Internet Appendix for "Liquidity Cycles and Make/Take Fees in Electronic Markets"*

Thierry Foucault, Ohad Kadan, and Eugene Kandel

In this appendix we provide additional results and proofs mentioned in the paper but unreported there for brevity. The appendix is organized as follows. In Section A we provide the proof of Corollary 1 in the paper. In Section B we provide a closed-form solutions for traders' monitoring levels and the optimal make and take fees when $M = N = 1$. In Section C we solve for the optimal level of the total fee, $\bar{c}$, charged by the trading platform in the case of a thick market. In Section D we provide the proof of Proposition 7. In Section E we show that, for fixed make and take fees, a reduction in trader's monitoring costs is a Pareto improvement. In Section F we provide a detailed analysis of the extension of the model discussed in Section A in the paper (finite arrival rate for market takers' trading needs). In Section G we provide the Matlab code used for the numerical simulations in the paper.

*Citation format: Foucault, Thierry, Ohad Kadan, and Eugene Kandel, Internet Appendix for "Cycle and Make/Take Fees in Electronic Markets," *Journal of Finance*, DOI: 10.1111/j.1540-6261.2012.01801.x.

A. Proof of Corollary 1

Recall that $\mathcal{V}^*$ is such that

$$\mathcal{V}^* = h(\mathcal{V}^*, M, N, z), \tag{IA.1}$$

where $h(\cdot)$ is defined in Equation (A12). It is immediate that $h(\cdot)$ increases in M, decreases in N, and increases in z. As $h(\cdot)$ decreases in $\mathcal{V}^*$, we have

$$\frac{\partial \mathcal{V}^*}{\partial M} > 0, \tag{IA.2}$$

$$\frac{\partial \mathcal{V}^*}{\partial N} < 0, \tag{IA.3}$$

$$\frac{\partial \mathcal{V}^*}{\partial z} > 0. \tag{IA.4}$$

Now, using equations (14) and (IA.3), we conclude that

$$\frac{\partial \mu_i^*}{\partial N} = \frac{-\frac{\partial \mathcal{V}^*}{\partial N} \cdot ((M+1) + (M-1)\mathcal{V}^*)}{(1+\mathcal{V}^*)^3} \left(\frac{\pi_m}{M\beta}\right) > 0.$$

Hence, $\frac{\partial \bar{\mu}^*}{\partial N} > 0$. Furthermore, since $\bar{\tau}^* = \frac{\bar{\mu}^*}{\mathcal{V}^*}$, we have (using (IA.3)) that $\frac{\partial \bar{\tau}^*}{\partial N} > 0$. A similar argument shows that $\frac{\partial \bar{\tau}^*}{\partial M} > 0$ and $\frac{\partial \bar{\mu}^*}{\partial M} > 0$.

Now consider the effect of a change in β on market takers' monitoring intensities. We have (see Proposition 2)

$$\tau_j^* = \zeta(\mathcal{V}^*) \left(\frac{\pi_t}{N\gamma}\right),$$

where

$$\zeta(\mathcal{V}^*) = \left(\frac{\mathcal{V}^* \left((1 + \mathcal{V}^*) N - 1 \right)}{(1 + \mathcal{V}^*)^2} \right).$$

Thus,

$$\frac{\partial \tau_j^*}{\partial \beta} = \left(\frac{\partial \zeta(\mathcal{V}^*)}{\partial \mathcal{V}^*} \frac{\partial \mathcal{V}^*}{\partial z} \frac{\partial z}{\partial \beta} \right) \left(\frac{\pi_t}{N\gamma} \right).$$

We have $\frac{\partial \zeta(\mathcal{V}^*)}{\partial \mathcal{V}^*} > 0$. Moreover $\frac{\partial \mathcal{V}^*}{\partial z} > 0$ and $\frac{\partial z}{\partial \beta} < 0$. Thus,

$$\frac{\partial \tau_j^*}{\partial \beta} < 0,$$

which implies that $\frac{\partial \bar{\tau}^*}{\partial \beta} < 0$. Now, since $\bar{\mu}^* = \mathcal{V}^* \bar{\tau}^*$, we have

$$\frac{\partial \bar{\mu}^*}{\partial \beta} = \mathcal{V}^* \frac{\partial \bar{\tau}^*}{\partial \beta} + \frac{\partial \mathcal{V}^*}{\partial z} \frac{\partial z}{\partial \beta} \bar{\tau}^* < 0,$$

which implies $\frac{\partial \mu_i^*}{\partial \beta} < 0$ for all i. The impact of make/take fees on traders' aggregate monitoring levels is obtained in the same way. The second part of the corollary follows directly from the first part and the definition of the trading rate (equation (7)).

B. The Case $M = N = 1$

When there is one market maker and one market taker, we can easily obtain closed-form solutions for traders' monitoring levels and the optimal make/take fee breakdown.

Claim 1: *When $M = N = 1$, the market maker's monitoring level is $\mu_1^* = \left(1 + z^{\frac{1}{3}}\right)^{-2} \cdot \left(\frac{\pi_m}{\beta}\right)$ and the market taker's monitoring level is $\tau_1^* = \left(1 + z^{-\frac{1}{3}}\right)^{-2} \cdot \left(\frac{\pi_t}{\gamma}\right)$.*

Proof of Claim 1**:** When $M = N = 1$, it is immediate that the solution to equation (16) in

Proposition 2 is $\mathcal{V}^* = z^{\frac{1}{3}}$. The expressions for traders' monitoring levels are then obtained by replacing $\mathcal{V}^*$ by its value in equations (14) and (15) in Proposition 2. ■

We now derive the optimal make/take fee breakdown for the platform when $M = N = 1$.

