Internet Appendix to "Financial Flexibility, Bank Capital Flows, and Asset Prices"*

A. *Financial Flexibility and Real Growth*

The financial development literature has identified two main channels through which the presence of financial development, and intermediaries in particular, can lead to growth. First, intermediaries can help investors make better investment decisions by reducing transaction costs; this could be by producing information or by providing a more efficient monitoring technology. For example, Greenwood and Jovanovic (1990) present a model in which financial intermediaries provide information that allows investors to earn a higher rate of return on capital, which promotes growth. The greater the growth, the more intermediaries invest in information production, reinforcing their benefits. Intermediaries also provide efficient information processing in Ramakrishnan and Thakor (1984). Similarly, in Bencivenga and Smith (1993), intermediaries reduce adverse selection costs and foster growth. In de la Fuente and Marin (1996), intermediaries reduce the cost of optimal monitoring and therefore increase investment efficiency.

The second channel in the financial development literature is risk-related: intermediaries, by pooling investments, can help investors achieve efficient diversification. Once diversified, investors may be willing to invest in riskier projects with higher returns, leading to growth. Papers in this vein include Bencivenga and Smith (1991), who illustrate that banks, by aggregating deposits, encourage investment in growth-enhancing illiquid assets by effectively allowing investors to diversify liquidity risk. Bencivenga, Smith, and Starr (1995) argue that liquid financial markets allow investors to supply capital to illiquid but productive investment opportunities, which spurs growth. Their work provides a natural link between stock market liquidity and economic growth. Acemoglu and Zilibotti (1997) present a model in which project indivisibility leads to inefficient investment as investors cannot diversify away all idiosyncratic risk. A developed financial system allows agents to hold a diversified portfolio of risky projects and therefore encourages more risky investment, leading to higher growth. Finally, Allen and Gale (1997) present a model in which, because financial markets are incomplete, there is not enough investment in reserves that could be used to smooth asset returns over time. A long-lived financial intermediary issuing less risky claims could improve social welfare.

Our model provides a framework that could be useful to this debate on the relationship between financial flexibility, intermediation, and real growth. To make our point cleanly, we consider an economy in which the risk for jump events is $p = 0$. As we have established, in this case capital flows will be equal to capital flexibility, that is, the solution to the control problem will be "bang-bang." Further, observe that in our model, the empirical proxy for GDP is $B + D$, or current consumption.

Our results fit into the long-running debate about the relationship between economic growth rates and financial innovation. Rather than view financial flexibility as a cause (Schumpeter (1911)) or a consequence (Robinson (1952)) of economic growth, we focus on economic growth as the natural consequence of the equilibrium risk appetite of a representative consumer. Specifically, the existence of high financial flexibility may induce a large banking sector and, consequently, a low stationary

*Citation format: Parlour, Christine A., Richard Stanton, and Johan Walden, 2012, Internet Appendix to "Financial Flexibility, Bank Capital Flows, and Asset Prices," *Journal of Finance*, 10.1111/j.1540-6261.2012.01770.x. Please note: Wiley-Blackwell is not responsible for the content or func- tionality of any supporting information supplied by the authors. Any queries (other than missing material) should be directed to the authors of the article.

growth rate. It is important in what follows to distinguish between the exogenous characteristics of the entrepreneurial sector, including the growth rate and volatility (μ and σ), and the overall growth rate of the economy, which depends only on the size of the entrepreneurial sector.

Within our framework, economies facing different levels of financial flexibility (λ) will have different optimal bank shares, z_*. Further, z_* depends on financial flexibility nonmonotonically. In fact, z_* increases in λ if the growth rate in the risky sector is sufficiently high and decreases in λ if the growth rate is sufficiently low. This suggests that there is not a simple causal link between the size of the intermediated sector and growth rates. To see this, consider Figure IA.1, which illustrates the relationship between z_* and λ. Consider first the case where $\hat{\mu} = 0.01$. If this is the growth rate of the risky sector, then the central planner optimally keeps half of the economy in the intermediated sector and half in the risky sector, irrespective of the speed with which capital moves between the sectors.

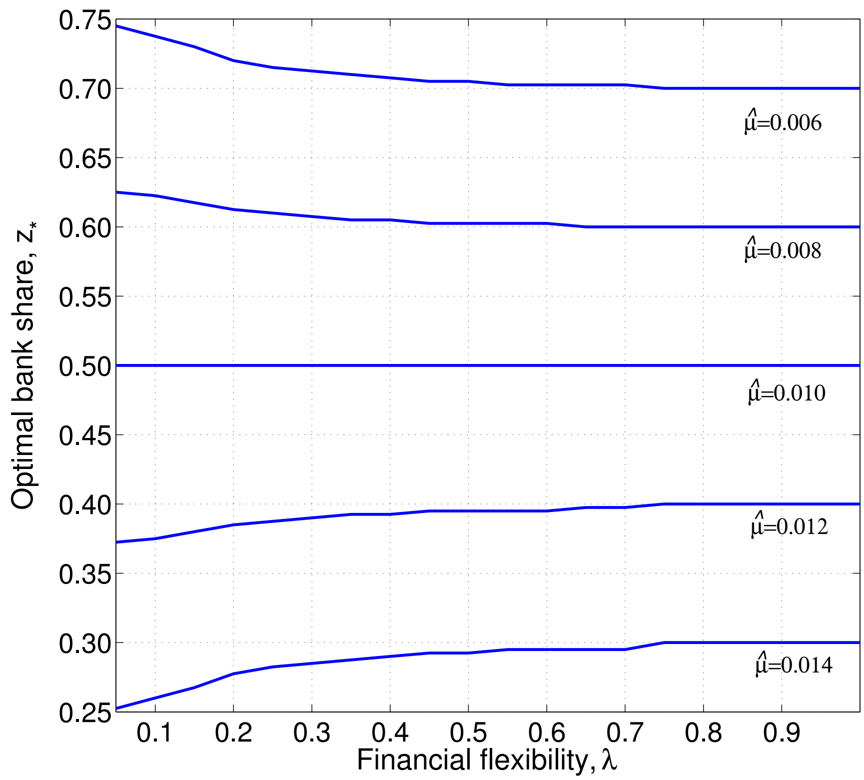

Figure IA.1. The optimal size of the banking sector, z_*, as a function of financial flexibility, λ, for different values of the growth rate of the risky asset, $\hat{\mu}$. Parameters: $\sigma = 0.141$, $\rho = 0.02$, $\gamma = 1$, and $T = 500$.

If the growth rate in the risky sector is high (say $\hat{\mu} = 0.014$), then increasing the rate at which capital moves increases the optimal size of the banking sector. In this case the social cost of having an inordinately large banking sector (and therefore forgone growth) is very high. Therefore, as insurance against this state, the central planner decreases the size of the banking sector to maintain a "buffer," compared with the fully flexible case, $\lambda = \infty$. Because of this, for very low λ, the size of the banking sector is smaller. As λ increases, the central planner is willing to increase the size of the banking sector (alternatively, to decrease the size of the buffer) because the chance of the economy spending a long time in the low-growth state is small. Thus, when the growth rate in the risky sector is high, the optimal size of the banking sector is increasing in the flexibility of capital, λ.

The situation is reversed when the growth rate of capital is quite low (say $\hat{\mu} = 0.006$). In this case, the cost to the central planner of ending up with too much capital in the risky sector is high because the return is low relative to the risk. Therefore, he hedges against this possibility by maintaining a somewhat larger banking sector. As the flexibility of capital increases, he is willing to reduce the size of the banking sector as he no longer needs a buffer against the possibility that the risky sector will become too large.

The relationship between the size of the banking sector and the flexibility of capital is therefore nontrivial. Specifically, financial innovation or government policy that increases the speed with which funds can be reallocated between sectors may, in equilibrium, either decrease the size of the banking sector or increase it. Moreover, increasing financial flexibility may decrease the growth rate of the economy.

These results can be formalized. Specifically, the key variable is $\kappa = \frac{\hat{\mu}}{\gamma\sigma^2}$. If $\kappa < 1/2$, then an increase in λ leads to a lower z_*, whereas if $\kappa > 1/2$, an increase in λ leads to a higher z_*. Since κ is increasing in $\hat{\mu}$ and decreasing in σ, this immediately leads to the following hypotheses regarding growth and growth volatility across economies:

PREDICTION 1: *All else equal,*

a) in low-growth economies, the growth rate increases with financial flexibility,

b) in high-growth economies, the growth rate decreases with financial flexibility,

c) in high-volatility economies, the growth rate increases with financial flexibility, and

d) in low-volatility economies, the growth rate decreases with financial flexibility.

Thus, in high growth economies, increasing λ, for example, through financial innovation, will actually decrease the growth rate of the economy. This suggests that cross-country regressions of economic performance (including growth rates) on proxies for financial innovation or variables that measure the speed with which capital flows between the banking and entrepreneurial sectors are complex to interpret. For example, the work of Levine (1998), drawing on that of La Porta et al. (1998), considers the effect of legal protections on the development of banks and subsequent growth rates. Our analysis suggests that unambiguous causal links are difficult to find because increasing the efficiency of the banking sector may lead to an overall larger or smaller sector, depending on the fundamentals of the economy.

