi}\right]}\left[\overline{x} + \nu_i\mu_{\underline{bo}}\theta\Sigma_i - \frac{\phi\sum_{j=1}^2 \frac{\lambda\mu_{sj}}{r+\overline{\kappa}+(1-\phi)\lambda\mu_{bj}}(\overline{x} + \nu_j\mu_{\underline{bo}}\theta\Sigma_j)}{1 + \phi\sum_{j=1}^2 \frac{\lambda\mu_{sj}}{r+\overline{\kappa}+(1-\phi)\lambda\mu_{bj}}}\right]. \tag{C20}$$

C.3. Short-Selling Surplus

Adding (C3) and (C8), and subtracting (C7) and (C2), we find

$$(r + \overline{\kappa} + \underline{\kappa} + \nu_i \mu_{\underline{bo}} \theta)\Sigma_i + \sum_{j=1}^{2} \nu_j \mu_{\overline{\ell}j}(1-\theta)\Sigma_j = \lambda\mu_{bi}(V_{\overline{n}\underline{n}i} + V_{\underline{n}i} + p_i - V_{\overline{n}\underline{s}i} - V_{\underline{s}i}). \tag{C21}$$

Adding (C4), (C9), and $rp_i = rp_i$, and subtracting (C3) and (C8), we find

$$(r+\overline{\kappa}+\underline{\kappa}+\lambda\mu_{bi})(V_{\overline{n}\underline{n}i}+V_{\underline{n}i}+p_i-V_{\overline{n}\underline{s}i}-V_{\underline{s}i}) = rp_i-\delta+\underline{x}-y+\overline{\kappa}(p_i-V_{\overline{s}i})+\underline{\kappa}(V_{\overline{n}\underline{b}i}+V_{\underline{b}i}+p_i-V_{\overline{\ell}i}). \tag{C22}$$

Adding (C5), (C10), and $rp_i = rp_i$, and subtracting (C2), we find

$$(r + \overline{\kappa} + \lambda\mu_{si})(V_{\overline{n}\underline{b}i} + V_{\underline{b}i} + p_i - V_{\overline{\ell}i}) = rp_i - \delta - y + \overline{\kappa}(p_i - V_{\overline{s}i}) - \nu_i\mu_{\underline{bo}}\theta\Sigma_i. \tag{C23}$$

Substituting $V_{\overline{n}\underline{b}i}+V_{\underline{b}i}+p_i-V_{\overline{\ell}i}$ from (C23) into (C22), and then substituting $V_{\overline{n}\underline{n}i}+V_{\underline{n}i}+p_i-V_{\overline{n}\underline{s}i}-V_{\underline{s}i}$ from (C22) into (C21), we find

$$\begin{aligned} &\left[r + \overline{\kappa} + \underline{\kappa} + \nu_i\mu_{\underline{bo}}\theta\left[1 + \frac{\lambda\mu_{bi}\underline{\kappa}}{(r+\overline{\kappa}+\underline{\kappa}+\lambda\mu_{bi})(r+\overline{\kappa}+\lambda\mu_{si})}\right]\right]\Sigma_i + \sum_{j=1}^{2}\nu_j\mu_{\overline{\ell}j}(1-\theta)\Sigma_j \\ = \quad & \frac{\lambda\mu_{bi}}{r+\overline{\kappa}+\underline{\kappa}+\lambda\mu_{bi}}\left[\underline{x} + \frac{r+\overline{\kappa}+\underline{\kappa}+\lambda\mu_{si}}{r+\overline{\kappa}+\lambda\mu_{si}}\left[rp_i - \delta - y + \overline{\kappa}(p_i - V_{\overline{s}i})\right]\right]. \end{aligned} \tag{C24}$$

To derive an equation involving only Σ_1 and Σ_2, we need to eliminate the price p_i. We have

$$\begin{aligned} & rp_i - \delta - y + \overline{\kappa}(p_i - V_{\overline{s}i}) \\ = \quad & -2y + rp_i - \delta + y + \overline{\kappa}\frac{rp_i - \delta + y}{r + \lambda\mu_{bi}} \\ = \quad & -2y + \frac{(1-\phi)(r+\overline{\kappa}+\lambda\mu_{bi})}{r+\overline{\kappa}+(1-\phi)\lambda\mu_{bi}}\left[\overline{x} + \nu_i\mu_{\underline{bo}}\theta\Sigma_i - \frac{\phi\sum_{j=1}^{2}\frac{\lambda\mu_{sj}}{r+\overline{\kappa}+(1-\phi)\lambda\mu_{bj}}(\overline{x}+\nu_j\mu_{\underline{bo}}\theta\Sigma_j)}{1+\phi\sum_{j=1}^{2}\frac{\lambda\mu_{sj}}{r+\overline{\kappa}+(1-\phi)\lambda\mu_{bj}}}\right], \end{aligned}$$

where the first step follows from (C17) and the second from (C20). Plugging back into (C24), we can write it as

$$a_i\Sigma_i + \sum_{j=1}^{2} f_j\Sigma_j + b_i\sum_{j=1}^{2} g_j\Sigma_j = c_i, \tag{C25}$$

where

$$a_i = r+\overline{\kappa}+\underline{\kappa}+\nu_i\mu_{\underline{bo}}\theta\left[\frac{r+\overline{\kappa}+\underline{\kappa}}{r+\overline{\kappa}+\underline{\kappa}+\lambda\mu_{bi}} + \frac{\phi(r+\overline{\kappa})\lambda\mu_{bi}(r+\overline{\kappa}+\underline{\kappa}+\lambda\mu_{si})}{(r+\overline{\kappa}+\underline{\kappa}+\lambda\mu_{bi})(r+\overline{\kappa}+\lambda\mu_{si})[r+\overline{\kappa}+(1-\phi)\lambda\mu_{bi}]}\right],$$

$$f_i = \nu_i \mu_{\overline{\ell} i}(1-\theta),$$

$$b_i = \frac{(1-\phi)\lambda\mu_{bi}(r+\overline{\kappa}+\underline{\kappa}+\lambda\mu_{si})(r+\overline{\kappa}+\lambda\mu_{bi})}{(r+\overline{\kappa}+\underline{\kappa}+\lambda\mu_{bi})(r+\overline{\kappa}+\lambda\mu_{si})\left[r+\overline{\kappa}+\lambda(1-\phi)\lambda\mu_{bi}\right]},$$

$$g_i = \phi\nu_i\mu_{\underline{bo}}\theta \frac{\frac{\lambda\mu_{si}}{r+\overline{\kappa}+(1-\phi)\lambda\mu_{bi}}}{1+\phi\sum_{j=1}^{2}\frac{\lambda\mu_{sj}}{r+\overline{\kappa}+(1-\phi)\lambda\mu_{bj}}},$$

$$c_i = \frac{\lambda\mu_{bi}}{r+\overline{\kappa}+\underline{\kappa}+\lambda\mu_{bi}}\left[\underline{x} - \frac{r+\overline{\kappa}+\underline{\kappa}+\lambda\mu_{si}}{r+\overline{\kappa}+\lambda\mu_{si}}\left[2y - \frac{(1-\phi)(r+\overline{\kappa}+\lambda\mu_{bi})}{r+\overline{\kappa}+(1-\phi)\lambda\mu_{bi}}\frac{\overline{x}}{1+\phi\sum_{j=1}^{2}\frac{\lambda\mu_{sj}}{r+\overline{\kappa}+(1-\phi)\lambda\mu_{sj}}}\right]\right].$$

The short-selling surpluses Σ_1 and Σ_2 are the solution to the linear system consisting of (C25) for $i \in \{1,2\}$.