Claim 2: *Assume $M = N = 1$ and suppose that $a - v_0 = s \cdot \Delta$. The trading platform optimally allocates its fee $\bar{c}$ between the market-making side and the market-taking side as follows:*

$$c^*_{ms} = s \cdot \Delta - \frac{\Gamma - \bar{c}}{1 + r^{\frac{1}{4}}} \quad \textit{and } c^*_{ts} = \bar{c} - c^*_m.$$

Proof of Claim **2.** There is a one-to-one mapping between the fees charged by the platform and the per trade trading profits obtained by the market-making side and the market-taking side, π_m and π_t. Thus, instead of using c_m and c_t as the decision variables of the platform, we can use π_m and π_t. Thus, for a fixed $\bar{c}$, we rewrite the platform's problem as

$$\begin{aligned} &Max_{\pi_m,\pi_t} \frac{\mu_1^* \tau_1^*}{\mu_1^* + \tau_1^*} \bar{c}, \\ &s.t \;\; \pi_t + \pi_m = \overline{\pi}. \end{aligned}$$

From Claim 1, we obtain,

$$\frac{\mu_1^*}{\tau_1^*} = z^{\frac{1}{3}} = (\frac{\pi_m}{\pi_t} \frac{\gamma}{\beta})^{\frac{1}{3}}$$

and

$$\mu_1^* = \frac{\pi_m}{\beta} \frac{1}{\left(1 + z^{\frac{1}{3}}\right)^2}.$$

Thus, we can rewrite the previous optimization problem as

$$Max_{\pi_m, z} \quad \frac{\mu_1^*}{1 + z^{\frac{1}{3}}} \bar{c} \tag{IA.5}$$

$$s.t \quad \pi_m \left(1 + \frac{\gamma}{\beta z}\right) = \Gamma - \bar{c}. \tag{IA.6}$$

$$\text{and } \mu_1^* = \frac{\Gamma - \bar{c}}{\beta \left(1 + z^{\frac{1}{3}}\right)^2 \left(1 + \frac{\gamma}{\beta z}\right)}. \tag{IA.7}$$

This problem is equivalent to finding z that minimizes

$$\left(1 + z^{\frac{1}{3}}\right)^3 \left(\beta + \frac{\gamma}{z}\right).$$

The FOC to this problem imposes

$$-\frac{1}{z^2} \left(\gamma - z^{\frac{4}{3}} \beta\right) \left(z^{\frac{1}{3}} + 1\right)^2 = 0.$$

Hence, the solution is

$$z = \left(\frac{\gamma}{\beta}\right)^{\frac{3}{4}} = r^{\frac{3}{4}}. \tag{IA.8}$$

Using the constraint (IA.6), we have

$$\pi_m^* = \frac{\Gamma - \bar{c}}{1 + r^{\frac{1}{4}}}. \tag{IA.9}$$

Since $\pi_m^* = a - v_0 - c_{ms}^*$ and $c_{ms}^* + c_{ts}^* = \bar{c}$, we obtained the required results.

C. The Optimal Level of Total Fees ($\bar{c}$)

The platform's optimization problem can be decomposed into two steps: (i) choose the optimal make/take fees for a given $\bar{c}$ (we solved this problem in the paper); and (ii) choose the optimal

$\bar{c}$. Observe that the optimal make/take fees, (c_m^*, c_t^*), increase in $\bar{c}$, and recall that the trading rate decreases in both the make fee and the take fee (Corollary 1). Thus, in the second step the trading platform faces the standard price-quantity trade-off: by raising $\bar{c}$, the trading platform gets a larger revenue per trade but it decreases the rate at which trades occur. The next result gives the optimal value of $\bar{c}$ for the trading platform in the case of a thick market.

Claim 3: *In the thick market case, the trading platform maximizes its expected profit by setting its total trading fee at $\bar{c} = \Gamma/2$ and by splitting this fee between the two sides as described in Proposition 3.*

In contrast to the make/take fees, the optimal total fee for the platform is independent of traders' relative monitoring costs and the relative size of the market-making side. Thus, our results regarding the effect of q and r hold even if $\bar{c}$ is optimally set by the trading platform.

Proof of Claim **3:** We fix $q > 0$ and let $N = \frac{M}{q}$. From (A26) and (A27), we obtain that when fees are set optimally in the thick market case, we have

$$z = \frac{\pi_m}{\pi_t} r = q^{-\frac{1}{3}} r^{\frac{2}{3}}$$

and

$$\mathcal{V}^{\infty} = (zq)^{\frac{1}{2}} = (rq)^{\frac{1}{3}}. \tag{IA.10}$$

Using Corollary 4 and equations (A26), (A27), and (IA.10) we obtain

$$\mu_i^{\infty} = \frac{\Gamma - \bar{c}}{\beta\left(1 + (qr)^{\frac{1}{3}}\right)^2} \text{ and } \tau_j^{\infty} = \frac{\Gamma - \bar{c}}{\gamma\left(1 + (qr)^{-\frac{1}{3}}\right)^2} \text{ for } i, j = 1, 2, ... \tag{IA.11}$$

Now, for any given M, maximizing $\mathcal{R}(\mu^*, \tau^*)\bar{c}$ is equivalent to maximizing $\frac{\mathcal{R}(\mu^*,\tau^*)}{M}\bar{c}$, which in turn is equivalent to maximizing $\frac{\mu_1^*}{1+\mathcal{V}^*}\bar{c}$ (using equation (A13) in the Proof of Proposition 3 and the fact that $\bar{\mu}^* = M\mu_1^*$) . Denote $\mathcal{H}(\bar{c}) \equiv \frac{\mu_1^*}{1+\mathcal{V}^*}$. Then, to find the optimal total fee $\bar{c}$ in the thick market we need to find the limit as M tends to infinity:

$$\arg\max_{\bar{c}\geq 0} \mathcal{H}(\bar{c})\,\bar{c}.$$

The FOC of this optimization problem for a given M is

$$\mathcal{H}(\bar{c}) + \mathcal{H}'(\bar{c})\,\bar{c} = 0. \tag{IA.12}$$

Note that $\mathcal{H}$ depends on $\bar{c}$ only through its dependence on μ_1^* and $\mathcal{V}^*$. It follows that

$$\mathcal{H}'(\bar{c}) = \frac{\partial \mathcal{H}}{\partial \mu_1^*}\frac{\partial \mu_1^*}{\partial \bar{c}} + \frac{\partial \mathcal{H}}{\partial \mathcal{V}^*}\frac{\partial \mathcal{V}^*}{\partial \bar{c}} = \frac{1}{1+\mathcal{V}^*}\frac{\partial \mu_1^*}{\partial \bar{c}} - \frac{\mu_1^*}{(1+\mathcal{V}^*)^2}\frac{\partial \mathcal{V}^*}{\partial \bar{c}}. \tag{IA.13}$$

Since equation (IA.12) holds for any M, we can take the limit as $M \to \infty$. We have

$$\lim_{M\to\infty} \mathcal{H}(\bar{c}) = \frac{\mu_1^\infty}{1+\mathcal{V}^\infty} = \frac{\Gamma - \bar{c}}{\beta\left(1+(qr)^{\frac{1}{3}}\right)^3} \quad \text{(using (IA.10) and (IA.11))}.$$