Our analysis also shows that there is a complex relation between the equilibrium size of the banking sector and economic fundamentals, an important consideration for regulators. For example, the equilibrium banking sector size is particularly small in high-growth economies with low financial flexibility, as can be seen in Figure IA.1.

B. Value for Extreme Cases, $\lambda = 0$ and $\lambda = \infty$

Lemma 1: *Suppose that capital is fully flexible, $\lambda \equiv \infty$, and that the central planner chooses a constant bank share, z. Then the expected utility of the representative agent is*

$$U^\infty(z) = \begin{cases} \frac{1}{1-\gamma} \times \frac{1}{\rho + (1-\gamma)((1-z)\hat{\mu} - \gamma(1-z)^2\sigma^2/2)} & \gamma > 1 \\ \frac{1}{\rho^2}\left((1-z)\hat{\mu} - (1-z)^2\frac{\sigma^2}{2}\right) & \gamma = 1, \end{cases}$$

which takes on its maximal value, $\frac{1}{1-\gamma} \times \frac{1}{\rho + \frac{\gamma-1}{\gamma} \times \frac{\hat{\mu}}{\sigma^2}}$ for $\gamma > 1$ and $\frac{\hat{\mu}^2}{2\rho^2\sigma^2}$ for $\gamma = 1$, respectively, at $z_* = 1 - \frac{\hat{\mu}}{\gamma\sigma^2}$.

Proof of Lemma 1: The optimal solution follows immediately from the unconstrained portfolio problem; see, for example, Merton (1969). ■

Lemma 2: *In the infinite horizon economy, $T = \infty$, define $q = \sqrt{\mu^2 + 2\rho\sigma^2}$. Suppose that*
(i) $\gamma = 1$. Then, if the initial bank share is $0 < z < 1$, the expected utility of the representative agent is

$$\begin{aligned} w(z) &= \frac{1}{2\rho}\Big(\left(2\mu^2 + \sigma^2(2\rho + q) + \mu(\sigma^2 + 2q)\right)\ {}_2F_1\left(1, \frac{q-\mu}{\sigma^2}, \frac{q-\mu}{\sigma^2} + 1, \frac{z}{z-1}\right) \\ &+ 2\frac{z-1}{z}\left(\mu^2 + \rho\sigma^2 - \mu q\right)\ {}_2F_1\left(1, \frac{q+\mu}{\sigma^2} + 1, \frac{q+\mu}{\sigma^2} + 2, \frac{z-1}{z}\right)\Big) \\ &/ \left(\mu^2 - \mu q + 2\rho(\sigma^2 + q)\right), \end{aligned}$$

where ${}_2F_1$ is the hypergeometric function. Also, $w(1) = 0$ and $w(0) = \frac{\mu}{\rho^2}$.
(ii) If $\gamma > 1$, then if the initial bank share is $0 < z < 1$, the expected utility of the representative agent is

$$\begin{aligned} w(z) &= \frac{z^{1-\gamma}}{q(1-\gamma)} \\ &\times \left[\left(\frac{z}{1-z}\right)^{\frac{\mu-q}{\sigma^2}}\left(V\left(\frac{z}{1-z}, \gamma + \frac{q-\mu}{\sigma^2}, 1-\gamma\right) + V\left(\frac{z}{1-z}, \gamma + \frac{q-\mu}{\sigma^2} - 1, 1-\gamma\right)\right)\right. \\ &+ \left.\left(\frac{1-z}{z}\right)^{-\frac{q+\mu}{\sigma^2}}\left(V\left(\frac{1-z}{z}, \frac{q+\mu}{\sigma^2}, 1-\gamma\right) + V\left(\frac{1-z}{z}, \frac{q+\mu}{\sigma^2} + 1, 1-\gamma\right)\right)\right]. \end{aligned}$$

Here, $V(y, a, b) \overset{\text{def}}{=} \int_0^y t^{a-1}(1+t)^{b-1}dt$ is defined for $a > 0$. Also, $w(1) = \frac{1}{\rho(1-\gamma)}$. Moreover, define $x \overset{\text{def}}{=} \rho + (\gamma - 1)\mu - (\gamma - 1)^2\frac{\sigma^2}{2}$. Then, if $x > 0$, $w(0) = -\frac{1}{x}$. If, on the other hand, $x \leq 0$, then $w(0) = -\infty$.

We note that the definition of z in Parlour et al. (2011) is as the risky share, which corresponds to $1 - z$ in our notation.

Proof of Lemma 1: See Parlour et al. (2011). ■

C. Prices

Define

$$Q(B,D,t) \equiv E_t\left[\frac{G_T}{(B_T+D_T)^\gamma}\middle| B_t=B, D_t=D\right]. \tag{IA.1}$$

From equation (19), we have

$$Q(B,D,t) = \frac{e^{\rho(T-t)}P(B,D,t)}{(B+D)^\gamma} - E_t\left[\int_t^T e^{\rho(T-s)}\frac{\delta_s}{(B_s+D_s)^\gamma}\,ds\right]. \tag{IA.2}$$

By iterated expectations,

$$E(dQ)=0. \tag{IA.3}$$

Also,

$$E_t\left[d\left(E\left[\int_t^T e^{\rho(T-s)}\frac{\delta_s}{(B_s+D_s)^\gamma}\,ds\right]\right)\right] = -\frac{e^{\rho(T-t)}\delta_t}{(B_t+D_t)^\gamma}dt,$$

so

$$E_t\left[d\left(\frac{e^{\rho(T-t)}P(B,D,t)}{(B+D)^\gamma}\right)\right] + \frac{e^{\rho(T-t)}\delta_t}{(B_t+D_t)^\gamma}dt = 0. \tag{IA.4}$$

Now,

$$\begin{aligned} E_t\left[d\left(\frac{e^{\rho(T-t)}P(B,D,t)}{(B+D)^\gamma}\right)\right] &= e^{\rho(T-t)}\left[-\rho\frac{P}{(B+D)^\gamma}dt + \frac{P_t}{(B+D)^\gamma}dt + \frac{P_B}{(B+D)^\gamma}dB\right.\\ &- \frac{\gamma P}{(B+D)^{\gamma+1}}dB + \frac{P_D}{(B+D)^\gamma}E[dD] - \frac{\gamma P}{(B+D)^{\gamma+1}}E[dD]\\ &+ \left.\frac{1}{2}\left(\frac{P_{DD}}{(B+D)^\gamma} - 2\gamma\frac{P_D}{(B+D)^{\gamma+1}} + \gamma(1+\gamma)\frac{P}{(B+D)^{\gamma+2}}\right)(dD)^2\right]. \end{aligned}$$

Substituting this into (IA.4), noting that when $\rho=0$ and $r=0$ equation (4) reduces to $dB = aB\,dt$, and multiplying by $e^{-\rho(T-t)}(B+D)^\gamma$ leads to the following PDE, which must be satisfied by P, subject to the terminal boundary condition $P(B,D,T)=G(B,D,T)$:

$$\begin{aligned} P_t + \frac{1}{2}\sigma^2D^2P_{DD} + \left[\hat{\mu}D - aB - \gamma\frac{\sigma^2D^2}{B+D}\right]P_D + aBP_B&\\ - \left(\rho + \gamma\hat{\mu}\frac{D}{B+D} - \frac{1}{2}\gamma(\gamma+1)\sigma^2\frac{D^2}{(B+D)^2}\right)P + \delta(B,D,t) &= 0. \end{aligned} \tag{IA.5}$$

For the special case where δ is of the form $\delta(B,D,t) = g(z,t)(B+D)$ and $G(B,D) = 0$, we have

$$\begin{aligned} P(B,D,t) &= P\left(\frac{z}{1-z}, 1, t\right)(B+D) \\ &\equiv p(z,t)(B+D); \\ P_t &= p_t(B+D); \\ P_B &= p_z\frac{\partial z}{\partial B}(B+D) + p = p_z\frac{D}{B+D} + p; \\ P_D &= p_z\frac{\partial z}{\partial D}(B+D) + p = p_z\frac{-B}{B+D} + p; \\ P_{DD} &= p_{zz}\frac{B^2}{(B+D)^3}. \end{aligned}$$

Plugging this into (IA.5) yields

$$\begin{aligned} p_t &+ \frac{1}{2}\sigma^2 z^2(1-z)^2 p_{zz} + \left[az - \hat{\mu}z(1-z) + \sigma^2\gamma z(1-z)^2\right] p_z \\ &- \left[\rho - \hat{\mu}(1-\gamma)(1-z) + \frac{1}{2}\sigma^2\gamma(1-\gamma)(1-z)^2\right] p + g(z) = 0. \end{aligned} \tag{IA.6}$$

We can use this to calculate the value of the dividends paid by the bank sector, using $g(z) = z$, and by the risky sector, using $g(z) = 1 - z$. Finally, plugging in $g(z) = 1$, we can calculate the value of the total economy, $P = p \times (B + D)$.