Note that the collateral C_i does not enter in (C25), and thus does not affect the short-selling surplus. It neither affects the price, from (C20). It affects only the lending fee because when lenders can seize more collateral they accept a lower fee. From now on (and as stated in Footnote 14), we set the collateral equal to the utility of a seller $\overline{s}i$, i.e.,

$$C_i = V_{\overline{s}i}. \tag{C26}$$

D. Proofs of Propositions 5-11

Proof of Proposition 5: From Appendix B we know that given the short-selling decisions $\nu_1 = \nu_2 = \nu$, the population measures are uniquely determined. From Appendix C we know that given any short-selling decisions and population measures, the utilities, prices, and lending fees are uniquely determined. Therefore, what is left to show is (i) the short-selling surplus Σ is positive, (ii) buyers' and sellers' reservation values are ordered as in (1), and (iii) agents' trading strategies are optimal. To show these results, we recall from Appendix B that when search frictions become small, i.e., λ goes to ∞ holding $n \equiv \nu/\lambda$ constant, μ_b converges to m_b, $\mu_{\overline{\ell}}$ converges to S, $\lambda\mu_s$ converges to g_s, and $\nu\mu_{\underline{bo}}$ converges to $g_{\underline{bo}}$.

We start by computing Σ, w, and p, thus proving Proposition 6. Eq. (C25) implies that when $\Sigma_1 = \Sigma_2 \equiv \Sigma$,

$$\Sigma = \frac{c}{a+2(f+bg)},$$

where we suppress the asset subscripts from a, b, c, f, g because of symmetry. When search frictions become small, a and b converge to positive limits, c converges to

$$\underline{x} - \frac{r + \overline{\kappa} + \underline{\kappa} + g_s}{r + \overline{\kappa} + g_s}(2y - \overline{x}), \tag{D1}$$

g converges to zero, and f converges to ∞, being asymptotically equal to $\nu S(1-\theta)$. Therefore, the surplus converges to zero, and its asymptotic behavior is as in Proposition 6.

Eqs. (C16) and (C26) imply that the lending fee is

$$w = \left[r + \overline{\kappa} + \underline{\kappa} \frac{(r+\overline{\kappa})(r + \overline{\kappa} + \underline{\kappa} + \lambda\mu_s) + \lambda\mu_s(\underline{\kappa} + \lambda\mu_b)}{(r + \overline{\kappa} + \lambda\mu_b)(r + \overline{\kappa} + \underline{\kappa}) + \lambda\mu_s(r + \overline{\kappa} + \underline{\kappa} + \lambda\mu_b)} + \nu\mu_{\underline{bo}} \right] \theta\Sigma.$$

Because the term in brackets converges to

$$r + \overline{\kappa} + \underline{\kappa} \frac{g_s}{r + \overline{\kappa} + \underline{\kappa} + g_s} + g_{\underline{bo}},$$

the lending fee converges to zero, and its asymptotic behavior is as in Proposition 6.

Eq. (C20) implies that the price is equal to

$$p = \frac{\delta - y}{r} + \frac{1}{r}\left[1 - \frac{\phi r + \overline{\kappa}}{(1-\phi)\lambda m_b} + o(1/\lambda)\right]\left[\overline{x} + g_{\underline{bo}}\theta\Sigma - \frac{2\phi g_s \overline{x}}{(1-\phi)\lambda m_b} + o(1/\lambda)\right].$$

Using this equation and the fact that Σ is in order $1/\lambda$, it is easy to check that the asymptotic behavior (i.e., order $1/\lambda$) of the price is as in Proposition 6.

To show that Σ is positive, we need to show that (D1) is positive. This follows because (4) implies that

$$\underline{x} > 2y + \frac{\underline{\kappa}}{r + \overline{\kappa} + g_s}(2y - \overline{x}) > 2y - \overline{x} + \frac{\underline{\kappa}}{r + \overline{\kappa} + g_s}(2y - \overline{x}) = \frac{r + \overline{\kappa} + \underline{\kappa} + g_s}{r + \overline{\kappa} + g_s}(2y - \overline{x}). \tag{D2}$$

We next show that reservation values are ordered as in (1), i.e., $\Delta_{\underline{b}} > \Delta_{\overline{b}}$ and $\Delta_{\overline{s}} > \Delta_{\underline{s}}$. For this, we need to compute $V_{\underline{b}}$ and $V_{\underline{n}} - V_{\underline{s}}$. Adding (C10) and $rp = rp$, and using (C26), we find

$$V_{\underline{b}} + p = \frac{rp - \delta - y - w + \overline{\kappa}(p - V_{\overline{s}})}{r + \overline{\kappa} + \lambda\mu_s}. \tag{D3}$$

Adding (C9) and $rp = rp$, and subtracting (C8), we similarly find

$$V_{\underline{n}} + p - V_{\underline{s}} = \frac{rp - \delta + \underline{x} - y + \underline{\kappa}(V_{\underline{b}} + p) + \overline{\kappa}(p - V_{\overline{s}})}{r + \overline{\kappa} + \underline{\kappa} + \lambda\mu_b}. \tag{D4}$$

Inequality $\Delta_{\underline{b}} > \Delta_{\overline{b}}$ is equivalent to

$$\begin{aligned}
& -V_{\underline{b}} - p > V_{\overline{\ell}} - p - V_{\overline{b}} \\
\Leftrightarrow \quad & \frac{\delta + y - rp + w - \overline{\kappa}(p - V_{\overline{s}})}{r + \overline{\kappa} + \lambda\mu_s} > \frac{\phi}{1-\phi}(p - V_{\overline{s}}) \\
\Leftrightarrow \quad & \frac{\delta + y - rp + w - \overline{\kappa}\frac{rp-\delta+y}{r+\lambda\mu_b}}{r + \overline{\kappa} + \lambda\mu_s} > \frac{\phi}{1-\phi}\frac{rp - \delta + y}{r + \lambda\mu_b}
\end{aligned} \tag{D5}$$

where the second step follows from (C12) and (D3), and the third from (C17). Because rp converges to $\delta + \overline{x} - y$, and w converges to zero, the LHS of (D5) converges to $(2y - \overline{x})/(r + \overline{\kappa} + g_s)$, which is positive from Assumption 1, while the RHS converges to zero. Inequality $\Delta_{\overline{s}} > \Delta_{\underline{s}}$ is equivalent to

$$\begin{aligned}
& V_{\underline{n}} + p - V_{\underline{s}} > p - V_{\overline{s}} \\
\Leftrightarrow \quad & \frac{\underline{x} + \frac{r+\overline{\kappa}+\underline{\kappa}+\lambda\mu_s}{r+\underline{\kappa}+\lambda\mu_s}[rp - \delta - y + \overline{\kappa}(p - V_{\overline{s}})] - \frac{\underline{\kappa}}{r+\overline{\kappa}+\lambda\mu_s}w}{r + \overline{\kappa} + \underline{\kappa} + \lambda\mu_b} > \frac{rp - \delta + y}{r + \lambda\mu_b},
\end{aligned}$$

where the second step follows from (C17), (D3), and (D4). When search frictions become small, this inequality holds if the limit of the numerator in the LHS exceeds that for the RHS, i.e.,

$$\underline{x} - \frac{r + \overline{\kappa} + \underline{\kappa} + g_s}{r + \underline{\kappa} + g_s}(2y - \overline{x}) > \overline{x}.$$

This holds because of the first inequality in (D2).

We finally show that trading strategies are optimal. The flow benefit that an average-valuation agent can derive from a long position in asset i is bounded above by $\delta - y + w$, and the flow benefit for a short position is bounded above by $-\delta - y$. Therefore, an average-valuation agent finds it optimal to establish no position, or to unwind a previously established one, if $(\delta - y + w)/r < \min\{p, C\}$ and $(\delta + y)/r > p$. These conditions are satisfied for small frictions because p converges to $(\delta + \overline{x} - y)/r$, w converges to zero, $C - p$ converges to zero, and $2y > \overline{x}$.