From (IA.10) and (IA.11) it also follows that

$$\begin{aligned}
\lim_{M\to\infty} \frac{\partial \mu_1^*}{\partial \bar{c}} &= \frac{\partial \mu_1^\infty}{\partial \bar{c}} = -\frac{1}{\beta\left(1+(qr)^{\frac{1}{3}}\right)^2}, \text{ and} \\
\lim_{M\to\infty} \frac{\partial \mathcal{V}^*}{\partial \bar{c}} &= \frac{\partial \mathcal{V}^\infty}{\partial \bar{c}} = 0.
\end{aligned}$$

Thus, from (IA.13),

$$\lim_{M\to\infty} \mathcal{H}'(\bar{c}) = -\frac{1}{\beta\left(1+(qr)^{\frac{1}{3}}\right)^3}.$$

Hence, in the limit (IA.12) becomes

$$\frac{\Gamma-\bar{c}}{\beta\left(1+(qr)^{\frac{1}{3}}\right)^3} - \frac{1}{\beta\left(1+(qr)^{\frac{1}{3}}\right)^3}\bar{c} = 0,$$

which gives $\bar{c} = \frac{\Gamma}{2}$.

D. Proof of Proposition 7

In this section we develop in detail the analysis that leads to the proof of Proposition 7. We also provide numerical examples that illustrate this proposition.

Proof of Part 1 of Proposition **7**. We first show that the ask price, $a^*(c_m, \theta)$, is an increasing step function of the make fee when there is a positive tick size.

Claim 4: *The optimal ask price is an increasing step function of the make fee c_m. Specifically, there exists a partition of the interval $[\hat{c}_{m1}, \hat{c}_{m1}+(\ell-1)\Delta]$ into $\ell-1$ segments given by $[\hat{c}_{ms}, \hat{c}_{ms+1}]$, where $\hat{c}_{m1}$ is the unique solution to the equation $\mathcal{O}(v_0, \hat{c}_{m1}) = \mathcal{O}(v_0+\Delta, \hat{c}_{m1})$, and $\hat{c}_{ms} = \hat{c}_{m1} + (s-1)\Delta$ for $s \in \{1, .., \ell\}$ such that:*

1. *When $c_m \in (\hat{c}_{ms}, \hat{c}_{ms+1})$, the optimal ask price is unique and given by $a^*(c_m, \theta) = v_0 + s\cdot\Delta$ for $s \in \{1, .., \ell-1\}$.*

2. *When $c_m = \hat{c}_{ms+1}$ ($s \in \{1, .., \ell-1\}$), both $v_0+s\Delta$ and $v_0+(s+1)\Delta$ are optimal ask prices, and we can set $a^*(c_m, \theta)$ to any of them.*

Proof of Claim **4:** Since θ is fixed throughout this proof, we omit the argument for θ in $\mathcal{O}(a, c_m, \theta)$ to save space. It is straightforward that the objective function $\mathcal{O}(a, c_m)$ is concave in a and that $\frac{\partial^2 \mathcal{O}(a,c_m)}{\partial a \partial c_m} > 0$. Thus, $\mathcal{O}(a, c_m)$ satisfies the Milgrom-Shannon (1994) single-crossing property (SCP): if $\mathcal{O}(a', c_m) \geq \mathcal{O}(a, c_m)$, then $\mathcal{O}(a', c_m') > \mathcal{O}(a, c_m')$ for all $a' > a$ and $c_m' > c_m$.

Let $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ be the floor and ceiling functions, respectively.[38] When the tick size is zero the solution to the optimization problem (30) is

$$a_0^*(c_m, \theta) = v_0 + c_m + (1 - \theta)(\Gamma - \bar{c}).$$

The objective function, $\mathcal{O}(\cdot, \cdot)$, is concave in a. Thus, when the tick size is strictly positive, the solution to (30) is $a_0^*(c_m, \theta)$ (the unconstrained solution) rounded up or down to the first feasible price on the grid. That is,

$$a^*(c_m, \theta) = v_0 + s^*(c_m, \theta) \cdot \Delta,$$

where

$$s^*(c_m, \theta) \in \left\{ \left\lfloor \frac{c_m + (1 - \theta)(\Gamma - \bar{c})}{\Delta} \right\rfloor, \left\lceil \frac{c_m + (1 - \theta)(\Gamma - \bar{c})}{\Delta} \right\rceil \right\}.$$

We first show that $s^*(c_m, \theta)$ and hence $a^*(c_m, \theta)$ weakly increase in c_m. We proceed by contradiction. Suppose that $c_m < c_m'$ and yet $s^*(c_m, \theta) > s^*(c_m', \theta)$. As $a_0^*(c_m, \theta) < a_0^*(c_m', \theta)$, this is possible only if

$$s^*(c_m, \theta) = \left\lceil \frac{c_m + (1 - \theta)(\Gamma - \bar{c})}{\Delta} \right\rceil = \left\lceil \frac{c_m' + (1 - \theta)(\Gamma - \bar{c})}{\Delta} \right\rceil$$

and

$$s^*(c_m', \theta) = \left\lfloor \frac{c_m + (1 - \theta)(\Gamma - \bar{c})}{\Delta} \right\rfloor = \left\lfloor \frac{c_m' + (1 - \theta)(\Gamma - \bar{c})}{\Delta} \right\rfloor. \qquad \text{(IA.14)}$$

Now, if

$$s^*(c_m,\theta) = \left\lceil \frac{c_m + (1-\theta)\left(\Gamma - \bar{c}\right)}{\Delta} \right\rceil,$$

then

$$\mathcal{O}\left(v_0 + \left\lceil \frac{c_m + (1-\theta)\left(\Gamma - \bar{c}\right)}{\Delta} \right\rceil \Delta, c_m\right) \geq \mathcal{O}\left(v_0 + \left\lfloor \frac{c_m + (1-\theta)\left(\Gamma - \bar{c}\right)}{\Delta} \right\rfloor \Delta, c_m\right).$$

Thus, using the SCP, we have

$$\mathcal{O}\left(v_0 + \left\lceil \frac{c_m + (1-\theta)\left(\Gamma - \bar{c}\right)}{\Delta} \right\rceil \Delta, c_m'\right) > \mathcal{O}\left(v_0 + \left\lfloor \frac{c_m + (1-\theta)\left(\Gamma - \bar{c}\right)}{\Delta} \right\rfloor \Delta, c_m'\right),$$

which implies

$$\mathcal{O}\left(v_0 + \left\lceil \frac{c_m' + (1-\theta)\left(\Gamma - \bar{c}\right)}{\Delta} \right\rceil \Delta, c_m'\right) > \mathcal{O}\left(v_0 + \left\lfloor \frac{c_m + (1-\theta)\left(\Gamma - \bar{c}\right)}{\Delta} \right\rfloor \Delta, c_m'\right).$$

But this contradicts (IA.14). Thus, $s^*(c_m,\theta)$ and $a^*(c_m,\theta)$ weakly increase in c_m.