For assets that pay dividends $\delta(z,t)$, with $G(B,D,T) = \hat{G}(z)$, we can make a similar argument. This is an interesting special case. For example, a zero-coupon bond is obtained when $\delta \equiv 0$, with $\hat{G}(z) \equiv 1$. By homogeneity, we can write

$$\begin{aligned} P(B,D,t) &= P\left(\frac{z}{1-z}, 1, t\right) \\ &\equiv p(z,t); \\ P_t &= p_t; \\ P_B &= p_z\frac{\partial z}{\partial B} = p_z\frac{D}{(B+D)^2}; \\ P_D &= p_z\frac{\partial z}{\partial D} = p_z\frac{-B}{(B+D)^2}; \\ P_{DD} &= p_{zz}\left(\frac{\partial z}{\partial D}\right)^2 + p_z\frac{\partial^2 z}{\partial D^2} = p_{zz}\frac{B^2}{(B+D)^4} + p_z\frac{2B}{(B+D)^3}. \end{aligned}$$

Substituting these into equation (IA.5) and simplifying, we obtain

$$\begin{aligned} p_t &+ \frac{1}{2}\sigma^2 z^2(1-z)^2 p_{zz} + \left[az - \hat{\mu}z(1-z) + \sigma^2(1+\gamma)z(1-z)^2\right] p_z \\ &- \left[\rho + \hat{\mu}\gamma(1-z) - \frac{1}{2}\sigma^2\gamma(1+\gamma)(1-z)^2\right] p + \delta(z,t) = 0. \end{aligned} \tag{IA.7}$$

We use this formula to calculate value of the bank sector, $P^B = p \times B$, by using $\delta(z,t) = 1$ and $\hat{G}(z) = 0$ in (IA.7). We note that p here is what in the paper we refer to as q^B.

Similarly, we would like to calculate the value of the equity sector, P^D. This sector grows according to

$$d\hat{D} = \hat{D}(\hat{\mu}\, dt + \sigma\, d\omega),$$

and the value of such a sector is, from (19),

$$P(B_t, D_t, \hat{D}_t, t) = (B_t + D_t)^\gamma E_t \left[\int_t^T e^{-\rho(s-t)} \frac{\hat{D}_t}{(B_s + D_s)^\gamma}\, ds \right]. \tag{IA.8}$$

An argument similar to that leading to (IA.4) shows that

$$E_t \left[d \left(\frac{e^{\rho(T-t)} P(B, D, \hat{D}, t)}{(B+D)^\gamma} \right) \right] + \frac{e^{\rho(T-t)} \hat{D}}{(B_t + D_t)^\gamma} dt = 0.$$

We can then expand

$$\begin{aligned}
E_t \left[d \left(\frac{e^{\rho(T-t)} P(B, D, \hat{D}, t)}{(B+D)^\gamma} \right) \right] &= e^{\rho(T-t)} \left[-\rho \frac{P}{(B+D)^\gamma} dt + \frac{P_t}{(B+D)^\gamma} dt + \frac{P_B}{(B+D)^\gamma} dB \right. \\
&+ \frac{P_{\hat{D}}}{(B+D)^\gamma} E[d\hat{D}] - \frac{\gamma P}{(B+D)^{\gamma+1}} dB + \frac{P_D}{(B+D)^\gamma} E[dD] \\
&- \frac{\gamma P}{(B+D)^{\gamma+1}} E[dD] + \frac{1}{2} \frac{P_{\hat{D}\hat{D}}}{(B+D)^\gamma} (d\hat{D})^2 \\
&+ 2\frac{1}{2} \left(\frac{P_{D\hat{D}}}{(B+D)^\gamma} - \gamma \frac{P_{\hat{D}}}{(B+D)^{\gamma+1}} \right) (d\hat{D})(dD) \\
&+ \left. \frac{1}{2} \left(\frac{P_{DD}}{(B+D)^\gamma} - 2\gamma \frac{P_D}{(B+D)^{\gamma+1}} + \gamma(1+\gamma) \frac{P}{(B+D)^{\gamma+2}} \right) (dD)^2 \right].
\end{aligned}$$

From (IA.8) and homogeneity, it follows that $P(B_t, D_t, \hat{D}_t, t) = p(z)\hat{D}$, for some function $p : [0,1] \to \mathbb{R}$, implying that

$$\begin{aligned}
P_t &= p_t \hat{D}; \\
P_B &= p_z \frac{D}{(B+D)^2} \hat{D}; \\
P_D &= p_z \frac{-B}{(B+D)^2} \hat{D}; \\
P_{\hat{D}} &= p; \\
P_{\hat{D}\hat{D}} &= 0; \\
P_{D\hat{D}} &= p_z \frac{-B}{(B+D)^2}; \\
P_{DD} &= \left(p_{zz} \frac{B^2}{(B+D)^4} + p_z \frac{2B}{(B+D)^3} \right) \hat{D}.
\end{aligned}$$

Substituting yields

$$p_t + \frac{1}{2}\sigma^2 z^2(1-z)^2 p_{zz} + \left[az - \hat{\mu}z(1-z) + \sigma^2(1+\gamma)z(1-z)^2\right] p_z$$
$$- \left[\rho + \hat{\mu}\gamma(1-z) - \frac{1}{2}\sigma^2\gamma(1+\gamma)(1-z)^2\right] p + (\hat{\mu} - \gamma(1-z)\sigma^2)p - z(1-z)\sigma^2 p_z + 1 = 0. \quad \text{(IA.9)}$$

Thus, since $\hat{B} = B$ at $t = 0$, $P^D = p(z) \times D$. We note that p here is what in the paper is referred to as q^D.

Now, to calculate P^I, we can use either $P^I = P - P^B - P^D$ or (7) to derive $P^I = p \times (B + D)$, where p solves

$$p_t + \frac{1}{2}\sigma^2 z^2(1-z)^2 w_{zz} + \left[az - \hat{\mu}z(1-z) + \sigma^2\gamma z(1-z)^2\right] p_z$$
$$- \left[\rho - \hat{\mu}(1-\gamma)(1-z) + \frac{1}{2}\sigma^2\gamma(1-\gamma)(1-z)^2\right] p = a\left(q^B - q^D\right). \quad \text{(IA.10)}$$

D. *Proofs*

Proof of Proposition 1: We first prove Corollary 1, and then show how the analysis generalizes to the general proposition. For completeness, we also analyze the case in which $\gamma = 1$, that is, in which the representative investor has log-utility.

Proof of Corollary 1: We proceed by characterizing the central planner's problem for a finite T by finding a locally optimal control or reallocation (a) that will also be globally optimal. The infinite horizon case follows immediately. Given the central planner's objective, for $\gamma > 1$, the Bellman equation for optimality is

$$\sup_{a \in \mathcal{A}} \left[V_t + \frac{1}{2}\sigma^2 D^2 V_{DD} + \left[\hat{\mu}D - aB\right] V_D + aBV_B - \rho V + \frac{(B+D)^{1-\gamma}}{1-\gamma}\right] = 0. \quad \text{(IA.11)}$$

Equation (IA.11) can be simplified by observing that, by homogeneity, we can write

$$V(B, D, t) = -\frac{(B+D)^{1-\gamma}}{1-\gamma} w(z, t), \quad \text{(IA.12)}$$

where the normalized value function $w(z,t) \equiv V(z, 1-z, t)$. The derivatives of V in terms of derivatives of w are given by

$$V_t = -\frac{(B+D)^{1-\gamma}}{1-\gamma} w_t, \quad \text{(IA.13)}$$

$$V_B = -\frac{(B+D)^{1-\gamma}}{1-\gamma}\left(w\frac{1-\gamma}{B+D} + w_z\frac{D}{(B+D)^2}\right), \quad \text{(IA.14)}$$

$$V_D = -\frac{(B+D)^{1-\gamma}}{1-\gamma}\left(w\frac{1-\gamma}{B+D} - w_z\frac{B}{(B+D)^2}\right), \quad \text{(IA.15)}$$

$$V_{DD} = -\frac{(B+D)^{1-\gamma}}{1-\gamma}\left(-w\frac{\gamma(1-\gamma)}{(B+D)^2} + w_z\frac{2\gamma B}{(B+D)^3} + w_{zz}\frac{B^2}{(B+D)^4}\right). \quad \text{(IA.16)}$$

This step allows us to write derivatives of V in terms of derivatives of w. Substituting these into equation (IA.11), we obtain

$$\sup_{a\in\mathcal{A}} w_t + \frac{1}{2}\sigma^2 z^2(1-z)^2 w_{zz} + \left[az - \hat{\mu}z(1-z) + \sigma^2\gamma z(1-z)^2\right] w_z$$
$$- \left[\rho - \hat{\mu}(1-\gamma)(1-z) + \frac{1}{2}\sigma^2\gamma(1-\gamma)(1-z)^2\right] w - 1 = 0. \quad \text{(IA.17)}$$