A high-valuation agent finds it optimal to buy asset i if $V_{\overline{\ell}} - p - V_{\overline{b}} \geq 0$. This condition is satisfied because

$$V_{\overline{\ell}} - p - V_{\overline{b}} = \frac{\phi}{1-\phi}(p - V_{\overline{s}}) = \frac{\phi}{1-\phi}\frac{rp - \delta + y}{r + \lambda\mu_b} \sim \frac{\phi}{1-\phi}\frac{\overline{x}}{\lambda\mu_b} \geq 0.$$

The agent finds it optimal to lend the asset because $V_{\overline{n}\underline{s}} - V_{\overline{\ell}} = \theta\Sigma > 0$. Likewise, a low-valuation agent finds it optimal to borrow asset i because $V_{\underline{s}} - V_{\underline{bo}} = (1-\theta)\Sigma > 0$, and to sell it because $V_{\underline{n}} + p - V_{\underline{s}} = p - \Delta_{\underline{s}} > p - \Delta_{\overline{s}} = p - V_{\overline{s}} > 0$. ∎

Proof of Proposition 6: See the proof of Proposition 5. ∎

Proof of Proposition 7: We need to show that (i) the short-selling surplus Σ_1 is positive and Σ_2 is negative, (ii) buyers' and sellers' reservation values are ordered as in (1), and (iii) agents' trading strategies are optimal. We recall from Appendix B that for small search frictions and given the short-selling decisions $\nu_1 = \nu$ and $\nu_2 = 0$, μ_{bi} converges to $\hat{m}_{bi}$, $\mu_{\overline{\ell}i}$ converges to S, $\lambda\mu_{si}$ converges to $\hat{g}_{si}$, and $\nu\mu_{\underline{bo}}$ converges to $\hat{g}_{\underline{bo}}$.

We start by computing Σ_1, w_1, p_1, and p_2, thus proving Proposition 8. Eq. (C25) implies that when $\nu_2 = 0$,

$$\Sigma_1 = \frac{c_1}{a_1 + f_1 + b_1 g_1}.$$

When search frictions become small, c_1 converges to

$$\underline{x} - \frac{r + \overline{\kappa} + \underline{\kappa} + \hat{g}_{s1}}{r + \overline{\kappa} + \hat{g}_{s1}}(2y - \overline{x}), \tag{D6}$$

and the dominant term in the denominator is $f_1 \sim \nu S(1-\theta)$. Therefore, the surplus converges to zero, and its asymptotic behavior is as in Proposition 8. To determine the asymptotic behavior of the lending fee and the price, we proceed as in the proof of Proposition 5.

To show that Σ_1 is positive, we need to show that (D6) is positive. This follows from (D2) and the fact that $\hat{g}_{s1} > g_s$, established in the proof of Proposition 9. To show that Σ_2 is negative, we note that from (C25),

$$\Sigma_2 = \frac{c_2 - (f_1 + b_2 g_1)\Sigma_1}{a_2} = \frac{c_2 - \frac{f_1 + b_2 g_1}{a_1 + f_1 + b_1 g_1} c_1}{a_2}.$$

When search frictions become small, the numerator converges to the same limit as $c_2 - c_1$. This limit is equal to

$$\left[\frac{r + \overline{\kappa} + \underline{\kappa} + \hat{g}_{s1}}{r + \overline{\kappa} + \hat{g}_{s1}} - \frac{r + \overline{\kappa} + \underline{\kappa} + \hat{g}_{s2}}{r + \overline{\kappa} + \hat{g}_{s2}}\right](2y - \overline{x}),$$

and is negative if $\hat{g}_{s1} > \hat{g}_{s2}$. Using (B36) and (B38), we can write this inequality as

$$\frac{\overline{\kappa}S + \frac{\overline{\kappa}+\underline{\kappa}}{\underline{\kappa}}\underline{F}}{\hat{m}_{b1}} > \frac{\overline{\kappa}S}{\hat{m}_{\overline{b}}}. \tag{D7}$$

Eqs. (B38)-(B40) imply that

$$\hat{m}_{\overline{b}} = \frac{\overline{F} - \overline{\kappa}S}{\overline{F} - \overline{\kappa}S + \underline{F}}\hat{m}_{b1}. \tag{D8}$$

Using this equation, we can write (D7) as

$$\frac{\overline{\kappa}S + \frac{\overline{\kappa}+\underline{\kappa}}{\underline{\kappa}}\underline{F}}{\overline{\kappa}S} > \frac{\overline{F} - \overline{\kappa}S + \underline{F}}{\overline{F} - \overline{\kappa}S}.$$

It is easy to check that this inequality holds because of Assumption 2.

To show that $\Delta_{\underline{b}i} > \Delta_{\overline{b}}$ and $\Delta_{\overline{s}i} > \Delta_{\underline{s}i}$, we proceed as in the proof of Proposition 5. The only change is that the condition for $\Delta_{\overline{s}i} > \Delta_{\underline{s}i}$ now is

$$\underline{x} - \frac{r + \overline{\kappa} + \underline{\kappa} + \hat{g}_s}{r + \underline{\kappa} + \hat{g}_s}(2y - \overline{x}) > \overline{x}.$$

This inequality is implied by the first inequality in (D2) and the fact that $\hat{g}_{s1} > g_s$. Finally, the arguments in the proof of Proposition 5 establish that trading strategies are optimal. ∎

Proof of Proposition 8: See the proof of Proposition 7. ∎

Proof of Proposition 9: We start with a lemma.

Lemma 1. *For $\chi < 1$, inequality $(1-\chi)\hat{m}_{b1} > m_b$ is equivalent to*

$$(1 - 2\chi)(\underline{F} - \chi\overline{\kappa}\hat{m}_{b1}) > \chi\overline{F}. \tag{D9}$$

Proof: Since m_b is the unique positive solution of (B35), whose RHS is decreasing in m_b, inequality $(1-\chi)\hat{m}_{b1} > m_b$ is equivalent to

$$\begin{aligned}
& 1 > \frac{\overline{F}}{\overline{\kappa}(1-\chi)\hat{m}_{b1} + 2\overline{\kappa}S + \frac{\overline{\kappa}+\underline{\kappa}}{\underline{\kappa}}\underline{F}} + \frac{\underline{F}}{2\overline{\kappa}(1-\chi)\hat{m}_{b1} + 2\overline{\kappa}S + \frac{\overline{\kappa}+\underline{\kappa}}{\underline{\kappa}}\underline{F}} \\
\Leftrightarrow \quad & 1 > \frac{\overline{F}}{\overline{F} + \underline{F} - \chi\overline{\kappa}\hat{m}_{b1}} + \frac{\underline{F}}{\overline{F} + \underline{F} + (1-2\chi)\overline{\kappa}\hat{m}_{b1}} \\
\Leftrightarrow \quad & \frac{\underline{F} - \chi\overline{\kappa}\hat{m}_{b1}}{\overline{F} + \underline{F} - \chi\overline{\kappa}\hat{m}_{b1}} > \frac{\underline{F}}{\overline{F} + \underline{F} + (1-2\chi)\overline{\kappa}\hat{m}_{b1}},
\end{aligned}$$

where the second step follows from (B40). The last inequality implies (D9). ∎

Result (i): We need to show that $\hat{m}_{b1} > m_b$ and $\hat{g}_{s1} > g_s$. Since (D9) holds for $\chi = 0$, Lemma 1 implies that $\hat{m}_{b1} > m_b$. Using (B33) and (B38), we can write inequality $\hat{g}_{s1} > g_s$ as

$$\frac{\overline{\kappa}S + \frac{\overline{\kappa}+\underline{\kappa}}{2\underline{\kappa}}\underline{F}}{\overline{\kappa}S + \frac{\overline{\kappa}+\underline{\kappa}}{\underline{\kappa}}\underline{F}}\hat{m}_{b1} < m_b.$$