Now let $\underline{c}_m = -(1-\theta)(\Gamma - \bar{c})$ and observe that

$$v_0 + \left\lfloor \frac{\underline{c}_m + (1-\theta)\left(\Gamma - \bar{c}\right)}{\Delta} \right\rfloor \Delta = v_0 + \left\lceil \frac{\underline{c}_m + (1-\theta)\left(\Gamma - \bar{c}\right)}{\Delta} \right\rceil \Delta = v_0.$$

Similarly, let $\bar{c}_m = \Gamma - (1-\theta)\left(\Gamma - \bar{c}\right)$. Then,

$$\begin{aligned} v_0 + \left\lfloor \frac{\bar{c}_m + (1-\theta)\left(\Gamma - \bar{c}\right)}{\Delta} \right\rfloor \Delta &= v_0 + \left\lceil \frac{\bar{c}_m + (1-\theta)\left(\Gamma - \bar{c}\right)}{\Delta} \right\rceil \Delta \\ &= v_0 + \ell\Delta = v_0 + \Gamma. \end{aligned}$$

As $a^*(c_m,\theta)$ weakly increases in c_m, $s^*\left(c_m,\theta\right)$ takes all the values between zero and ℓ as c_m goes from $\underline{c}_m$ to $\bar{c}_m$. For $s \in \{1, 2, ...\ell - 1\}$, let $\hat{c}_{ms} \in (\underline{c}_m, \bar{c}_m)$ be the smallest value of c_m such that $a^*(c_m, \theta) = v_0 + s\Delta$. Then, for $c_m \in (\hat{c}_{ms}, \hat{c}_{ms+1})$, we have $a^*(c_m, \theta) = v_0 + s\Delta$. Moreover, by the continuity of $\mathcal{O}(\cdot, \cdot)$ we have that

$$\mathcal{O}(v_0 + s\Delta, \hat{c}_{ms+1}) = \mathcal{O}(v_0 + (s+1)\Delta, \hat{c}_{ms+1})$$

for all $s \in \{1, 2, ...\ell - 1\}$. Thus, at the partition points $\{\hat{c}_{ms}\}_{s=2}^{\ell-1}$, the optimal ask price can be chosen as either $v_0 + s\Delta$ or $v_0 + (s+1)\Delta$. Furthermore, the first partition point $\hat{c}_{m1}$ is determined uniquely by the indifference condition

$$\mathcal{O}(v_0, \hat{c}_{m1}) = \mathcal{O}(v_0 + \Delta, \hat{c}_{m1}).$$

That is, $\hat{c}_{m1}$ is the unique solution to

$$(\Gamma - \bar{c} + \hat{c}_{m1})^{\theta} (-\hat{c}_{m1})^{1-\theta} = (\Gamma - \Delta - \bar{c} + \hat{c}_{m1})^{\theta} (\Delta - \hat{c}_{m1})^{1-\theta},$$

which implies that $\hat{c}_{m1} < 0$.

Now, note that for any number u,

$$\mathcal{O}(a, c_m) = \mathcal{O}(a + u, c_m + u).$$

In particular, setting $u = \Delta$,

$$\mathcal{O}(v_0 + s\Delta, c_m) = \mathcal{O}(v_0 + (s+1)\Delta, c_m + \Delta).$$

Thus, setting $a^*(c_m, \theta) = v_0 + s\Delta$ is optimal given $c_m \in [\hat{c}_{ms}, \hat{c}_{ms+1}]$ if and only if setting

$a^*(c_m, \theta) = v_0 + (s+1)\,\Delta$ is optimal given $c_m \in [\hat{c}_{ms} + \Delta, \hat{c}_{ms+1} + \Delta]$. Hence, $\hat{c}_{ms+1} = \hat{c}_{ms} + \Delta$. In particular, $\hat{c}_{ms} = \hat{c}_{m1} + (s-1)\Delta$ for all $s = 1, ..., \ell$. ■

Thus, the optimal ask price is an increasing step function of the make fee when the tick size is positive and this function has jumps at the partition points $\hat{c}_{m1}$,..., $\hat{c}_{ms}$,..., $\hat{c}_{m\ell-1}$. At the partition points, the ask price is not uniquely determined. For instance, if $c_m = \hat{c}_{ms+1}$, the ask price $a = v_0 + s\Delta$ and the ask price that is immediately above on the grid, that is, $a = v_0 + (s+1)\Delta$, yield the same value for the objective function $\mathcal{O}(a, c_m)$. In this case, we assume that traders choose the smallest of the two prices.

The first part of Proposition 7 follows immediately. Indeed, if the platform sets its fee in $[\hat{c}_{ms}, \hat{c}_{ms+1}]$, then market makers and market takers will choose to trade at $a^*(c_m, \theta) = v_0 + s\Delta$. Note that Claim 4 only covers the case in which the make fee is in the interval $[\hat{c}_{m1}, \hat{c}_{m1} + (\ell-1)\Delta]$. In this case, $a^*(c_m, \theta)$ is in the range of traders' valuations $[v_0, v_0 + \Gamma]$. For make fees outside this range (e.g., very large rebates for one side), the price $a^*(c_m, \theta)$ will be outside the range of traders' valuations $[v_0, v_0 + \Gamma]$.[39] Our results hold in this case as well, but it is natural to focus attention on parameters such that the ask price is always in the range $[v_0, v_0 + \Gamma]$.[40]

As an illustration of the first part of Proposition 7, consider the following numerical example. Set $v_0 = 800$, $\Gamma = 50$, and $\bar{c} = 1/10$ (all monetary amounts are in cents). Moreover, $\theta = 0.5$. Hence, market makers and market takers split the gains from trade equally when the tick size is zero. Finally, $\ell = 4$, that is, the tick size is $\Delta(4) = \frac{\Gamma}{4} = \$\frac{1}{8}$ (12.5 cents). The partition points in this case are $\hat{c}_{m1} = -18.7$, $\hat{c}_{m2} = -6.2$, and $\hat{c}_{m3} = 6.3$.

The solid step-function in the top graph of Figure 1 depicts the optimal ask price $a^*(c_m, \theta)$ as a function of c_m when the tick size is $\$\frac{1}{8}$ ($\ell = 4$). For the values of the make fee within the partition points the optimal price is unique, while at the partition points one can choose either the left or the right price on the grid.[41] The 45° thin line in the graph depicts the optimal ask price as as a function of c_m when the tick size is zero.