The derivation for $\gamma = 1$ is slightly different. Define

$$V(B, D, t) \equiv \sup_{a\in\mathcal{A}} E_t\left[\int_t^T e^{-\rho(s-t)} \log(B+D)\, ds\right].$$

The Bellman equation for optimality is

$$\sup_{a\in\mathcal{A}} \left[V_t + \frac{1}{2}\sigma^2 D^2 V_{DD} + [\hat{\mu}D - aB]\, V_D + aBV_B - \rho V + \log(B+D)\right] = 0. \quad \text{(IA.18)}$$

By homogeneity, we can write V and its derivatives in terms of D and z:

$$\begin{aligned}
V(B, D, t) &= \frac{\log(B+D)\left(1-e^{-\rho(T-t)}\right)}{\rho} + V\left(z, 1-z, t\right) \\
&\equiv \frac{\log(B+D)\left(1-e^{-\rho(T-t)}\right)}{\rho} + w(z, t). \\
V_t &= -e^{-\rho(T-t)} \log(B+D) + w_t, &\text{(IA.19)} \\
V_B &= \frac{1-e^{-\rho(T-t)}}{\rho(B+D)} + w_z \frac{D}{(B+D)^2}, &\text{(IA.20)} \\
V_D &= \frac{1-e^{-\rho(T-t)}}{\rho(B+D)} - w_z \frac{B}{(B+D)^2}, &\text{(IA.21)} \\
V_{DD} &= -\frac{1-e^{-\rho(T-t)}}{\rho(B+D)^2} + w_z \frac{2B}{(B+D)^3} + w_{zz} \frac{B^2}{(B+D)^4}. &\text{(IA.22)}
\end{aligned}$$

Substituting these into equation (IA.18), we obtain

$$w_t + \frac{1}{2}\sigma^2 z^2(1-z)^2 w_{zz} + \left[az - \hat{\mu}z(1-z) + \sigma^2 z(1-z)^2\right] w_z - \rho w$$
$$+ \frac{1-e^{-\rho(T-t)}}{\rho}\left[\hat{\mu}(1-z) - \frac{\sigma^2(1-z)^2}{2}\right] = 0.$$

In sum, we therefore have

$$\sup_{a\in\mathcal{A}} w_t + \frac{1}{2}\sigma^2 z^2(1-z)^2 w_{zz} + \left[az - \hat{\mu}z(1-z) + \sigma^2\gamma z(1-z)^2\right] w_z$$
$$- \left[\rho - \hat{\mu}(1-\gamma)(1-z) + \frac{1}{2}\sigma^2\gamma(1-\gamma)(1-z)^2\right] w + F_\gamma(t, z) = 0, \quad \text{(IA.23)}$$

where

$$F_\gamma(t, z) = \begin{cases} -1, & \gamma > 1, \\ \frac{1-e^{-\rho(T-t)}}{\rho}\left(\hat{\mu}(1-z) - \frac{\sigma^2(1-z)^2}{2}\right), & \gamma = 1. \end{cases} \quad \text{(IA.24)}$$

We study the case $\gamma = 1$. The case $\gamma > 1$ can be treated in an identical way. We first note that $azw_z = \lambda(z)\,\mathrm{sign}(w_z)w_z = \lambda(z)|w_z|$, so (16) is the same as (IA.23). We define a solution to the central planner's optimization to be interior if $a(t,0) > 0$ and $a(t,1) < 0$ in a neighborhood of the boundaries for all $t < T$, where the radii of the neighborhoods do not depend on t. A solution is thus interior if it is always optimal for the central planner to stay away from the boundaries, $z = 0$ and $z = 1$. From our previous argument, we know that any smooth interior solution must satisfy (16). What remains to be shown is that the solution to the central planner's problem is indeed interior, and that, given that the solution is interior, equations (16) and (17) have a unique, smooth solution, that is, that (16) and (17) provide a *well-posed* PDE (Egorov and Shubin (1992)).[1]

We begin with the second part, that is, the well-posedness of the equation, given that the solution is interior. As usual, we first study the Cauchy problem, that is, the problem without boundaries, on the entire real line $z \in \mathbb{R}$ (or, equivalently, with periodic boundary conditions). We then extend the analysis to the bounded case, $z \in [0,1]$. Equation (16) has the structure of a generalized KPZ equation, which has been extensively studied in recent years (see Kardar, Parisi, and Zhang (1986), Gilding, Guedda, and Kersner (2003), Ben-Artzi, Goodman, and Levy (1999), Hart and Weiss (2005), Laurencot and Souplet (2005), and references therein). The Cauchy problem is well posed, that is, given bounded, regular initial conditions, there exists a unique, smooth solution. Specifically, given continuous, bounded initial conditions, there is a unique solution that is bounded, twice continuously differentiable in space, and once continuously differentiable in time, that is, $w \in C^{2,1}[0,T] \times \mathbb{R}$ (see, for example, Ben-Artzi et al. (1999)).

Given that the Cauchy problem is well posed and that the solution is smooth, it is clear that $az = \lambda(z)\,\mathrm{sign}(w_z)$ will have a finite number of discontinuities on any bounded interval at any point in time. Moreover, given that the solution is interior, a is continuous in a neighborhood of $z = 0$ and also in a neighborhood of $z = 1$. The PDE

$$0 \;=\; w_t - \rho w + (az - z(1-z)\hat{\mu} + z(1-z)^2\sigma^2)w_z + \frac{\sigma^2}{2}z^2(1-z)^2 w_{zz} + q(t,z)$$

is parabolic in the interior but hyperbolic at the boundaries, since the $\frac{\sigma^2}{2}z^2(1-z)^2 w_{zz}$-term vanishes at boundaries. For example, at the boundary $z = 1$, using the transformation $\tau = T - t$, the equation reduces to

$$w_\tau \;=\; -\rho w - \lambda(1) w_z.$$

Similarly, at $z = 0$, the equation reduces to

$$w_\tau \;=\; -\rho w + \lambda(0) w_z + q(t,0).$$

Both these equations are hyperbolic and, moreover, both correspond to outflow boundaries. Specifically, the characteristic lines at $z = 0$ are $\tau + z/\lambda(0) = const$, and at $z = 1$ they are $\tau - z/\lambda(1) = const$. For outflow boundaries to hyperbolic equations, no boundary conditions are needed, that is, if the Cauchy problem is well posed, then the initial boundary value with an outflow boundary is well posed without a boundary condition (Kreiss and Lorenz (1989)). This suggests that no boundary conditions are needed.

To show that this is indeed the case, we use the energy method to show that the operator $Pw \overset{\text{def}}{=} \rho w + (a - z(1-z)\hat{\mu} + z(1-z)^2\sigma^2)w_z + \frac{\sigma^2}{2}z^2(1-z)^2 w_{zz}$ is maximally semi-bounded, that is,

[1] The concept of well-posedness additionally requires the solution to depend continuously on initial and boundary conditions. This requirement is natural, since we cannot hope to numerically approximate the solution if it fails.

we use the L_2 inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$ and the norm $\|w\|^2 = \langle w, w\rangle$ to show that for any smooth function, w, $\langle w, Pw\rangle \le \alpha\|w\|^2$ for some $\alpha > 0$.[2] This allows us to bound the growth of $\frac{d}{d\tau}\|w(t,\cdot)\|^2$ by $\frac{d}{d\tau}\|w(t,\cdot)\|^2 \le \alpha\|w\|^2$, since $\frac{1}{2} \times \frac{d}{d\tau}\|w(t,\cdot)\|^2 = \langle w, Pw\rangle$. Such a growth bound, in turn, ensures well-posedness (see Kreiss and Lorenz (1989), Gustafsson, Kreiss, and Oliger (1995)).