Using Lemma 1, we then need to show that

$$(1-2\chi)(\underline{F} - \chi\overline{\kappa}\hat{m}_{b1}) < \chi\overline{F}, \tag{D10}$$

for

$$\chi = \frac{\frac{\overline{\kappa}+\underline{\kappa}}{2\underline{\kappa}}\underline{F}}{\overline{\kappa}S + \frac{\overline{\kappa}+\underline{\kappa}}{\underline{\kappa}}\underline{F}}.$$

Plugging for χ, we can write (D10) as

$$\overline{\kappa}S(\underline{F} - \chi\overline{\kappa}\hat{m}_{b1}) < \frac{\overline{\kappa}+\underline{\kappa}}{2\underline{\kappa}}\underline{F}\overline{F},$$

which holds because of Assumption 2 and $\hat{m}_{b1} > 0$.

Result (ii): We need to show that $\hat{m}_{b2} < m_b$ and $\hat{g}_{s2} < g_s$. Using (D8) and $\hat{m}_{b2} = \hat{m}_{\overline{b}}$, we can write inequality $\hat{m}_{b2} < m_b$ as

$$\frac{\overline{F} - \overline{\kappa}S}{\overline{F} - \overline{\kappa}S + \underline{F}}\hat{m}_{b1} < m_b.$$

Using Lemma 1, we then need to show (D10) for

$$\chi = \frac{\underline{F}}{\overline{F} - \overline{\kappa}S + \underline{F}}.$$

Plugging for χ, we can write (D10) as

$$\frac{\overline{F} - \overline{\kappa}S - \underline{F}}{\overline{F} - \overline{\kappa}S + \underline{F}}\left(\overline{F} - \overline{\kappa}S + \underline{F} - \overline{\kappa}\hat{m}_{b1}\right) < \overline{F},$$

which holds because $\hat{m}_{b1} > 0$. Using (B33), (B36), and (D8), we can write inequality $\hat{g}_{s2} < g_s$ as

$$\frac{\overline{F} - \overline{\kappa}S}{\overline{F} - \overline{\kappa}S + \underline{F}}\frac{\overline{\kappa}S + \frac{\overline{\kappa}+\underline{\kappa}}{2\underline{\kappa}}\underline{F}}{\overline{\kappa}S}\hat{m}_{b1} > m_b.$$

Using Lemma 1, we then need to show (D9) for

$$\chi = \frac{\underline{F}}{\overline{F} - \overline{\kappa}S + \underline{F}} \left(1 - \frac{\overline{\kappa} + \underline{\kappa}}{2\underline{\kappa}} \frac{\overline{F} - \overline{\kappa}S}{\overline{\kappa}S}\right).$$

Assumption 2 implies that

$$\chi < \frac{\underline{F}}{\overline{F} - \overline{\kappa}S + \underline{F}} \left(1 - \frac{\overline{\kappa} + \underline{\kappa}}{2\underline{\kappa}}\right) < \frac{\underline{F}}{2(\overline{F} - \overline{\kappa}S + \underline{F})} \equiv \hat{\chi}.$$

Because $\hat{\chi}, \hat{m}_{b1} > 0$, (D9) holds for χ if it holds for $\hat{\chi}$. The latter is easy to check using Assumption 2.

Result (iii): Eqs. (7), (11), and $\hat{g}_{s1} > g_s$, imply that Σ_i in the symmetric equilibrium is smaller than Σ_1 in the asymmetric equilibrium. Since, in addition, $\hat{g}_{\underline{bo}} > g_{\underline{bo}}$ (from (B32) and (B37)), (6) and (10) imply that the lending fee w_i in the symmetric equilibrium is smaller than w_1 in the asymmetric equilibrium.

Result (iv): For $\phi = 0$, the result follows from (5), (8), $\hat{m}_{b1} > m_b > \hat{m}_{b2}$, $\hat{g}_{\underline{bo}} > g_{\underline{bo}}$, and the fact that Σ_i in the symmetric equilibrium is smaller than Σ_1 in the asymmetric equilibrium. An example where the prices of both assets are higher in the asymmetric equilibrium is $S = 0.5$, $\overline{F} = 3$, $\underline{F} = 5.7$, $\overline{\kappa} = 1$, $\underline{\kappa} = 3$, $\phi = \theta = 0.5$, $r = 4\%$, $\delta = 1$, $\overline{x} = 0.4$, $\underline{x} = 1.6$, $y = 0.5$, and any ν/λ. ∎

Proof of Proposition 10: We show that buying asset 2 and shorting asset 1 is unprofitable under

$$p_1 - p_2 < \frac{w_1}{r} + \frac{\overline{x}}{\lambda \hat{m}_{b1}} + \frac{\overline{\kappa x}}{r(\nu S + \lambda \hat{m}_{b2})}. \tag{D11}$$

(which is implied by (12)), while buying asset 1 and shorting asset 2 is unprofitable under (13). We then show that (12) and (13) are satisfied if ν/λ is in an interval (n_1, n_2).

Buy asset 2, short asset 1

Because trading opportunities arrive one at a time, an arbitrageur cannot set up the two legs of the position simultaneously. The arbitrageur can, for example, buy asset 2 first, then borrow asset 1, and then sell asset 1. Alternatively, he can borrow asset 1 first, then buy asset 2, and then sell asset 1. The final possibility, which is to sell asset 1 before buying asset 2 is suboptimal. Indeed, for small search frictions the time to meet a buyer converges to zero while the time to meet a seller does not. Therefore, the cost of being unhedged converges to zero only when asset 2 is bought before asset 1 is sold.

Suppose now that the arbitrage strategy is profitable. Because the payoff of the strategy is decreasing in asset 1's lending fee, there exists a fee $\overline{w}_1 > w_1$ for which the arbitrageur is indifferent between following the strategy and holding no position. If for this fee it is optimal to initiate the strategy by buying asset 2, the arbitrageur can be in three possible states:

- Long position in asset 2. State $n2$ with utility V_{n2}.
- Long position in asset 2 and borrowed asset 1. State $s1n2$ with utility V_{s1n2}.
- Long position in asset 2 and short in asset 1. State $n1n2$ with utility V_{n1n2}.

The utilities are characterized by the following flow-value equations:

$$rV_{n2} = \delta - y + \nu\mu_{\overline{\ell}1}(V_{s1n2} - V_{n2}) \tag{D12}$$

$$rV_{s1n2} = \delta - y - \overline{w}_1 + \lambda\mu_{b1}(V_{n1n2} + p_1 - V_{s1n2}) + \overline{\kappa}(V_{n2} - V_{s1n2}) \tag{D13}$$

$$rV_{n1n2} = -\overline{w}_1 + \overline{\kappa}(V_{n2} - C_1 - V_{n1n2}). \tag{D14}$$

Solving (D12)-(D14), we find

$$rV_{n2} = \delta - y + \frac{\nu\mu_{\overline{\ell}1}}{r + \overline{\kappa} + \nu\mu_{\overline{\ell}1}}\left[-\overline{w}_1 + \frac{\lambda\mu_{b1}}{r + \overline{\kappa} + \lambda\mu_{b1}}\left[rp_1 - \delta + y + \overline{\kappa}(p_1 - C_1)\right]\right].$$

The arbitrageur is indifferent between initiating the strategy and holding no position if V_{n2} is equal to p_2. Using this condition, and substituting C_1 from (C17) and (C26), we find

$$\overline{w}_1 = \frac{\lambda\mu_{b2}}{r + \lambda\mu_{b2}}(rp_1 - \delta + y) - \frac{r + \overline{\kappa} + \nu\mu_{\overline{\ell}2}}{\nu\mu_{\overline{\ell}2}}(rp_2 - \delta - y).$$

For small search frictions, this equation becomes

$$\overline{w}_1 = r(p_1 - p_2) - \frac{r\overline{x}}{\lambda\hat{m}_{b1}} - \frac{(r + \overline{\kappa})\overline{x}}{\nu S},$$

and is inconsistent with (D11) since $w_1 < \overline{w}_1$.