Suppose that $c_m = 0.03$ cents. With a zero tick size the ask price is $a_0^*(0.03, 0.5) = 824.98$ and market makers obtain 50% of the gains from trade. In contrast, with a \$1/8 tick size the ask price is $a_2^*(0.03, 0.5) = 825$. Indeed, this is the price on the grid that yields the division of gains from trade that is the closest to that obtained in the absence of a minimum price variation (with this price market makers obtain 49.95% of the total gains from trade). Now suppose that the make fee decreases by 0.01 cents. With no minimum price variation, the ask price would decrease by the same amount, leaving unchanged the 50/50 split of the gains from trade between makers and takers. However, when the tick size is greater than 0.01 cents, the change in the make fee cannot be fully neutralized and an ask price of 825 is still the price that yields the division of gains from trade that is the closest to that obtained in absence of the tick size. More generally, traders keep trading at 825 as long as the make fee does not exceed 6.3 cents. At this point, traders are indifferent between an ask price of 825 and an ask price of 837.5, and so both are optimal. Once the make fee exceeds 6.3 cents, the optimal ask price becomes 837.5 cents. To sum up, the required minimum price variation renders the ask price less elastic to a change in the make fee, as can be seen from the top graph of Figure 1.

As a consequence, when the tick size is strictly positive, market makers' profits depend on the make fee charged by the platform, as shown in the bottom graph of Figure 1. Consider again the previous numerical example and suppose that $c_m = 0.03$. In this case, as explained previously, the ask price is $a_4^*(0.03, 0.5) = 825$ cents. Market makers earn a surplus of $\pi_m = 24.77$ cents and market takers earn a surplus of $\pi_t = 25.13$ cents. If instead the platform offers a rebate of one cent to the market makers ($c_m = -1$), the price remains unchanged but the market makers now have a higher surplus (25.8 cents) and market takers a lower surplus (24.1 cents). Thus, market

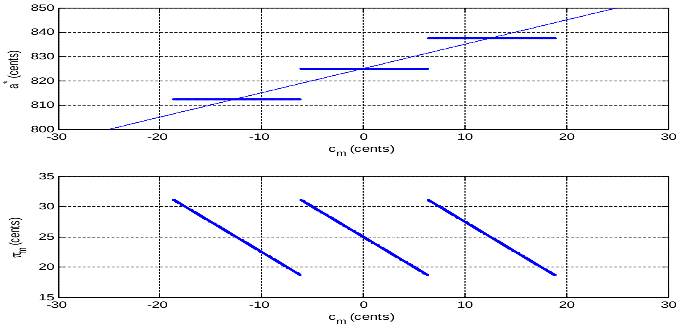

Figure 1: **Effect of the make fee on the ask price and market makers' per trade profit.**

makers (market takers) will monitor the market more (less) intensively in the second case and the trading rate will be affected, as in the baseline model. More generally, as long as the tick size is not zero, the trading platform can affect traders' monitoring intensities with its make/take fees.

Proof of Part 2 of Proposition **7**: The first step is to show that there are multiple make fees that result in the same trading rate.

Claim 5: *The trading rate when the platform chooses a make fee $c_m \in [\hat{c}_{ms}, \hat{c}_{ms+1}]$ is equal to the trading rate when the platform sets the make fee at $c_m' = c_m + n\Delta$ for all integers n.*

The intuition for this result is straightforward. Suppose that initially the platform picks its fee in $[\hat{c}_{ms}, \hat{c}_{ms+1}]$. If the platform adds an integer number of ticks to its make-fee, then traders neutralize the effect of this shift in the make fee by adjusting the transaction price by exactly the same number of ticks (Claim 4). As a consequence, the division of gains from trade between makers and takers is unchanged, and traders' monitoring intensities are unaffected as well. Thus, the trading rate is identical in both situations. The bottom graph of Figure 1 illustrates Claim 5: there are multiple values of the make fee that yields the same trading profits for the makers (e.g., both a make fee of zero or 12.5 cents yield a trading profit of 25 cents to the makers).

Proof of Claim **5:** With a make fee $c_m \in [\hat{c}_{ms},\, \hat{c}_{ms+1}]$, traders' profits per trade are

$$\begin{aligned}
\pi_m(a^*(c_m,\theta),c_m) &= s\Delta - c_m, \\
\pi_t(a^*(c_m,\theta),c_m) &= \Gamma - \bar{c} - s\Delta + c_m
\end{aligned}$$

Now consider the following make fee: $c'_m = c_m + n\Delta$ for some integer n. We have $c'_m \in [\hat{c}_{ms+n},$ $\hat{c}_{ms+n+1})$ (using $\hat{c}_{ms} = \hat{c}_{m1} + (s-1)\Delta$). Thus, $a^*(c'_m,\theta) = v_0 + (s+n)\Delta$. We conclude that traders' per trade profits with a fee equal to c'_m are

$$\begin{aligned}
\pi_m(a^*(c'_m,\theta),c'_m) &= s\Delta - c_m, \\
\pi_t(a^*(c'_m,\theta),c'_m) &= \Gamma - \bar{c} - s\Delta + c_m.
\end{aligned}$$

Thus, the fees c_m and c'_m result in exactly the same division of gains from trade between market makers and market takers. As a consequence, monitoring intensities are identical in both cases and both fees result in the same trading rate. ■

From Claim 5 it follows immediately that when traders can choose the price on the grid at which they trade, the optimal make fee is not unique. Thus, in studying the platform's optimization problem, we can arbitrarily choose the interval $[\hat{c}_{ms},\, \hat{c}_{ms+1}]$ in which the platform picks its make fee. To fix things, suppose that the platform chooses its fee in the interval $[\hat{c}_{ms},$ $\hat{c}_{ms+1}]$ for some $s \in \{1, 2, ..., \ell - 1\}$. That is, the ask price at which transactions take place is $a^* = v_0 + s\Delta$. The optimization problem of the platform is then identical to its problem in the baseline case, with the additional constraint that $c_m \in [\hat{c}_{ms},\, \hat{c}_{ms+1}]$, where $\hat{c}_{ms+1} = \hat{c}_{ms} + \Delta$. Hence, if the optimal solution in the baseline case satisfies the constraint, then it also solves the constrained problem. This implies that:

$$c_m^*(\Delta, r, q) = c_{ms}^* \quad if \quad \hat{c}_{ms} < c_{ms}^* < \hat{c}_{ms+1}.$$

If instead, the optimal solution in the baseline case does not satisfy the constraint, then the constraint binds and the optimal solution for the platform is a corner solution, which is either $\hat{c}_{ms+1}$ or $\hat{c}_{ms}$ (depending on whether the constraint binds because $c_{ms}^* > \hat{c}_{ms+1}$ or because $c_{ms}^* < \hat{c}_{ms}$. The second part of Proposition 7 follows immediately from this observation.