We define $I = [\epsilon, 1-\epsilon]$. Here, $\epsilon > 0$ is chosen such that w_z is nonzero outside of I for all $\tau > 0$. By integration by parts, we have

$$\begin{aligned}
\langle w, Pw\rangle &= -\rho\|w\|^2 + \langle w, cw_z\rangle + \langle w, dw_{zz}\rangle \\
&= -\rho\|w\|^2 + \frac{1}{2}\left(\langle w, cw_z\rangle - \langle w_z, cw\rangle - \langle w, c_z w\rangle + [w^2 c]_0^1\right) - \langle w_z, dw_z\rangle - \langle w, d_z w_z\rangle + [w\, dw_z]_0^1 \\
&= -\rho\|w\|^2 - \langle w, c_z w\rangle - \lambda(1)w(t,1)^2 - \lambda(0)w(0,t)^2 - \langle w_z, dw_z\rangle - \langle w, d_z w_z\rangle \\
&\le (r-\rho)\|w\|^2 + \gamma \max_{z\in I} w(z)^2 - \langle w_z, dw_z\rangle - \langle w, d_z w_z\rangle \\
&\le \left(r + \sigma^2 - \rho\right)\|w\|^2 + \gamma \max_{z\in I} w(z)^2 - \frac{\sigma^2}{2}\int_0^1 z^2(1-z)^2 w_z^2 dz,
\end{aligned}$$

where $c(t,z) = az - \hat{\mu}z(1-z) + \sigma^2 z(1-z)^2$ and $d(z) = \sigma^2 z^2(1-z)^2/2$. Also, $\gamma = 2\max_{z\in I}\lambda(z)$ and $r = \max_{0\le z\le 1}|\hat{\mu}z(1-z) - \sigma^2 z(1-z)^2|$. The last inequality follows from

$$\begin{aligned}
-\langle w_z, dw_z\rangle - \langle w, d_z w_z\rangle &= \frac{\sigma^2}{2}\int_0^1 z(1-z)\left(-z(1-z)w_z^2 - (2-4z)ww_z\right)dz \\
&\le \frac{\sigma^2}{2}\int_0^1 z(1-z)\left(-z(1-z)w_z^2 + 2|w||w_z|\right)dz \\
&\le \frac{\sigma^2}{2}\int_0^1 z(1-z)\left(-z(1-z)w_z^2 + \frac{z(1-z)}{2}w_z^2 + \frac{2}{z(1-z)}w^2\right)dz \\
&= \sigma^2\|w\|^2 - \frac{\sigma^2}{2}\int_0^1 z^2(1-z)^2 w_z^2 dz,
\end{aligned}$$

where we use the relation $|u||v| \le \frac{1}{2}(\delta|u| + |v|/\delta)$ for all u, v for all $\delta > 0$. Finally, a standard Sobolev inequality implies that

$$\gamma \max_{z\in I} w(z)^2 \le \gamma\left(\xi\int_I w_z(z)^2 dz + \left(\frac{1}{\xi}+1\right)\int_I w(z)^2 dz\right)$$

for arbitrary $\xi > 0$. Specifically, we can choose $\xi = \epsilon^2(1-\epsilon)^2/(2\gamma)$ to bound

$$\gamma \max_{z\in I} w(z)^2 - \frac{\sigma^2}{2}\int_0^1 z^2(1-z)^2 w_z^2 dz \le \gamma\left(\frac{1}{\xi}+1\right)\|w\|^2,$$

and the final estimate is then

$$\frac{d}{d\tau}\|w\|^2 \le \left(r + \sigma^2 - \rho + \frac{\gamma}{\xi} + \gamma\right)\|w\|^2.$$

We have thus derived an energy estimate for the growth of $\|w\|^2$, and well-posedness follows from the theory in Kreiss and Lorenz (1989) and Gustafsson et al. (1995). Notice that we also used that $a(0,\cdot) > 0$ and $a(1,\cdot) < 0$ in the first equation, to ensure the negative sign in front of the $\lambda(0)$ and

[2] Since we impose no boundary conditions, it immediately follows that P is maximally semi-bounded if it is semi-bounded.

$\lambda(1)$ terms.

What remains is to show that if Condition 1 is satisfied, then the solution is indeed an interior one. We first note that an argument identical to that behind Proposition 1 in Longstaff (2001) implies that the central planner will never choose to be in the region $z < 0$ or $z > 1$, since the nonzero probability of ruin in these regions always makes such strategies inferior. Since any solution will be smooth, the only way in which the solution can fail to be interior is thus if $a = 0$ for some t, either at $z = 0$ or at $z = 1$.

We note that close to time T, the solution to (IA.23) will always be an interior one, since $\hat{\mu}(1-z) - \frac{\sigma^2}{2}(1-z)^2$ is strictly concave, with an optimum in the interior of $[0,1]$ and

$$w_z(T-\tau, z) = \int_0^\tau q_z(T-s, z)ds + O(\tau^3) = \frac{\tau^2}{2}\left(-\hat{\mu} + \sigma^2(1-z)\right) + O(\tau^3),$$

so the solution to $w_z = 0$ lies at $z_* = 1 - \frac{\hat{\mu}}{\sigma^2} + O(\tau)$, which from Condition 1 lies strictly inside the unit interval for small τ. Thus, if a solution degenerates into a non-interior one, it must happen after some time.

We next note that for the benchmark case in which $\lambda(z) \equiv 0$, that is, for the case with no flexibility, the solution is increasing in z at $z = 0$ and decreasing in z at $z = 1$ for all t. For example, at $z = 0$, by differentiating (16) with respect to z and once again using the transformation $\tau = T - t$, it is clear that w_z satisfies the ODE

$$(w_z)_\tau \quad = \quad -(\rho + \hat{\mu} - \sigma^2)w_z + q_z(T-\tau, 0), \qquad \text{(IA.25)}$$

and since $q_z(T-\tau, 0) > 0$ and $(w_z)(0,0) = 0$, it is clear that $(w_z) > 0$ for all $\tau > 0$. In fact, the solution to (IA.25) is

$$\frac{e^{-(\hat{\mu}+\rho)\tau}\left(-e^{-\tau\sigma^2}\rho + e^{\tau\hat{\mu}}(\hat{\mu}+\rho-\sigma^2) + e^{\tau(\hat{\mu}+\rho)}(-\hat{\mu}+\sigma^2)\right)}{\rho(\hat{\mu}+\rho-\sigma^2)},$$

which is strictly increasing in τ as long as Condition 1 is satisfied. An identical argument can be made at the boundary $z = 1$, showing that $w_z(\tau, 1) < 0$, for all $\tau > 0$. Now, standard theory of PDEs implies that, for any finite τ, w depends continuously on parameters for the lower-order terms, so $w_z \neq 0$ at boundaries for small but positive $\lambda(z)$.

For large τ, we know that w converges to the steady-state benchmark case, which has $w_z \neq 0$ in a neighborhood of the boundaries. Moreover, for small τ it is clear that $w_z \neq 0$ in a neighborhood of the boundaries according to the previous argument. Since the solution is smooth in $[0,T] \times [0,1]$, and $w_z \neq 0$ at the boundaries for all $\tau > 0$, it is therefore clear that there exists an $\epsilon > 0$ such that $w_z(t,z) > 0$ for all $\tau > 0$, for all $z < \epsilon$, and $w_z(t,z) < 0$ for all $z > 1-\epsilon$. Thus, for $\lambda \equiv 0$, and for λ close to zero by continuity, the solution is interior.

Next, it is easy to show that for any λ, the central planner will not choose to stay at the boundary for a very long time. To show this, we will use the obvious ranking of value functions implied by their control functions: $\lambda^1(z) \leq \lambda^2(z)$ for all $z \in [0,1] \Rightarrow w^1(\tau, z) \leq w^2(\tau, z)$ for all $\tau \geq 0$, $z \in [0,1]$, where w^1 is the solution to the central planner's problem with control constraint λ^1, and similarly for w^2.

Specifically, let's assume that $\lambda^1 \equiv 0$, and $\lambda^2 > 0$. Next, let's assume that for all $\tau > \tau_0$, the optimal strategy in the case with some flexibility (λ^2) is for the central planner to stay at the boundary $z = 1$ for some $\tau_0 > 0$. From (16), it is clear that $w^2(\tau, 0) = e^{-\rho(\tau-\tau_0)}w^2(\tau_0, 0)$, which will become arbitrarily small over time. Specifically, it will become smaller than $w^1(1-\epsilon,\tau)$ for arbitrarily small $\epsilon > 0$, in line with the previous argument, since $w^1(\tau,0) \equiv 0$ for all τ and $w^1_z(\tau,0) < -\nu$ for large τ, for some $\nu > 0$. It cannot therefore be optimal to stay at the boundary for arbitrarily large τ, since $w^2(\tau,1-\epsilon) \geq w^1(\tau,1-\epsilon) > w^2(\tau,0)$. A similar argument can be made for the boundary $z=0$.

