Internet Appendix to "Technological Growth and Asset Pricing"*

This appendix complements the paper in a number of ways. Section A contains the proofs of the formal results stated in the paper. Section B discusses several of the model's assumptions and offers details concerning the introduction of tree-specific schocks in Section III.C of the paper. Section C contains the calculation of the covariance between excess returns and subsequent long-run consumption growth in the models of Campbell and Cochrane (1999) and Bansal and Yaron (2004). Finally, Section D provides details concerning the covariance decomposition (equation (24)) inside our model.

A. Proofs

Proof of Proposition 1. Equation (3) can be rewritten as

$$P^o_{j,t} \equiv \sup_{\tau_j} E_t \left[\frac{H_{\tau_j}}{H_t} \left(\beta \zeta(i_j) \theta_{\tau_j} - q \right) \right], \tag{IA.1}$$

where β is defined as

$$\begin{aligned} \beta &\equiv E_t \left[\int_t^\infty \frac{H_s}{H_t} \frac{\theta_s}{\theta_t} ds \right] \\ &= \int_t^\infty E_t e^{-\left(r-\mu+\frac{\sigma^2}{2}+\frac{\kappa^2}{2}\right)(s-t)+(\sigma-\kappa)(B_s-B_t)} ds. \\ &= \frac{1}{r+\kappa\sigma-\mu}. \end{aligned} \tag{IA.2}$$

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The second line in equation (IA.2) follows from solving the stochastic differential equations (6) and (1), while the third line of (IA.2) follows from the properties of lognormal variables.

To, solve (IA.1), we choose to to work under the risk-neutral measure and re-write it as

$$P_{j,t}^{o} \equiv \sup_{\tau_j} E_t^Q \left\{ e^{-r(\tau_j - t)} \left(\beta\zeta(i_j)\theta_{\tau_j} - q \right) \right\}, \tag{IA.3}$$

where the Brownian motions under the risk-neutral and the natural probability measurea are related by $dB_t^Q = dB_t + \kappa dt$, so that the dynamics of θ_t are given by

$$\frac{d\theta_t}{\theta_t} = (\mu - \kappa\sigma)\, dt + \sigma dB_t^Q.$$

To solve problem (IA.3), we use the fact that in the continuation region $P_{j,t}^o$ satisfies the ODE

$$\frac{\sigma^2}{2}\theta_t^2 P_{\theta\theta}^o + (\mu - \kappa\sigma)\,\theta_t P_\theta^o - rP^o = 0,$$

which has the general solution $P_{j,t}^o = D_1\theta^{\phi^+} + D_2\theta^{\phi^-}$ with

$$\phi^{\pm} = \frac{-\left(\mu - \kappa\sigma - \frac{\sigma^2}{2}\right) \pm \sqrt{\left(\mu - \kappa\sigma - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2 r}}{\sigma^2}.$$

For our purposes, given that $\phi^- < 0$ we can set $D_2 = 0$, since otherwise $\lim_{\theta_t \to 0} P_{j,t}^o = \pm\infty$, which clearly is not a solution that corresponds to the economic problem (IA.3). We also observe that the assumption $r + \kappa\sigma > \mu$ implies that $\phi^+ > 1$.[1]

At the level $\overline{\theta}_{(i_j)}$ where the firm plants a new tree, the function $P_{j,t}^o$ satisfies the value-matching and smooth-pasting conditions

$$P^o\left(\overline{\theta}_{(i_j)}\right) = \beta\zeta(i_j)\overline{\theta}_{(i_j)} - q, \tag{IA.4}$$

$$P_\theta^o\left(\overline{\theta}_{(i_j)}\right) = \beta\zeta(i_j). \tag{IA.5}$$

Using $P_{j,t}^o = D_1\theta_t^{\phi^+}$ inside (IA.4) and (IA.5) gives a solution of two equations in the two unknown constants D_1 and $\overline{\theta}_{(i_j)}$. The solution of this system is given by

$$\overline{\theta}_{(i_j)} = \frac{\phi^+}{\phi^+ - 1}\frac{q}{\beta\zeta(i_j)}, \tag{IA.6}$$

$$D_1 = \left[\overline{\theta}_{(i_j)}\right]^{-\phi^+}\left(\beta\zeta(i_j)\overline{\theta}_{(i_j)} - q\right).$$

[1] $\phi^{\pm}$ are the roots of the quadratic $\frac{\sigma^2}{2}\phi\,(\phi - 1) + (\mu - \kappa\sigma)\,\phi - r = 0$. The value of this quadratic at $\phi = 1$ is $\mu - \kappa\sigma - r < 0$. Hence, the positive root ϕ^+ must be larger than one.