Suppose instead that it is optimal to initiate the strategy by borrowing asset 1. The arbitrageur then starts from a state $s1$, in which he has borrowed asset 1 but holds no position in asset 2. The utility V_{s1} in this state is characterized by

$$rV_{s1} = -w_1 + \lambda\mu_{s2}(V_{s1n2} - p_2 - V_{s1}). \tag{D15}$$

The utility in states $s1n2$ and $n1n2$ is given by (D13) and (D14), respectively. The utility in state $n2$, however, is given by

$$rV_{n2} = \delta - y + \nu\mu_{\bar{\ell}1}(V_{s1n2} - V_{n2}) + \lambda\mu_{b2}(p_2 - V_{n2}) \tag{D16}$$

instead of (D12). Indeed, since it suboptimal to initiate the strategy by buying asset 2, buying that asset is dominated by holding no position. Therefore, if the arbitrageur finds himself with a long position in asset 2, he prefers to unwind it upon meeting a seller. Eqs. (D13), (D14), and (D16) imply that

$$V_{s1n2} = \frac{\frac{r+\bar{\kappa}+\nu\mu_{\bar{\ell}1}+\lambda\mu_{b2}}{r+\nu\mu_{\bar{\ell}1}+\lambda\mu_{b2}}(\delta - y) + \frac{\bar{\kappa}\lambda\mu_{b2}}{r+\nu\mu_{\bar{\ell}1}+\lambda\mu_{b2}}p_2 - \overline{w}_1 + \frac{\lambda\mu_{b1}}{r+\bar{\kappa}+\lambda\mu_{b1}}\left[rp_1 - \delta + y + \bar{\kappa}(p_1 - C_1)\right]}{\frac{r(r+\bar{\kappa}+\nu\mu_{\bar{\ell}1}+\lambda\mu_{b2})+\bar{\kappa}\lambda\mu_{b2}}{r+\nu\mu_{\bar{\ell}1}+\lambda\mu_{b2}}}.$$

Plugging into (D15), and using (C17), (C26), and the indifference condition which now is $V_{s1} = 0$, we find

$$\overline{w}_1 = \frac{\frac{\lambda\mu_{b1}}{r+\lambda\mu_{b1}}(rp_1 - \delta + y) - \frac{r+\bar{\kappa}+\nu\mu_{\bar{\ell}1}+\lambda\mu_{b2}}{r+\nu\mu_{\bar{\ell}1}+\lambda\mu_{b2}}(rp_2 - \delta + y)}{1 + \frac{r(r+\bar{\kappa}+\nu\mu_{\bar{\ell}1}+\lambda\mu_{b2})+\bar{\kappa}\lambda\mu_{b2}}{\lambda\mu_{s2}(r+\nu\mu_{\bar{\ell}1}+\lambda\mu_{b2})}}.$$

For small search frictions, this equation becomes

$$\overline{w}_1 = \frac{r(p_1 - p_2) - \frac{r\bar{x}}{\lambda\hat{m}_{b1}} - \frac{\bar{\kappa}\bar{x}}{\nu S+\lambda\hat{m}_{b2}}}{1 + \frac{r(nS+\hat{m}_{b2})+\bar{\kappa}\hat{m}_{b2}}{\hat{g}_{s2}(nS+\hat{m}_{b2})}},$$

and is inconsistent with (D11) since $w_1 < \overline{w}_1$.

Buy asset 1, short asset 2

We consider a "relaxed" problem where asset 1 can be bought instantly and asset 2 can be borrowed instantly at a lending fee of zero. Clearly, if the arbitrage strategy is unprofitable in the relaxed problem, it is also unprofitable when more frictions are present.

Suppose that the arbitrage strategy is profitable. Because the payoff of the strategy is increasing in asset 1's lending fee, there exists a fee $\overline{w}_1 < w_1$ for which the arbitrageur is indifferent between following the strategy and holding no position. When following the strategy, the arbitrageur is always in a state where he holds asset 1 and has borrowed asset 2, because these can be done instantly. If the arbitrageur has not sold asset 2, he can be in four possible states:

- Seeking to lend asset 1. State $\ell1s2$ with utility $V_{\ell1s2}$.

- Lent asset 1 to an agent $\underline{s}1$. State $n\underline{s}1s2$ with utility $V_{n\underline{s}1s2}$.
- Lent asset 1 to an agent $\underline{n}1$. State $n\underline{n}1s2$ with utility $V_{n\underline{n}1s2}$.
- Lent asset 1 to an agent $\underline{b}1$. State $n\underline{b}1s2$ with utility $V_{n\underline{b}1s2}$.

If the arbitrageur has sold asset 2, he can be in the four corresponding states that we denote with $n2$ instead of $s2$.

For brevity, we skip the eight flow-value equations, but note that they have a simple solution. To each outcome concerning asset 1 ($\ell 1$, $n\underline{s}1$, $n\underline{n}1$, $n\underline{b}1$) and to each outcome concerning asset 2 ($s2$, $n2$), we can associate a separate utility that we denote by $\hat{V}$. We can then write the utility of a state (which is a "joint" outcome) as the sum of the two separate utilities. For example, the utility $V_{\ell 1s2}$ is equal to $\hat{V}_{\ell 1}+\hat{V}_{s2}$. This decomposition is possible because the outcomes concerning each asset evolve independently.

The utilities $\hat{V}_{\ell 1}$, $\hat{V}_{n\underline{s}1}$, $\hat{V}_{n\underline{n}1}$, and $\hat{V}_{n\underline{b}1}$ are characterized by the flow-value equations

$$
\begin{aligned}
r\hat{V}_{\ell 1} &= \nu\mu_{\underline{bo}}(\hat{V}_{n\underline{s}1}-\hat{V}_{\ell 1}) \\
r\hat{V}_{n\underline{s}1} &= \overline{w}_1+\lambda\mu_{b1}(\hat{V}_{n\underline{n}1}-\hat{V}_{n\underline{s}1}) \\
r\hat{V}_{n\underline{n}1} &= \overline{w}_1+\underline{\kappa}(\hat{V}_{n\underline{b}1}-\hat{V}_{n\underline{n}1}) \\
r\hat{V}_{n\underline{b}1} &= \overline{w}_1+\lambda\mu_{s1}(\hat{V}_{\ell 1}-\hat{V}_{n\underline{b}1}).
\end{aligned}
$$

and the utilities $\hat{V}_{s2}$, $\hat{V}_{n2}$ are characterized by

$$
\begin{aligned}
r\hat{V}_{s2} &= \delta-y+\lambda\mu_{b2}(\hat{V}_{n2}+p_2-\hat{V}_{s2}) \\
r\hat{V}_{n2} &= \overline{\kappa}(\hat{V}_{s2}-C_2-\hat{V}_{n2}).
\end{aligned}
$$