In fact, a similar argument shows that the condition $w_z = 0$ can *never* occur at boundaries. For example, focusing on the boundary $z=0$, assume that $w_z = 0$ at $z=0$ for some τ and define $\tau_* = \inf_{\tau>0} w_z(\tau,0) = 0$. Similarly to the argument leading to (IA.25), the space derivative of (IA.23) at the boundary $z=0$ is

$$(w_z)_\tau \quad = \quad -(\hat{\mu}+\rho-\sigma^2)w_z + q_z + aw_{zz}, \tag{IA.26}$$

where $q_z = (-\hat{\mu}+\sigma^2)\frac{1-e^{-\rho\tau}}{\rho}$ is strictly positive for all $\tau > 0$. Since, per definition, $w_z(\tau_*^-,0) > 0$ and $w_z(\tau_*,0) = 0$, it must therefore be the case that $q_z + aw_{zz} \leq 0$, which, since $a(\tau,0) > 0$, for $\tau < \tau_*$, implies that w is strictly concave in a neighborhood of τ_* and $z=0$. Moreover, just before τ_*, say at $\tau_* - \Delta\tau$, w_z is zero at an interior point close to $z=0$ because of the strict convexity of w, that is, $w_z(\tau_*-\Delta\tau,\Delta z) = 0$. However, at Δz, w_z satisfies the following PDE, which follows directly from (IA.23):

$$(w_z)_\tau \quad = \quad -(\hat{\mu}+\rho-\sigma^2+O(\Delta z))w_z + (1+O(\Delta z))q_z + O((\Delta z)^2), \tag{IA.27}$$

and, since $w_z = 0$, this implies that

$$(w_z)_\tau \quad = \quad q_z + O((\Delta z)^2) > 0, \tag{IA.28}$$

so at time τ_*, $w_z(\tau_*,\Delta z) = q_z(\tau_*-\Delta\tau,\Delta z)\Delta\tau + O((\Delta z)^2\Delta\tau) + O((\Delta\tau)^2) > 0$. However, since w_{zz} is strictly concave on $z \in [0,\Delta z]$, it cannot be the case that $w_z(\tau_*,0) = 0$ and $w_z(\tau_*,\Delta z) > 0$, so we have a contradiction. A similar argument can be made at the boundary at $z=1$.

We have thus shown that the solution to (IA.23) must be an interior one and that, given that the solution is interior, the formulation as an initial value problem with no boundary conditions, (16) and (17), is well posed. We are done.

Since a is a bang-bang control, $az = \lambda(z)\mathrm{sign}(w_z)$. ■

Full statement of Proposition 1: Suppose that Condition 1 is satisfied. Then for a solution to the social planner's problem $V(B,D,t) \in C^2\left(\mathbb{R}^2_+ \times [0,T]\right)$, *with control* $a : [0,1]\times[0,T] \to [-1,1]$, *we have:*
a) if $\gamma = 1$,

$$V(B,D,t) = \frac{\log(B+D)}{\rho} + w\left(\frac{B}{B+D}, t\right),$$

where $w : [0,1]\times[0,T] \to \mathbb{R}$ *solves the PDE*

$$\begin{aligned} 0 \quad = \quad & w_t + \frac{1}{2}\sigma^2 z^2(1-z)^2 w_{zz} + \left(az - \hat{\mu}z(1-z) + \sigma^2\gamma z(1-z)^2\right) w_z \\ & -(\rho+p)w + \frac{1-e^{-\rho(T-t)}}{\rho}\left(\hat{\mu}(1-z) - \frac{\sigma^2(1-z)^2}{2}\right) \\ & +p\left[\frac{\log(1-|a|z)(1-e^{-\rho(T-t)})}{\rho} + w\left(\frac{(1-|a|)z}{1-|a|z}, t\right)\right], \end{aligned} \tag{IA.29}$$

where $a(z,t) = \alpha(z,t)\,\mathrm{sign}(w_z)$ *and, for each* z *and* t,

$$\alpha = \underset{\alpha\in[0,1]}{\arg\max}\ \alpha|w_z| + p\left[\frac{\log(1-\alpha z)(1-e^{-\rho(T-t)})}{\rho} + w\left(\frac{(1-\alpha)z}{1-\alpha z}, t\right)\right]; \tag{IA.30}$$

b) if $\gamma > 1$,

$$V(B,D,t) = -\frac{(B+D)^{1-\gamma}}{1-\gamma} w\left(\frac{B}{B+D}, t\right),$$

where $w : [0,1]\times[0,T] \to \mathbb{R}_-$ *solves the PDE*

$$\begin{aligned}
0 \;=\; & w_t + \frac{1}{2}\sigma^2 z^2(1-z)^2 w_{zz} + \left(az - \hat{\mu}z(1-z) + \sigma^2\gamma z(1-z)^2\right) w_z \\
& - \left[\rho + p - \hat{\mu}(1-\gamma)(1-z) + \frac{1}{2}\sigma^2\gamma(1-\gamma)(1-z)^2\right] w \\
& -1 + p\left[1 - (1-|a|z)^{1-\gamma} + w\left(\frac{(1-|a|)z}{1-|a|z}, t\right)\right],
\end{aligned} \tag{IA.31}$$

where $a(z,t) = \alpha(z,t)\,\mathrm{sign}(w_z)$ *and, for each* z *and* t,

$$\alpha(z,t) = \underset{\alpha\in[0,1]}{\arg\max}\ \alpha|w_z| + p\left[(1-\alpha z)^{1-\gamma} + w\left(\frac{(1-\alpha)z}{1-\alpha z}, t\right)\right]. \tag{IA.32}$$

For all $\gamma \geq 1$, *the terminal condition is*

$$w(z,T) = 0.$$

Proof of Proposition 1: We have

$$\begin{aligned}
dB &= aB\,dt - \alpha B\,dJ, \\
dD &= -aB\,dt + D\left(\hat{\mu}\,dt + \sigma\,d\omega\right).
\end{aligned}$$

For $\gamma = 1$, define

$$V(B,D,t) \equiv \sup_{a\in\mathcal{A}} E_t\left[\int_t^T e^{-\rho(s-t)}\log(B+D)\,ds\right].$$

The Bellman equation for optimality with jump-diffusion processes is then

$$\sup_{a\in\mathcal{A}}\left[V_t + \frac{1}{2}\sigma^2 D^2 V_{DD} + [\hat{\mu}D - aB]\,V_D + aBV_B - (\rho+p)V + \log(B+D) + pV((1-|a|)B, D, t)\right] = 0.$$

As before, by homogeneity, we can write V and its derivatives in terms of D and z:

$$V(B,D,t) = \frac{\log(B+D)\left(1-e^{-\rho(T-t)}\right)}{\rho} + w(z,t).$$

Using (IA.19) to (IA.22), and substituting into (IA.11), we obtain

$$0 = w_t + \frac{1}{2}\sigma^2 z^2(1-z)^2 w_{zz} + \left[az - \hat{\mu}z(1-z) + \sigma^2 z(1-z)^2\right] w_z - (\rho+p)w$$
$$+ \frac{1-e^{-\rho(T-t)}}{\rho}\left[\hat{\mu}(1-z) - \frac{\sigma^2(1-z)^2}{2}\right] + p\left[\frac{1-e^{-\rho(T-t)}}{\rho}\log(1-|a|z)) + w\left(\frac{(1-|a|)z}{1-|a|z}, t\right)\right].$$

A similar argument as in the proof of Proposition 1 implies that no boundary conditions are needed, and the natural terminal condition is $w(z,T) = 0$.

For $\gamma > 1$, define

$$V(B,D,t) \equiv \sup_{a\in\mathcal{A}} E_t\left[\int_t^T e^{-\rho(s-t)}\frac{(B(s)+D(s))^{1-\gamma}}{1-\gamma}\,ds\right].$$

Th Bellman equation for optimality is

$$0 \;\; = \;\; \sup_{a\in\mathcal{A}}\left[V_t + \frac{1}{2}\sigma^2 D^2 V_{DD} + \left[\hat{\mu}D - aB\right]V_D + aBV_B - (\rho+p)V + \frac{(B+D)^{1-\gamma}}{1-\gamma} + pV((1-|a|)B+D)\right].$$

By homogeneity, we can write

$$V(B,D,t) = -\frac{(B+D)^{1-\gamma}}{1-\gamma}w(z,t),$$

which, using (IA.13) to (IA.16), leads to

$$\begin{aligned}
0 \;\; = \;\; & \frac{1}{2}\sigma^2 z^2(1-z)^2 w_{zz} + \left(az - \hat{\mu}z(1-z) + \sigma^2\gamma z(1-z)^2\right) w_z \\
& - \left[\rho + p - \hat{\mu}(1-\gamma)(1-z) + \frac{1}{2}\sigma^2\gamma(1-\gamma)(1-z)^2\right] w \\
& -1 + p\left[(1-|a|z)^{1-\gamma} - 1 + w\left(\frac{(1-|a|)z}{1-|a|z}, t\right)\right].
\end{aligned}$$

A similar argument as in the proof of Proposition 1 implies that no boundary conditions are needed, and the natural terminal condition is $w(z,T) = 0$. ■

Proof of Proposition 2: We prove the proposition for the case $\gamma > 1$. A similar argument applies to the case $\gamma = 1$. It follows directly from the proof of Proposition 1 that w is increasing close to $z = 0$ and decreasing close to $z = 1$, given that Condition 1 is satisfied. To prove that w is concave, we first note that given the linear constraints in the optimization problem, the social planner could divide the initial capital B_0 and D_0, corresponding to an initial bank share $z_0 = \frac{B_0}{B_0+D_0}$, into $B_0 = B_0^1 + B_0^2$ and $D_0 = D_0^1 + D_0^2$, and treat (B_0^1, D_0^1) and (B_0^2, D_0^2) as two separate allocation problems, with $z_0^1 = \frac{B_0^1}{B_0^1+D_0^1}$ and $z_0^2 = \frac{B_0^2}{B_0^1+D_0^2}$.