E. Algorithmic Trading and Welfare

In Section III.C in the paper, we claim that, for fixed trading fees, a reduction in traders' monitoring costs is a Pareto improvement (that is, results in a higher expected profit for each type of participant). We now provide a proof of this claim.

Claim 6: *For fixed trading fees, the total expected profit of each participant (market makers, market takers, and the trading platform), and therefore, aggregate welfare increases when β or γ decreases.*

Proof of Claim **6:** Consider first the aggregate expected profit for market takers. We have:

$$\Pi_t(\tau_1^*, .., \tau_j^*, ..., \tau_N^*, \overline{\mu}^*; \gamma, \beta, c_m, c_t) = \sum_j \Pi_{jt}(\tau_j^*, \bar{\mu}^*; \gamma, \beta, M, N).$$

Thus,

$$\begin{aligned}
\frac{d\Pi_t}{d\gamma} &= \sum_j \left(\frac{\partial \Pi_{jt}}{\partial \tau_j^*} \frac{\partial \tau_j^*}{\partial \gamma} + \frac{\partial \Pi_{jt}}{\partial \bar{\mu}^*} \frac{\partial \bar{\mu}^*}{\partial \gamma} + \frac{\partial \Pi_{jt}}{\partial \gamma} \right), \\
\frac{d\Pi_t}{d\beta} &= \sum_j \left(\frac{\partial \Pi_{jt}}{\partial \tau_j^*} \frac{\partial \tau_j^*}{\partial \beta} + \frac{\partial \Pi_{jt}}{\partial \bar{\mu}^*} \frac{\partial \bar{\mu}^*}{\partial \beta} + \frac{\partial \Pi_{jt}}{\partial \beta} \right).
\end{aligned}$$

The envelope theorem implies that $\frac{\partial \Pi_{jt}}{\partial \tau_j^*} = 0$ for all j. In addition, cross-side complementarity implies $\frac{\partial \Pi_{jt}}{\partial \bar{\mu}^*} > 0$ for all j, and Corollary 1 yields $\frac{\partial \bar{\mu}^*}{\partial \gamma} < 0$ and $\frac{\partial \bar{\mu}^*}{\partial \beta} < 0$. Last, for all j, $\frac{\partial \Pi_{jt}}{\partial \gamma} = -\frac{1}{2}\left(\tau_j^*\right)^2 < 0$ and $\frac{\partial \Pi_{jt}}{\partial \beta} = 0$. Thus, $\frac{d\Pi_t}{d\gamma} < 0$ and $\frac{d\Pi_t}{d\beta} < 0$. This establishes the first part of Claim 6 for the market-taking side. The proof for the market-making side is parallel. Further, we have proved in Corollary 1 in the paper that the trading rate decreases when β or γ increases. It follows that the expected profit of the platform decreases with β or γ.

F. Finite Arrival Rate for Market Takers

We now provide a detailed analysis of our model when the rate at which market takers receive a new buy order after a trade is finite, as described in Section V.A of the paper. Recall that in this case, we must assume $M = N = 1$ for tractability.

Monitoring Decisions with Fixed Fees

As in the baseline model, the first step is to analyze traders' monitoring decisions for fixed trading fees. As $M = N = 1$, we denote the market maker's monitoring intensity by μ and the market taker's monitoring intensity by τ.

Using the expression for the trading rate given in equation (28) in the paper, we have that the objective function of the market maker is

$$\Pi_m = \pi_m \cdot \mathcal{R}(\mu, \tau, \kappa) - \frac{1}{2}\beta\mu^2 = \frac{\pi_m}{\frac{1}{\mu} + \frac{1}{\tau} + \frac{1}{\kappa}} - \frac{1}{2}\beta\mu^2$$

and the objective function of the market taker is

$$\Pi_t = \frac{\pi_t}{\frac{1}{\mu} + \frac{1}{\tau} + \frac{1}{\kappa}} - \frac{1}{2}\gamma\tau^2.$$

Hence, the first order conditions for the market maker and market taker are

$$\beta\mu^3 = \frac{\pi_m}{\left(\frac{1}{\mu} + \frac{1}{\tau} + \frac{1}{\kappa}\right)^2} \tag{IA.15}$$

$$\gamma\tau^3 = \frac{\pi_t}{\left(\frac{1}{\mu} + \frac{1}{\tau} + \frac{1}{\kappa}\right)^2}. \tag{IA.16}$$

We conclude that

$$\frac{\mu}{\tau} = \left(\frac{\pi_m}{\pi_t}\frac{\gamma}{\beta}\right)^{\frac{1}{3}} = z^{\frac{1}{3}}. \tag{IA.17}$$

Using this observation and equation (IA.15), we have that the equilibrium monitoring intensity for the market maker, μ^*, solves

$$\beta(\mu^*)^3 \left(\frac{1}{\mu^*}\left(1 + z^{\frac{1}{3}}\right) + \frac{1}{\kappa}\right)^2 = \pi_m. \tag{IA.18}$$

That is

$$\mu^*\left(1 + z^{\frac{1}{3}}\right)^2 + \frac{2(\mu^*)^2}{\kappa}\left(1 + z^{\frac{1}{3}}\right) + \frac{(\mu^*)^3}{\kappa^2} = \frac{\pi_m}{\beta}. \tag{IA.19}$$

Similarly, the equilibrium intensity for the market taker, τ^*, solves

$$\tau^*\left(1+z^{-\frac{1}{3}}\right)^2+\frac{2(\tau^*)^2}{\kappa}\left(1+z^{-\frac{1}{3}}\right)+\frac{(\tau^*)^3}{\kappa^2}=\frac{\pi_t}{\gamma}. \tag{IA.20}$$

It is easy to see that the cubic equations (IA.19) and (IA.20) have a unique positive solution, (μ^*,τ^*). Furthermore, as κ gets larger, traders' monitoring levels converge to their values in the baseline model. Specifically, (IA.19) and (IA.20) yield

$$\begin{aligned}\lim_{\kappa\to\infty}\mu^* &= \left(1+z^{\frac{1}{3}}\right)^{-2}\cdot\left(\frac{\pi_m}{\beta}\right), \text{ and}\\ \lim_{\kappa\to\infty}\tau^* &= \left(1+z^{-\frac{1}{3}}\right)^{-2}\cdot\left(\frac{\pi_t}{\gamma}\right).\end{aligned}$$

Applying the implicit function theorem, we can use equations (IA.19) and (IA.20) to show that all of the main conclusions in the baseline model (with $M=N=1$) still hold in this more general case for any $\kappa>0$. To start, we illustrate that the complementarity results in Corollary 1 hold in this case.