Solving these equations, we find

$$
\begin{aligned}
rV_{\ell 1s2} &= r\hat{V}_{\ell 1}+r\hat{V}_{s2} \\
&= \frac{\frac{\nu\mu_{\underline{bo}}}{r+\nu\mu_{\underline{bo}}}\left(1-\frac{\lambda\mu_{b1}}{r+\lambda\mu_{b1}}\frac{\underline{\kappa}}{r+\underline{\kappa}}\frac{\lambda\mu_{s1}}{r+\lambda\mu_{s1}}\right)}{1-\frac{\nu\mu_{\underline{bo}}}{r+\nu\mu_{\underline{bo}}}\frac{\lambda\mu_{b1}}{r+\lambda\mu_{b1}}\frac{\underline{\kappa}}{r+\underline{\kappa}}\frac{\lambda\mu_{s1}}{r+\lambda\mu_{s1}}}\overline{w}_1+\left[\delta-y+\frac{\lambda\mu_{b2}}{r+\overline{\kappa}+\lambda\mu_{b2}}\left[rp_2-\delta+y+\overline{\kappa}(p_2-C_2)\right]\right].
\end{aligned}
$$

The arbitrageur is indifferent between initiating the strategy and holding no position if $V_{\ell 1s2}$ is equal to p_1. Using this condition, and substituting C_1 from (C17) and (C26)

$$\frac{\frac{\nu\mu_{\underline{bo}}}{r+\nu\mu_{\underline{bo}}}\left(1-\frac{\lambda\mu_{b1}}{r+\lambda\mu_{b1}}\frac{\underline{\kappa}}{r+\underline{\kappa}}\frac{\lambda\mu_{s1}}{r+\lambda\mu_{s1}}\right)}{1-\frac{\nu\mu_{\underline{bo}}}{r+\nu\mu_{\underline{bo}}}\frac{\lambda\mu_{b1}}{r+\lambda\mu_{b1}}\frac{\underline{\kappa}}{r+\underline{\kappa}}\frac{\lambda\mu_{s1}}{r+\lambda\mu_{s1}}}\overline{w}_1 = rp_1 - \delta + y - \frac{\lambda\mu_{b2}}{r+\lambda\mu_{b2}}(rp_2 - \delta + y).$$

For small search frictions, this equation becomes

$$\frac{\hat{g}_{\underline{bo}}}{r+\underline{\kappa}\frac{\hat{g}_{s1}}{r+\underline{\kappa}+\hat{g}_{s1}}+\hat{g}_{\underline{bo}}}\overline{w}_1 = r(p_1-p_2)+\frac{r\overline{x}}{\lambda\hat{m}_{b2}},$$

and is inconsistent with (13) since $w_1 > \overline{w}_1$.

Eqs. (12) and (13) are jointly satisfied

The two equations are jointly satisfied if

$$\frac{\hat{g}_{\underline{bo}}}{r+\underline{\kappa}\frac{\hat{g}_{s1}}{r+\underline{\kappa}+\hat{g}_{s1}}+\hat{g}_{\underline{bo}}}\frac{w_1}{r} < p_1 - p_2 < \frac{w_1}{r}.$$

Substituting p_1 and p_2 from (8) and (9), we can write this equation as

$$A_1\frac{w_1}{r} < \frac{B}{\lambda} + A_2\frac{w_1}{r} < \frac{w_1}{r}, \tag{D17}$$

where

$$A_2 \equiv \frac{\hat{g}_{\underline{bo}}}{r+\overline{\kappa}+\underline{\kappa}\frac{\hat{g}_{s1}}{r+\overline{\kappa}+\underline{\kappa}+\hat{g}_{s1}}+\hat{g}_{\underline{bo}}} < A_1 \equiv \frac{\hat{g}_{\underline{bo}}}{r+\underline{\kappa}\frac{\hat{g}_{s1}}{r+\underline{\kappa}+\hat{g}_{s1}}+\hat{g}_{\underline{bo}}} < 1$$

and

$$B \equiv \frac{(\phi r+\overline{\kappa})}{(1-\phi)}\left[\frac{1}{\hat{m}_{b2}}-\frac{1}{\hat{m}_{b1}}\right]\frac{\overline{x}}{r} > 0.$$

Eq. (D17) is satisfied if

$$\frac{B}{A_1-A_2} > \frac{\lambda w_1}{r} > \frac{B}{1-A_2}.$$

In this inequality, n enters only through the product λw_1. Therefore, the inequality is satisfied for n in some interval (n_1, n_2). ∎

Proof of Proposition 11: Generalizing the analysis of Section B.2, we can show that a solution for $\varepsilon = 0$ exists, and is close to that for small ε. The limiting equations are (B28)-(B31), but with the asset supplies depending on i. For the asymmetric equilibrium, (B36)-(B40) generalize to

$$\hat{g}_{\underline{bo}} = \frac{(\overline{\kappa}+\underline{\kappa})\underline{F}}{\underline{\kappa}S_1} \tag{D18}$$

$$\hat{g}_{s1} = \frac{\overline{\kappa}S_1 + \frac{\overline{\kappa}+\underline{\kappa}}{\underline{\kappa}}\underline{F}}{\hat{m}_{b1}} \tag{D19}$$

$$\hat{g}_{s2} = \frac{\overline{\kappa}S_2}{\hat{m}_{\overline{b}}}, \tag{D20}$$

$$\hat{m}_{b1} = \frac{\overline{F}}{\overline{\kappa}} - \sum_{i=1}^{2} S_i - \frac{\underline{F}}{\underline{\kappa}} \tag{D21}$$

$$\hat{m}_{\overline{b}} = \frac{\overline{F}-\overline{\kappa}S_2}{\overline{F}-\overline{\kappa}S_2+\underline{F}}\hat{m}_{b1} \tag{D22}$$

Result (i): An equilibrium where $\nu_1 = \nu$ and $\nu_2 = 0$ can exist if $\Sigma_1 > 0$ and $\Sigma_2 < 0$. Condition $\Sigma_1 > 0$ can be ensured by (4). For small search frictions, condition $\Sigma_2 < 0$ is equivalent to $\hat{g}_{s1} > \hat{g}_{s2}$, as shown in the proof of Proposition 7. Using (D19), (D20) and (D22), we can write condition $\hat{g}_{s1} > \hat{g}_{s2}$ as

$$\left[\overline{\kappa}(S_1 - S_2) + \frac{\overline{\kappa}+\underline{\kappa}}{\underline{\kappa}}\underline{F}\right](\overline{F} - \overline{\kappa}S_2) > \overline{\kappa}S_2\underline{F}. \tag{D23}$$

This equation holds for all values of $S_1 \geq S_2$ because Assumption 2 implies that $\overline{F} - \overline{\kappa}S_2 > \overline{\kappa}S_1 \geq \overline{\kappa}S_2$.

Result (ii): The existence condition is now (D23), but with S_1 and S_2 reversed. It does not hold, for example, when S_1 is large enough to make the term in square brackets negative.