Specifically, the social planner could choose a control a_1 for the (B_0^1, D_0^1) problem and another, a_2, for the (B_0^2, D_0^2) problem. In fact, he could choose the *optimal* controls a_1^* and a_2^*, respectively, for these two subproblems. Such a strategy, although feasible, would obviously be dominated compared with the one achieved by the optimal control for the global problem, a^*, that is,

$$\frac{1}{1-\gamma}E\left[\int_0^T (B_t+D_t)^{1-\gamma}e^{-\rho t}dt\Big|a^*\right] \geq \frac{1}{1-\gamma}E\left[\int_0^T (B_t^1+D_t^1+B_t^2+D_t^2)^{1-\gamma}e^{-\rho t}dt\Big|a_1^*,a_2^*\right], \tag{IA.33}$$

or equivalently

$$\frac{1}{1-\gamma}E\left[\int_0^T \left(\frac{B_t+D_t}{B_0+D_0}\right)^{1-\gamma}e^{-\rho t}dt\Big|a^*\right] \geq \frac{1}{1-\gamma}E\left[\int_0^T \left(\frac{B_t^1+D_t^1+B_t^2+D_t^2}{B_0+D_0}\right)^{1-\gamma}e^{-\rho t}dt\Big|a_1^*,a_2^*\right], \tag{IA.34}$$

Now define

$$\kappa = \frac{B_0^1+D_0^1}{B_0+D_0}.$$

It then follows that

$$\frac{B_t^1+D_t^1+B_t^2+D_t^2}{B_0+D_0} = \kappa\frac{B_t^1+D_t^1}{B_0^1+D_0^1}+(1-\kappa)\frac{B_t^1+D_t^1}{B_0^1+D_0^1}, \tag{IA.35}$$

and given the concavity of the utility function $U(x)=\frac{1}{1-\gamma}x^{1-\gamma}$, $U(\kappa x+(1-\kappa)y)\geq \kappa U(x)+(1-\kappa)U(y)$ for arbitrary x and y, and more specifically for $x=\frac{B_t^1+D_t^1}{B_0+D_0}$ and $y=\frac{B_t^2+D_t^2}{B_0+D_0}$, it then further follows that

$$\begin{aligned}\frac{1}{1-\gamma}E\left[\int_0^T \left(\frac{B_t^1+D_t^1+B_t^2+D_t^2}{B_0+D_0}\right)^{1-\gamma}e^{-\rho t}dt\Big|a_1^*,a_2^*\right] \quad &\geq \quad \frac{\kappa}{1-\gamma}E\left[\int_0^T \left(\frac{B_t^1+D_t^1}{B_0^1+D_0^1}\right)^{1-\gamma}e^{-\rho t}dt\Big|a_1^*\right] \\ &+ \quad \frac{1-\kappa}{1-\gamma}E\left[\int_0^T \left(\frac{B_t^2+D_t^2}{B_0^2+D_0^2}\right)^{1-\gamma}e^{-\rho t}dt\Big|a_2^*\right].\end{aligned}$$

Using the definition of the normalized value function, $w=\frac{\gamma-1}{(B_0+D_0)^{1-\gamma}}V(B_0,D_0,t)$, this can be rewritten as

$$(\gamma-1)w(z_0)\geq(\gamma-1)\kappa w(z_0^1)+(\gamma-1)(1-\kappa)w(z_0^2),$$

or

$$w(z_0)\geq \kappa w(z_0^1)+(1-\kappa)w(z_0^2). \tag{IA.36}$$

Finally, note that $z_0=\kappa z_0^1+(1-\kappa)z_0^2$, so (IA.36) is indeed the condition that ensures that w is a concave function.

The form of the capital flow function follows immediately from the shape of w. Specifically, given that $w_z>0$ on some interval $z\in(0,z^\ell)$, it follows from the social planner's optimization problem that he will only consider risking a crash by moving capital into the bank sector in this region, that is, $az\geq 0$. A similar argument implies that for $z\in(z^r,1)$, the social planner will only consider moving capital out of the bank sector, that is, $az\leq 0$. Finally, since w is smooth, there will be an interval of positive length to the left of a point z_* where w reaches its maximum, $(z_*-\epsilon,z_*)$, where w_z is close to zero, and the term $p(1-\alpha z)^{1-\gamma}$ in (15)—which is decreasing in α—therefore dominates the positive effect of the $\alpha|w_z|$ term. The third term, $pw\left(\frac{(1-\alpha)z}{1-\alpha z},t\right)$, in the equation is also nonpositive on this interval and the optimal choice for the social planner is therefore to choose $\alpha=0$, immediately implying that $a=0$. ■

Proof of Proposition 3: First, we note from Proposition 5 that the risk-free rate is

$$r_s = \rho + \gamma(1-z)\hat{\mu} - \gamma(\gamma+1)(1-z)^2\frac{\sigma^2}{2}. \tag{IA.37}$$

Now, the expected return of the bank tree is

$$r^B dt = E\left[\frac{dq^B + dt}{q^B}\right],$$

so

$$r^B q^B dt = E\left[q_z^B dz + \frac{1}{2}q_{zz}^B (dz)^2 + dt\right].$$

From equation (10),

$$\begin{aligned} r^B q^B = & \left[az - \hat{\mu}z(1-z) + \sigma^2 z(1-z)^2\right] q_z^B \\ & + \frac{1}{2}\sigma^2 z^2(1-z)^2 q_{zz}^B + 1, \end{aligned}$$

so

$$\begin{aligned} r^B q^B - r_s q^B &= \left[az - \hat{\mu}z(1-z) + \sigma^2 z(1-z)^2\right] q_z^B \\ &+ \frac{1}{2}\sigma^2 z^2(1-z)^2 q_{zz}^B - \left[\rho + \gamma(1-z)\hat{\mu} - \gamma(\gamma+1)(1-z)^2\frac{\sigma^2}{2}\right] q^B + 1. \end{aligned}$$

Now from (IA.7), given that $p_t = 0$ (since $T = \infty$) and $p \equiv q^B$ when $\delta \equiv 1$, it follows that

$$(r^B - r_s)q^B = -\gamma z(1-z)^2\sigma^2 q_z^B,$$

so the expected excess return on the bank tree is given by

$$r^B - r_s = -\gamma z(1-z)^2\sigma^2\frac{q_z^B}{q^B}.$$

To calculate the bank tree's volatility, we use the formula

$$(\sigma^B)^2 = \langle dP^B, dP^B\rangle/(P^B)^2 = \langle dq^B, dq^B\rangle/(q^B)^2 = (q_z^B)^2(dz)^2/(q^B)^2,$$

and since $dz^2 = \sigma^2 z^2(1-z)^2$, we get

$$\sigma^B = \sigma z(1-z)\frac{|q_z^B|}{q^B}.$$

It follows that the Sharpe ratio is

$$S^B = \frac{r^B - r_s}{\sigma^B} = -\gamma\sigma(1-z)\,\mathrm{sign}(q_z^B).$$

A similar argument applies to the equity tree. We begin by observing that

$$r^D dt = E\left[\frac{d(q^D\hat{D}) + \hat{D}dt}{q^D\hat{D}}\right] = E\left[\frac{dq^D\,\hat{D} + q^D\,d\hat{D} + dq^D\,d\hat{D} + \hat{D}dt}{q^D\hat{D}}\right],$$

where $d\hat{D} = \hat{D}(\hat{\mu}dt + \sigma d\omega)$ is the dynamics of equity capital without reallocation (see Internet Appendix Section C), so

$$r^D q^D dt = E\left[q_z^D dz + \frac{1}{2}q_{zz}^D(dz)^2 + q^D\frac{d\hat{D}}{\hat{D}} + dq^D\frac{d\hat{D}}{\hat{D}} + dt\right],$$

which from (10) and the expression for r_s leads to

$$\begin{aligned}(r^D - r_s)q^D &= \left[az - \hat{\mu}z(1-z) + \sigma^2 z(1-z)^2\right]q_z^D + \frac{1}{2}\sigma^2 z^2(1-z)^2 q_{zz}^D \\ &+ q^D\hat{\mu} - z(1-z)\sigma^2 q_z^D + 1 - \left[\rho + \gamma(1-z)\hat{\mu} - \gamma(\gamma+1)(1-z)^2\frac{\sigma^2}{2}\right]q^D.\end{aligned}$$

Using (IA.9), we arrive at

$$r^D - r_s = \gamma\sigma^2(1-z)\left(1 - z(1-z)\frac{q_z^D}{q^D}\right).$$

For the variance of returns of the equity sector, we write

$$(\sigma^D)^2 = \langle dP^D, dP^D\rangle/(P^D)^2 = \langle d(q^D D), d(q^D D)\rangle/(q^D D)^2 = \langle q_z^D dzD + q^D dD, q_z^D dzD + q^D dD\rangle/(q^D D)$$

(including reallocation, $az\,dt$, here does not change the results since its quadratic variation is zero, which motivates using D instead of $\hat{D}$), which leads to

$$\sigma^D = \sigma\left|1 - z(1-z)\frac{q_z^D}{q^D}\right|.$$

It follows that the Sharpe ratio is

$$S^D = \frac{r^D - r_s}{\sigma^D} = \gamma\sigma(1-z)\,\mathrm{sign}\left(1 - z(1-z)\frac{q_z^D}{q^D}\right).$$

■

Proof of Proposition 4: In general, a central planner's problem, possibly including consumption/investment trade-offs, is

$$\max\ E_t\left[\int_t^T e^{-\rho(s-t)}u(C_s)ds\right],$$

subject to constraints. With CRRA utility, this can be rewritten as

$$\frac{1}{1-\gamma}\min\ E_t\left[\int_t^T e^{-\rho(s-t)}u'(C_s)C_s ds\right].$$

In general, C_t can be chosen by the central planner. In our exchange-like economy with reallocation, however, $C_t = B_t + D_t$ is fixed, and the central planner can only influence future consumption.