Using the definition of υ and the fact that firm i_j plants a tree the first time that θ_t reaches $\overline{\theta}_{(i_j)}$ leads to (7). Furthermore, the value of a growth option is

$$P^o_{j,t} = D_1 \theta_t^{\phi^+} = \left(\frac{\theta_t}{\overline{\theta}_{(i_j)}} \right)^{\phi^+} \left(\beta \zeta(i_j) \overline{\theta}_{(i_j)} - q \right) = \left(\frac{\theta_t}{\overline{\theta}_{(i_j)}} \right)^{\phi^+} \left(\frac{q}{\phi^+ - 1} \right). \tag{IA.7}$$

Since the first time that θ_t reaches $\overline{\theta}_{(i_j)}$ is also the first time that the running maximum of θ_t reaches a new level, the running maximum of θ_t can be used to recover the mass of trees K_t that have been planted. Indeed, equation (IA.6) implies that $\max_{0 \le s \le t} \theta_s = \frac{\phi^+}{\phi^+ - 1} \frac{q}{\beta \zeta(K_t)}$. Using this equation along with the definition of $\zeta(K_t)$ to solve for K_t leads to (8). ■

Proof of Lemma 1. Since the expected excess return of any claim equals the product of its quadratic variation times the volatility of the pricing kernel (κ), it suffices to show that $\frac{\partial \log P^A_t}{\partial \log \theta_t} < \frac{\partial \log P^o_t}{\partial \log \theta_t}$. The quadratic variation of assets in place is $\frac{\partial \log P^A_t}{\partial \log \theta_t} \sigma = \sigma$, and similarly the quadratic variation of growth options is $\frac{\partial \log P^o_t}{\partial \log \theta_t} \sigma = \phi^+ \sigma$. Since $\phi^+ > 1$ (see the proof of Proposition 1), the result follows. ■

Proof of Lemma 2. Using the definition

$$X_{j,t} \equiv \sum_{n<N} A_n \zeta(i_{n,j}) 1_{\{\widetilde{\chi}_{n,j}=1\}}, \tag{IA.8}$$

a firm's total output of consumption goods is given by $X_{j,t}\theta_t$, and accordingly total output of consumption goods is $\left(\int_0^1 X_{j,t} dj \right) \theta_t$. A straightforward computation gives

$$\begin{aligned}
\int_0^1 X_{j,t} dj &= \sum_{n \le N-1} A_n \int_0^{K_{n,\tau_{n+1}}} \zeta(i) di + A_N \int_0^{K_{N,t}} \zeta(i) di \\
&= \sum_{n \le N-1} (A_{n+1} - A_n) + A_N \int_0^{K_{N,t}} \zeta(i) di \\
&= A_N \left(1 - \lim_{n \to -\infty} (A_n / A_N) + F(K_{N,t}) \right).
\end{aligned}$$

If $\Pr\left(K_{n,\tau^-_{n+1}} > \varepsilon \right) = p > 0$ for some $\varepsilon > 0$, then $\lim_{n \to -\infty} (A_n/A_N) = 0$ with probability one, and (13) follows. ■

Next we prove that there exists an appropriate constant $\upsilon = \upsilon^*$ such that if a firm perceives the equilibrium process for $K_{N,t}$ to be given by (21) and the stochastic discount factor to be given by $H_t = e^{-\rho t} U_c(c_t, M^C_t)$, with $c_t = C_t$ given by (13), then that firm will optimally plant a tree the first time that θ_t reaches the threshold value given by equation (19). We also provide closed-form expressions for the equilibrium value of any firm j in round N at time t.