Result (iii): We proceed by contradiction, assuming that for a given $S_1 - S_2 > 0$ there exists an equilibrium where $\nu_1 = \nu_2 = \nu$, even when search frictions converge to zero. Since the parameters a_i, b_i, c_i, and g_i in (C25) converge to finite limits, while f_i converges to ∞, Σ_i must converge to zero, and $f_i\Sigma_i$ to a finite limit. But then (C25) implies that the limits of c_1 and c_2 must be the same. This, in turn, implies that $g_{s1} = g_{s2} \equiv g_s$, which from (B30) and (B31) means that

$$\frac{\overline{\kappa}S_i + g_{\underline{bo}}S_i}{m_{\overline{b}} + \frac{\underline{\kappa}g_{\underline{bo}}S_i}{(\overline{\kappa}+\underline{\kappa})(\overline{\kappa}+g_s)}}$$

is independent of i, a contradiction when asset supplies differ. ∎

Proof of Proposition 12: The expected search time for buying asset i is $1/(\lambda\mu_{si})$ and for selling asset i is $1/(\lambda\mu_{bi})$. Thus, our liquidity measure is $\lambda^2\mu_{bi}\mu_{si} = \lambda(\mu_{bi}\gamma_{si})$. Dropping the multiplicative constant λ and assuming small search frictions, this is equal to $\Lambda_i \equiv \hat{m}_{bi}\hat{g}_{si}$. Eqs. (D19) and (D20) imply that

$$\Lambda_1 = \overline{\kappa}S_1 + \frac{\overline{\kappa}+\underline{\kappa}}{\underline{\kappa}}\underline{F} \tag{D24}$$

$$\Lambda_2 = \overline{\kappa}S_2. \tag{D25}$$

Eqs. (8)-(11), generalized to the case where asset supplies depend on i, imply that the lending fee is

$$w_1 = \theta\left(r+\overline{\kappa}+\underline{\kappa}\frac{\hat{g}_{s1}}{r+\overline{\kappa}+\underline{\kappa}+\hat{g}_{s1}}+\hat{g}_{\underline{bo}}\right)\frac{\underline{x}-\frac{r+\overline{\kappa}+\underline{\kappa}+\hat{g}_{s1}}{r+\overline{\kappa}+\hat{g}_{s1}}(2y-\overline{x})}{\nu(1-\theta)S_1} \tag{D26}$$

and the price premium is

$$p_1-p_2 = \frac{(\phi r+\overline{\kappa})}{\lambda(1-\phi)}\left[\frac{1}{\hat{m}_{b2}}-\frac{1}{\hat{m}_{b1}}\right]\frac{\overline{x}}{r}+\theta\hat{g}_{\underline{bo}}\frac{\underline{x}-\frac{r+\overline{\kappa}+\underline{\kappa}+\hat{g}_{s1}}{r+\overline{\kappa}+\hat{g}_{s1}}(2y-\overline{x})}{\nu(1-\theta)S_1 r}. \tag{D27}$$

Result (i): An increase in $\underline{F}$ increases Λ_1 by (D24) and leaves Λ_2 constant by (D25). It increases $\hat{g}_{\underline{bo}}$ by (D18), decreases $\hat{m}_{b1}$ by (D21), increases $\hat{m}_{b1}/\hat{m}_{b2}(=\hat{m}_{b1}/\hat{m}_{\overline{b}})$ by (D22), and increases $\hat{g}_{s1}$ by (D19). Eq. (D26) then implies that w_1 increases, and (D27) implies that p_1-p_2 increases. For small search frictions w_1/p_1 varies in the same direction as w_1 since p_1 is close to the limit $(\delta+\overline{x}-y)/r$ while w_1 is close to zero.

Result (ii): A decrease in S_1 decreases Λ_1 by (D24) and leaves Λ_2 constant by (D25). Numerical calculations indicate that w_1 and p_1-p_2 increase if $S_1 = S_2 = 0.5$, $\overline{F} = 3$, $\underline{F} = 5.7$, $\overline{\kappa} = 1$, $\underline{\kappa} = 3$, $\phi = \theta = 0.5$, $r = 4\%$, $\delta = 1$, $\overline{x} = 0.4$, $\underline{x} = 1.6$, $y = 0.5$, $\nu/\lambda = 0.25$. If, however, S_1 and S_2 are changed to 1.3, and $\underline{F}$ to 1, while other parameters stay the same, then w_1 and p_1-p_2 decrease. ∎

E. The CARA Setting

Agents can invest in a riskless asset with return r and in two risky assets paying the same cash flow. Cash flow is described by the cumulative dividend process

$$dD_t = \delta dt + \sigma dB_t,$$

where δ and σ are positive constants, and B_t is a standard Brownian motion. Agents derive utility from the consumption of a numéraire good, and have a CARA utility function

$$-E\left[\int_0^\infty \exp\left(-\alpha c_t - \beta t\right)\, dt\right]. \tag{E1}$$

Each agent receives a cumulative endowment process

$$de_t = \sigma_e \left[\rho_t dB_t + \sqrt{1-\rho_t^2} dZ_t\right],$$

where σ_e is a positive constant, Z_t a standard Brownian motion independent of B_t, and ρ_t the instantaneous correlation between the dividend process and the endowment process. The process ρ_t can take three values: $\rho_t = -\overline{\rho} < 0$ for high-valuation agents, $\rho_t = \underline{\rho} > 0$ for low-valuation agents, and $\rho_t = 0$ for average-valuation agents. The processes (ρ_t, Z_t) are pairwise independent across agents. We set $A \equiv r\alpha$, $y \equiv A\sigma^2/2$, $\overline{x} \equiv A\overline{\rho}\sigma\sigma_e$, and $\underline{x} \equiv A\underline{\rho}\sigma\sigma_e$.

E.1. Walrasian Equilibrium

Under Assumptions 1 and 2, the Walrasian equilibrium is identical to that in Proposition 3. This is true even when agents are allowed to invest in integer multiples of one share and in both assets simultaneously, provided that we make the additional assumption

Assumption 3. $4y > \overline{x} + \underline{x}$.

Proposition 13. *Suppose that agents have CARA preferences and can hold any position* $(q_1, q_2) \in \mathbb{Z}^2$ *in the two assets. In a Walrasian equilibrium both assets trade at the same price*

$$p = \frac{\delta + \overline{x} - y}{r}$$

and the lending fee w is zero. Moreover, high-valuation agents buy one share or stay out of the market, low-valuation agents short one share, and average-valuation agents stay out of the market.

Proof: The lending fee is zero by the same argument as in Proposition 3. An agent maximizes (E1) subject to the budget constraint

$$dW_t = \left[rW_t - c_t + \sum_{i=1}^{2}(\delta - rp_i)q_{it}\right] dt + \left[\sigma \sum_{i=1}^{2} q_{it} + \rho_t \sigma_e\right] dB_t + \sigma_e \sqrt{1-\rho_t^2} dZ_t$$

and the transversality condition

$$\lim_{T\to\infty} E\left[\exp(-AW_T - \beta T)\right] = 0, \tag{E2}$$

where W_t is the wealth and q_{it} is the number of shares invested in asset $i \in \{1, 2\}$. The agent's controls are the consumption $c \in \mathbb{R}$ and the investments $(q_1, q_2) \in \mathbb{Z}^2$. Obviously, if $p_1 \neq p_2$ the agent can achieve infinite utility by demanding an infinite amount of assets, contradicting equilibrium. Thus, in equilibrium p_1 and p_2 must be equal. Denoting their common value by p and the aggregate investment in the risky assets by $q \equiv q_1 + q_2$, we can write the budget constraint as

$$dW_t = [rW_t - c_t + (\delta - rp)q_t]\, dt + [\sigma q_t + \rho_t \sigma_e]\, dB_t + \sigma_e \sqrt{1-\rho_t^2} dZ_t.$$