Therefore, the optimization problem is identical to

$$\frac{u'(C_t)C_t}{1-\gamma}\min\ \frac{1}{u'(C_t)C_t}E_t\left[\int_t^T e^{-\rho(s-t)}u'(C_s)C_s ds\right],$$

but the Euler equations imply that

$$\frac{1}{u'(C_t)} E_t \left[\int_t^T e^{-\rho(s-t)} u'(C_s) C_s ds \right] = P_B + P_D,$$

so the central planner's problem is to solve

$$\frac{u'(C_t)C_t}{1-\gamma} \min \frac{P_B + P_D}{B+D}, \qquad \text{that is,}$$

$$\frac{(B_t + D_t)^{1-\gamma}}{1-\gamma} \min \frac{P_B + P_D}{B+D}.$$

Thus, the central planner's problem is to minimize the market price-dividend ratio. In fact, we have

$$\begin{aligned}
\frac{P_D + P_B}{B+D} &= (B+D)^{\gamma-1}(1-\gamma) E_t \left[\int_t^T e^{-\rho(s-t)} \frac{(B_s + D_s)^{1-\gamma}}{1-\gamma} ds \right] \\
&= -(B+D)^{\gamma-1}(1-\gamma) \frac{(B+D)^{1-\gamma}}{1-\gamma} w(z,t) = -w(z,t).
\end{aligned}$$

■

Proof of Proposition 5: The value function at $t = 0$ is

$$V_0 = E\left[\int_0^\infty e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma} \, dt \right] = -\frac{C_0^{1-\gamma}}{1-\gamma} E \left[-\int_0^\infty e^{-\rho t} \left(\frac{C_t}{C_0} \right)^{1-\gamma} dt \right] \stackrel{\text{def}}{=} -\frac{C_0^{1-\gamma}}{1-\gamma} w(z),$$

where the last definition of $w(z)$ is possible because the terms within the expectation do not depend on either C or T in the infinite horizon economy. A similar expression is, of course, valid at arbitrary t because of the time-homogeneity of the problem.

In a complete-market equilibrium, the price of the market claim—through the Euler conditions—is

$$\begin{aligned}
P_0 &= \frac{1}{u'(C_0)} E \left[\int_0^\infty e^{-\rho t} C_t^{-\gamma} C_t \, dt \right] \\
&= -C_0 C_0^{\gamma-1} E \left[-\int_0^\infty e^{-\rho t} C_t^{1-\gamma} \, dt \right] \\
&= -C_0 E \left[-\int_0^\infty e^{-\rho t} \left(\frac{C_t}{C_0} \right)^{1-\gamma} dt \right] \\
&= -C_0 w(z).
\end{aligned}$$

Again, the formula extends to arbitrary t, $P_t = -C_t w(z)$.

The pricing kernel is $M_t = e^{-\rho t} C_t^{-\gamma}$ and, from a standard argument, the formula for the market risk premium is $(r_e - r_s)dt = -\operatorname{cov}\left(\frac{dM_t}{M_t}, \frac{dP_t + C_t dt}{P_t}\right)$. We decompose the risk premium into

$$\begin{aligned}
(r_e - r_s)dt &= -E\left[\frac{dM_t - E[dM_t]}{M_t}, \frac{dP_t + C_t dt}{P_t}\right] \\
&= -(1 - p\,dt)E\left[\frac{dM_t - E[dM_t]}{M_t}, \frac{dP_t + C_t dt}{P_t}\Big|\text{No Jump}\right] \\
&\quad -p\,dtE\left[\frac{dM_t - E[dM_t]}{M_t}, \frac{dP_t + C_t dt}{P_t}\Big|\text{Jump}\right].
\end{aligned}$$

Disregarding the higher-order term in the first term leads to

$$\begin{aligned}
-(1 - p\,dt)E\left[\frac{dM_t - E[dM_t]}{M_t}, \frac{dP_t + C_t dt}{P_t}\Big|\text{No Jump}\right] &= \gamma\operatorname{cov}\left(\frac{dC}{C}, \left(\frac{dC}{C} + \frac{w_z}{w}dz\right)\right) \\
&= \gamma\sigma_c^2 dt + \gamma\frac{w_z}{w}\sigma_c\sigma_z dt \\
&= \gamma\sigma_c^2 dt + \gamma\frac{X_z}{X}\sigma_c\sigma_z dt \\
&= \gamma\sigma_c^2 dt + \gamma\frac{d[\log X(z)]}{dz}\sigma_c\sigma_z dt \\
&= r_m(1 + g(z))dt.
\end{aligned}$$

Similarly, disregarding higher-order terms in the second term leads to

$$\begin{aligned}
-p\,dtE\left[\frac{dM_t - E[dM_t]}{M_t}, \frac{dP_t + C_t dt}{P_t}\Big|\text{Jump}\right] &= p\,dtE\left[\left(\frac{C_t^{-\gamma} - C_{t-}^{-\gamma}}{C_t^{-\gamma}}\right)\left(\frac{w_t C_t - w_{t-}C_{t-}}{w_{t-}C_{t-}}\right)\right] \\
&= p\,dtE\left[\left(1 - \left(\frac{C_{t-}}{C_t}\right)^{\gamma}\right)\left(1 - \frac{w_t C_t}{w_{t-}C_{t-}}\right)\right].
\end{aligned}$$

Defining $C' \stackrel{\text{def}}{=} C_t = C_{t-}(1 - j_c(z))$, $z' \stackrel{\text{def}}{=} z_t = z_{t-} - j_z(z)$, and $q_J(z) \stackrel{\text{def}}{=} \left(1 - \left(\frac{C_t}{C'}\right)^{\gamma}\right)\left(1 - \frac{w(z')C'}{w(z)C}\right) = \left(1 - \left(\frac{C_t}{C'}\right)^{\gamma}\right)\left(1 - \frac{X(z')C'}{X(z)C}\right)$ therefore gives $pq_J(z)dt$ for the second term. In sum, we have the following expression for the market risk premium:

$$r_e - r_s = (1 + g(z))\,r_m + p\,q_J(z).$$

A standard argument for the short risk-free rate in the economy reveals that it is

$$\begin{aligned}
r_s dt &= -E\left[\frac{dM_t}{M_t}\right] \\
&= -(1 - p\,dt)E\left[\frac{dM_t}{M_t}\Big|\text{No jump}\right] - p\,dtE\left[\frac{dM_t}{M_t}\Big|\text{Jump}\right].
\end{aligned}$$

Ignoring higher-order terms, the first term reduces to

$$-(1 - p\,dt)E\left[\frac{dM_t}{M_t}\Big|\text{No jump}\right] = \left(\rho + \gamma\mu_c(z) - \gamma(\gamma + 1)\frac{\sigma_c^2(z)}{2}\right)dt.$$

Further, a similar argument as when deriving the formula for the market price of risk leads to

$-p\,dtE\left[\frac{dM_t}{M_t}\middle|\text{Jump}\right] = -p\,dtE\left[\left(\frac{C}{C'}\right)^\gamma - 1\right]$. We therefore have

$$r_s = \rho + \gamma\mu_c(z) - \gamma(\gamma+1)\frac{\sigma_c^2(z)}{2} - pE\left[\left(\frac{C}{C'}\right)^\gamma - 1\right].$$

A similar argument for the instantaneous variance of market returns, $\sigma^2 dt = \mathrm{cov}\left(\frac{dP}{P}, \frac{dP}{P}\right)$, once again using $P = XC$ and decomposing the return into jump and no-jump components, leads to

$$\sigma^2 = \sigma_c^2(1+g(z))^2 + p\left(1 - \frac{C'}{C}\frac{X(z')}{X(z)}\right)^2.$$

We are done. ■

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