Claim 7: *Assume $M=N=1$. The results of Corollary 1 hold for any $\kappa>0$.*

Proof of Claim **7:** We provide the proof that $\frac{\partial\mu^*}{\partial\gamma}<0$ and $\frac{\partial\tau^*}{\partial\gamma}<0$. The proof of the other results in Corollary 1 is similar.

Denote

$$\psi\equiv 1+z^{\frac{1}{3}}.$$

Then

$$\begin{aligned}\frac{\partial\psi}{\partial\pi_m} &= \frac{\partial}{\partial\pi_m}\left(1+\left(\frac{\pi_m}{\pi_t}\frac{\gamma}{\beta}\right)^{\frac{1}{3}}\right) = \frac{1}{3}\left(\frac{\pi_m}{\pi_t}\frac{\gamma}{\beta}\right)^{-\frac{2}{3}}\frac{\gamma}{\beta}\frac{1}{\pi_t} > 0, \text{ and}\\ \frac{\partial\psi}{\partial\pi_t} &= \frac{\partial}{\partial\pi_t}\left(1+\left(\frac{\pi_m}{\pi_t}\frac{\gamma}{\beta}\right)^{\frac{1}{3}}\right) = -\frac{1}{3}\left(\frac{\pi_m}{\pi_t}\frac{\gamma}{\beta}\right)^{-\frac{2}{3}}\frac{\gamma}{\beta}\frac{\pi_m}{\pi_t^2} < 0.\end{aligned}$$

Furthermore,

$$\frac{\partial\psi}{\partial\gamma} = \frac{1}{3}z^{-\frac{1}{3}}\frac{\pi_m}{\pi_t}\frac{1}{\beta} > 0. \tag{IA.21}$$

We can rewrite (IA.19) as

$$\mu\psi^2 + \frac{2\mu^2}{\kappa}\psi + \frac{\mu^3}{\kappa^2} - \frac{\pi_m}{\beta} = 0. \tag{IA.22}$$

Implicitly differentiating (IA.22) by γ gives

$$\frac{\partial\mu}{\partial\gamma}\psi^2 + 2\psi\mu\frac{\partial\psi}{\partial\gamma} + \frac{4\mu}{\kappa}\psi\frac{\partial\mu}{\partial\gamma} + \frac{2\mu^2}{\kappa}\frac{\partial\psi}{\partial\gamma} + \frac{3\mu^2}{\kappa^2}\frac{\partial\mu}{\partial\gamma} = 0.$$

Hence,

$$\frac{\partial\mu^*}{\partial\gamma} = -\frac{2\psi\mu^*\frac{\partial\psi}{\partial\gamma} + \frac{2\mu^{*2}}{\kappa}\frac{\partial\psi}{\partial\gamma}}{\psi^2 + \frac{4\mu^*}{\kappa}\psi + \frac{2\mu^{*2}}{\kappa}} < 0,$$

where the inequality follows from (IA.21). Also, using (IA.17),

$$\frac{\partial\tau^*}{\partial\gamma} = \frac{\partial\mu^*}{\partial\gamma}z^{-\frac{1}{3}} - \frac{1}{3}\mu^* z^{-\frac{2}{3}}\frac{\partial z}{\partial\gamma} < 0,$$

as required.

Now we derive the optimal pricing policy of the platform for all values of κ; that is, we provide a proof of Proposition 5 in the paper.

Claim 8: *Let $a - v_0 = s \cdot \Delta$. The optimal make/take fee breakdown does not depend on κ. In addition, in the thick market case, the optimal make and take fees are*

$$c^*_{ms} = s \cdot \Delta - \frac{\Gamma - \bar{c}}{1 + r^{\frac{1}{4}}} \quad \textit{and } c^*_{ts} = \bar{c} - c^*_{ms}.$$

Proof of Claim **8:** Remember that $\pi_m = a - v_0 - c_m$ and $\pi_t = v_0 + \Gamma - a - c_t$. Thus, there is a one-to-one mapping between traders' profits when a trade takes place and the make/take fees. For this reason, as in the baseline model, for a fixed $\bar{c}$ we can write the platform's problem as

$$\begin{aligned} & Max_{\pi_m, \pi_t} \quad \mathcal{R}(\mu^*, \tau^*)\bar{c} \\ & s.t \;\; \pi_t + \pi_m = \overline{\pi}, \end{aligned} \tag{IA.23}$$

where $\overline{\pi} \equiv \Gamma - \bar{c}$. Let w be the fraction of the total gains from trade that accrues to the market taker, that is,

$$w = \frac{\pi_t}{\overline{\pi}}. \tag{IA.24}$$

In equilibrium, $\mathcal{R}(\mu^*, \tau^*) = \frac{1}{\frac{1}{\mu^*} + \frac{1}{\tau^*} + \frac{1}{\kappa}} = \left(\frac{1}{\mu^*}\left(1 + z^{\frac{1}{3}}\right) + \frac{1}{\kappa}\right)^{-1}$, where the second equality follows from equation (IA.17). In addition, using equation (IA.18), we deduce that

$$\left(\frac{1}{\mu^*}\left(1 + z^{\frac{1}{3}}\right) + \frac{1}{\kappa}\right)^2 = \frac{\pi_m}{\beta \mu^{*3}} = \frac{(1-w)\,\overline{\pi}}{\beta \cdot \mu^{*3}} \tag{IA.25}$$

We conclude that the optimization problem of the platform, equation (IA.23), is therefore equivalent to

$$Min_w \; \frac{(1-w)}{\mu^{*3}}. \tag{IA.26}$$

Recall that μ^* is the unique solution of equation (IA.18). Clearly, this solution is a function of

z. Alternatively it can be written as a function of w since

$$z = \frac{1-w}{w}\frac{\gamma}{\beta}.$$

Using this observation and writing the first order condition for the optimization problem (IA.26), we deduce that the optimal w for the platform solves

$$\mu^* + 3\left(1-w^*\right)\frac{\partial \mu^*}{\partial w}|_{w=w^*} = 0. \tag{IA.27}$$