The agent's value function $J(W_t, \rho_t)$ satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$0 = \sup_{(c,q)\in\mathbb{R}\times\mathbb{Z}} \left\{ -\exp(-\alpha c) + \mathcal{D}^{(c,q)} J(W,\rho) - \beta J(W,\rho) \right\}, \tag{E3}$$

where

$$\begin{aligned} \mathcal{D}^{(c,q)} J(W,\rho) \quad &\equiv \quad J_W(W,\rho)\left[rW - c + (\delta - rp)q\right] + \frac{1}{2} J_{WW}(W,\rho)\left[\sigma^2 q^2 + 2\rho\sigma\sigma_e q + \sigma_e^2\right] \\ &\quad +\kappa(\rho)\left[J(W,0) - J(W,\rho)\right], \end{aligned}$$

and where the transition intensity $\kappa(\rho)$ is zero for $\rho = 0$, $\overline{\kappa}$ for $\rho = \overline{\rho}$, and $\underline{\kappa}$ for $\rho = \underline{\rho}$. We guess that $J(W,\rho)$ takes the form

$$J(W,\rho) = -\frac{1}{r}\exp\left[-A[W + V(\rho)] + \frac{r - \beta + \frac{A^2\sigma_e^2}{2}}{r} \right],$$

for some function $V(\rho)$. Replacing into (E3), we find that the optimal consumption is

$$c(\rho) = r[W + V(\rho)] - \frac{r - \beta + \frac{A^2\sigma_e^2}{2}}{A}$$

and the optimal investment satisfies

$$q(\rho) \in \mathrm{argmax}_{q\in\mathbb{Z}}\{C(\rho,q) - rpq\} \equiv Q(\rho),$$

where $C(\rho,q)$ is the incremental certainty equivalent of holding q shares relative to holding none. Using the definitions of y, $\overline{x}$, $\underline{x}$, we can write the certainty equivalent as $C(\overline{\rho}, z) = (\delta + \overline{x})q - yq^2$

for high-valuation agents, $C(\underline{\rho}, z) = (\delta - \underline{x})q - yq^2$ for low-valuation agents, and $C(0, z) \equiv \delta q - yq^2$ for average-valuation agents.

Plugging $c(\rho)$ back into (E3), we find that (E3) is satisfied iff

$$0 = -rV(\rho) + \max_{q \in \mathbb{Z}} \{C(\rho, q) - rpq\} + \kappa(\rho) \frac{1 - e^{-A(V(0) - V(\rho))}}{A}. \tag{E4}$$

Eq. (E4) implies that $V(0) = \max_q \{C(0, q) - rpq\}/r$. Moreover, given $V(0)$, the equations for $V(\overline{\rho})$ and $V(\underline{\rho})$ are in only one unknown, and it is easy to check that they have a unique solution.

We next determine the equilibrium value of p. Because each type-ρ agent holds a position $q(\rho) \in Q(\rho)$, the average position $q_m(\rho)$ of these agents is in the convex hull of $Q(\rho)$. Market clearing requires that $q_m(0) = 0$ because average-valuation agents are in infinite measure. It also requires that

$$\frac{\overline{F}}{\overline{\kappa}} q_m(\overline{\rho}) + \frac{\underline{F}}{\underline{\kappa}} q_m(\underline{\rho}) = 2S. \tag{E5}$$

Because the function $q \to C(\rho, q) - rpq$ is strictly concave, the set $Q(\rho)$ consists of either one or two elements. If there exists a q such that

$$C(\rho, q) - rpq > \max \{C(\rho, q + 1) - rp(q + 1), C(\rho, q - 1) - rp(q - 1)\}, \tag{E6}$$

then this q is unique and $Q(\rho) = \{q\}$. Otherwise, there exists a unique q such that

$$C(\rho, q) - rpq = C(\rho, q + 1) - rp(q + 1), \tag{E7}$$

and $Q(\rho) = \{q, q+1\}$. Using Assumptions 1 and 3 and the first-order conditions (E6) and (E7), it is easy to check that for $p = (\delta + \overline{x} - y)/r$, we have $Q(\overline{\rho}) = \{0, 1\}$, $Q(\underline{\rho}) = \{-1\}$, and $Q(0) = \{0\}$. Eq. (E5) then follows from Assumption 2, implying that $p = (\delta + \overline{x} - y)/r$ is an equilibrium price. It is the unique equilibrium price because if $p > (\delta + \overline{x} - y)/r$, then no agent would choose $q > 0$, and if $p < (\delta + \overline{x} - y)/r$ then high-valuation agents would choose $q \geq 1$, while other agents would choose at least as much as for $p = (\delta + \overline{x} - y)/r$. ∎

E.2. Search Equilibrium

Proposition 14 studies agents' optimization problem in a general Poisson setting, and shows that the value function is of the form

$$J(W,\tau) = -\frac{1}{r}\exp\left[-A[W+V(\tau)] + \frac{r-\beta+\frac{A^2\sigma_e^2}{2}}{r}\right], \tag{E8}$$

where $V(\tau)$ is a function characterized by (E9). Using (E9), it is easy to check that when α converges to zero, holding $(y,\overline{x},\underline{x})$ constant, $V(\tau)$ satisfies the flow-value equations derived under the utility specification of Section I. Therefore, if the trading strategies in the equilibria of Propositions 5 and 7 involve strict preferences (which is the case generically), they are also optimal under CARA preferences for small α. This means that the equilibria of Propositions 5 and 7 are also equilibria under CARA preferences.

Proposition 14. *Suppose that*

(i) An agent can be of finitely many types $\tau \in \mathcal{T}$.

(ii) While being of type τ, the agent receives a payoff described by the cumulative process

$$dX(\tau,t) = m(\tau)dt + \sqrt{\sigma(\tau_t)^2 + \sigma_e^2}\, d\tilde{B}_t,$$

where $\tilde{B}_t$ is a standard Brownian motion.

(iii) Transitions across types occur at the arrival times of a K-dimensional counting process N_t, with intensity associated to dimension k equal to a constant $\gamma(k)$. At the arrival times associated to dimension k, the agent can choose between types $\tau' \in \mathcal{T}'(\tau,k) \subseteq \mathcal{T}$.

(iv) Transition to type τ' brings an instant payoff $P(\tau,\tau')$.

Then, the value function is given by (E8) with

$$rV(\tau) = m(\tau) - \frac{A}{2}\sigma(\tau)^2 + \sum_{k=1}^{K}\gamma(k) \max_{\tau' \in \mathcal{T}'(\tau,k)} \frac{1 - e^{-A\left[V(\tau') - V(\tau) + P(\tau,\tau')\right]}}{A}. \tag{E9}$$

Proof: The agent's wealth evolves according to the SDE

$$dW_t = [rW_t + m(\tau_t) - c_t]dt + \sqrt{\sigma(\tau_t)^2 + \sigma_e^2}\, d\tilde{B}_t + \sum_{k=1}^{K} P(\tau_{t-},\tau_t)\, dN_t(k).$$

The agent chooses a transition and consumption policy to maximize (E1) subject to the transversality condition (E2). We also impose the boundedness condition

$$E\left[\int_0^T \exp(-zW_t)\,dt\right] < \infty$$

for all $T \geq 0$ and $z \in \{r\alpha, 2r\alpha\}$, because it is needed for the verification argument. The HJB equation is

$$0 \quad = \quad \sup_{c\in\mathbb{R},\tau'\in\mathcal{T}'(\tau,k)} \left\{-\exp\big[-\alpha\, c(\tau)\big] + \mathcal{D}^{(c,\tau')}J(W,\tau) - \beta J(W,\tau)\right\}, \tag{E10}$$

where

$$\begin{aligned}\mathcal{D}^{(c,\tau')}J(W,\tau) \quad &\equiv \quad J_W(W,\tau)\big[rW - c(\tau) + m(\tau)\big] + \frac{1}{2}\left[\sigma(\tau)^2 + \sigma_e^2\right] J_{WW}(W,\tau) \\ &\quad + \sum_{k=1}^{K} \gamma(k)\Big[J\big[W + P(\tau,\tau'),\tau'\big] - J(W,\tau)\Big].\end{aligned}$$

Substituting (E8) in (E10) and maximizing with respect to consumption, we find that (E8) is a solution iff $V(\tau)$ solves (E9). ∎