The LHS of this equation depends on κ since μ^* is also a function of κ (see equation (IA.19)). Yet, we now show that the value of w that solves equation (IA.27) does not depend on κ. To see this, we first rewrite equation (IA.18) as

$$\left(\frac{1+z^{\frac{1}{3}}}{\mu^*} + \frac{1}{\kappa}\right)^2 - \frac{\left(1-w\right)\overline{\pi}}{\beta\mu^{*3}} = 0.$$

Then, using implicit differentiation with respect to w, we deduce that

$$2\left(\frac{1+z^{\frac{1}{3}}}{\mu^*} + \frac{1}{\kappa}\right)\frac{\frac{1}{3}z^{-\frac{2}{3}}\frac{\partial z}{\partial w}\mu^* - \left(1+z^{\frac{1}{3}}\right)\frac{\partial \mu^*}{\partial w}}{\mu^{*2}} = -\frac{\overline{\pi}}{\beta}\cdot\frac{\mu^* + 3\left(1-w\right)\frac{\partial \mu^*}{\partial w}}{\mu^{*4}}. \tag{IA.28}$$

Using equation (IA.27), we deduce that the RHS of this equation is zero when $w = w^*$. Hence, at w^*, equation (IA.28) simplifies to

$$\frac{\partial \mu^*}{\partial w}|_{w=w^*} = \frac{\frac{1}{3}z^{-\frac{2}{3}}\frac{\partial z}{\partial w}\mu^*}{1+z^{\frac{1}{3}}}. \tag{IA.29}$$

Replacing $\frac{\partial \mu^*}{\partial w}\mid_{w=w^*}$ by this expression in Equation (IA.27) we have

$$\mu^* + 3\left(1 - w^*\right)\frac{\frac{1}{3}z^{-\frac{2}{3}}\frac{\partial z}{\partial w}\mu^*}{1 + z^{\frac{1}{3}}} = 0.$$

That is,

$$1 + \left(1 - w^*\right)\frac{z^{-\frac{2}{3}}\frac{\partial z}{\partial w}}{1 + z^{\frac{1}{3}}} = 0.$$

As $z = \frac{1-w^*}{w^*}\frac{\gamma}{\beta}$, this equation implicitly characterizes w^*. It does not depend on κ. We conclude that the optimal make/take fee breakdown for the platform (which is fixed by w^*) does not depend on κ, as claimed in Proposition 5.

Hence, the optimal make/take fee breakdown when κ is finite is identical to the optimal breakdown when κ is infinite. We have derived this breakdown in Claim 2 in this Internet Appendix when $M = N = 1$ (the case considered here). We conclude from Claim 2 that for all κ,

$$c^*_{ms} = \Delta \cdot s - \pi_m = \Delta \cdot s - \frac{\Gamma - \bar{c}}{1 + r^{\frac{1}{4}}},$$

and the optimal take fee follows from the fact that $c^*_{ms} + c^*_{ts} = \bar{c}$.

G. Matlab Code for Numerical Simulations

In section III of the paper we indicate that we have checked through numerical simulations that the results of Corollary 5 are robust even when the market is not thick. Below we provide the Matlab code used for these robustness checks. Running this code yields the optimal make/take fees for the platform by finding the fees that minimize the *negative* of the trading rate.[1] The main program is comp_stat. This program uses the function nvolume(), which calculates the negative of the trading rate for given parameter values. This function in turn uses the function cubic(), which is applied to find a zero of equation (14) in the paper. The program is set to perform comparative statics in *r*. It can be used to perform comparative statics in any other parameter (such as *q*). The output of this program is a graph showing the optimal make fee, take fee and take/make spread for various levels of *r*.

To run this program one needs to place comp_stat.m, nvolume.m, and cubic.m in separate m-files located in the same folder, and run comp_stat.m.

```
%comp_stat.m
%This program presents comparative statics of the fee structure using
numerical simulations.
%The code below performs comparative statics in r.
%Comparative statics in q (or any other parameter) can be done by
%introducing the obvious changes.

clear
M=20  %number of market makers
N=10  %number of market takers
beta=.4; %unit cost of monitoring to mm
gamma=.4; %unit cost of monitoring to mt
q=M/N;
v0=300;  %fundamental value
s=8; %makers' spread in ticks

del=1 %tick size in pennies
a=v0+s*del %price in which transactions are executed

G=20; %gains from trade in cents - corresponds to GAMMA in the paper

c_bar=.1;   %total fees in pennies

NUM=50; %number of sample points for the numerical analysis and plot


rr=linspace(1.2,2,NUM); % an array of q values.
for i=1:NUM
    r=rr(i);
    gamma=beta*r;
    N=M/q;
```

[1] We use the Matlab function fminbnd(), which numerically finds the minimum of a function. Thus, finding the optimal fee breakdown requires minimizing the negative of the trading rate.

```
    pi_t_star=fminbnd(@(pi_t)
nvolume(del,G,c_bar,M,N,beta,gamma,pi_t),0.03,G-c_bar-0.01); %Numerical
solution of the platform's problem of the optimal split of gains from
trade
    pi_m_star=G-c_bar-pi_t_star;
    c_t(i)=v0+G-a-pi_t_star;
    c_m(i)=a-v0-pi_m_star;
    spread(i)=c_t(i)-c_m(i); %take-make spread
end

plot(rr,c_m,'b-',rr,c_t,'b:',rr,spread,'b--','LineWidth',1)
xlabel('r')
legend('Make-Fee (cents)','Take-Fee (cents)','Take/Make Spread
(cents)','Location','northeast')
ylabel ('Fees (cents)')



function res=nvolume(del,L,c_bar,M,N,beta,gamma,pi_t)
%This function returns the negative of the trading rate

NUM=200;

pi_m=L-c_bar-pi_t;  %profit to mm
z=(pi_m*gamma)/(pi_t*beta);
guess=z^(1/3); %we set the initial guess to the solution in the simple
M=N=1 case
om_star = fzero(@(x) cubic(x,M,N,z),guess);
den=(1+om_star)^2;
lamlam=(M+(M-1)*om_star)*(pi_m/beta)/den;    %aggregate monitoring by
market makers
mumu=(om_star*((1+om_star)*N-1))*(pi_t/gamma)/den;   %aggregate
monitoring by market takers
res=-(lamlam*mumu)/(lamlam+mumu);    %the negative of trading rate



function om=cubic(x,M,N,z);

om=x^3*N+(N-1)*x^2-(M-1)*z*x-M*z;
```

REFERENCES

Milgrom, Paul, and Chris Shannon, 1994, Monotone comparative statics, *Econometrica* 62, 157-180.