We start by defining some constants and functions that appear repeatedly in the proof. Specifically, let constants γ_1 and γ_1^* be defined as

$$\begin{aligned}
\gamma_1 &\equiv \frac{\sqrt{\left(\mu-\frac{\sigma^2}{2}\right)^2+2\sigma^2\left(\rho+\lambda\right)}-\left(\mu-\frac{\sigma^2}{2}\right)}{\sigma^2}>0,\\
\gamma_1^* &\equiv \frac{\sqrt{\left(\mu-\frac{\sigma^2}{2}\right)^2+2\sigma^2\rho}-\left(\mu-\frac{\sigma^2}{2}\right)}{\sigma^2}>0,
\end{aligned}$$

and constants β_1 and β_2 be defined as

$$\begin{aligned}
\beta_1 &\equiv \frac{1}{\rho+\lambda-\mu\left(1-\gamma\right)+\gamma\left(1-\gamma\right)\frac{\sigma^2}{2}},\\
\beta_2 &\equiv \frac{-\nu p}{1-p+\nu p}-\nu-\gamma_1<0.
\end{aligned}$$

We assume that $\beta_1>0$. Furthermore, let the functions $g_1\left(x\right)$ and $\tilde{g}_1\left(x\right)$ be given by

$$\begin{aligned}
g_1\left(x\right) &\equiv \beta_2+\gamma_1+\gamma+x,\\
\tilde{g}_1\left(x\right) &\equiv \frac{\alpha\left(1-\gamma\right)p}{1-p+\nu p}+\left(\gamma-1\right)\left(1-\alpha\right)+x,
\end{aligned}$$

the functions $g_2\left(x\right)$ and $\tilde{g}_2\left(x\right)$ be defined as

$$\begin{aligned}
g_2\left(x\right) &\equiv \frac{\left(\gamma-1\right)\left(1-\alpha\right)+x}{\alpha\gamma-\alpha+\gamma_1}+\frac{g_1\left(x\right)}{1+\beta_2},\\
\tilde{g}_2\left(x\right) &\equiv \frac{\left(\gamma-1\right)\left(1-\alpha\right)+x}{\alpha\gamma-\alpha+\gamma_1}+\frac{\tilde{g}_1\left(x\right)}{1+\beta_2+\frac{p}{1-p+\nu p}},
\end{aligned}$$

and $g_3\left(x\right)$ be given by

$$g_3\left(x\right)\equiv\frac{\lambda}{\left(\rho+\lambda\right)+\frac{\sigma^2}{2}\left(\gamma+x\right)\left(1-\gamma-x\right)-\mu\left(1-\gamma-x\right)}.$$

A useful first result is contained in the following lemma.

LEMMA 3 *Fix a constant $\upsilon\geq bp$ and suppose that $K_{N,t}$ is given by*

$$K_{N,t}=K\left(\frac{M_t}{M_{\tau_N}}\right)=\min\left\{\max\left[\frac{\left[\left(\frac{bp}{\upsilon}\right)\frac{M_t}{M_{\tau_N}}\right]^{\frac{1}{1-p+\nu p}}-1}{b},0\right],1\right\}. \tag{IA.9}$$

C_t is given by $C_t=\theta_t X_{\tau_n}\left(1+bK_{N,t}\right)^p$, and H_t is given by $H_t=e^{-\rho t}C_t^{-\gamma+(\gamma-1)(1-\alpha)}\left(\frac{\theta_t}{M_t}\right)^{(1-\gamma)(1-\alpha)}$. Define $m_t\equiv\frac{M_t}{M_{\tau_{N_t}}}$, and also let

$$\begin{aligned}
g_4\left(x\right) &\equiv g_3\left(x\right)\left[\frac{\left(\gamma-1\right)\left(1-\alpha\right)+x}{\alpha\left(\gamma-1\right)+\gamma_1}+\left(\frac{bp}{\upsilon}\right)^{\alpha(\gamma-1)+\gamma_1}g_2\left(x\right)\left[\left(1+b\right)^{(1-p+\nu p)(1+\beta_2)}-1\right]\right],\\
\tilde{g}_4\left(x\right) &\equiv g_3\left(x\right)\left[\frac{\left(\gamma-1\right)\left(1-\alpha\right)+x}{\alpha\left(\gamma-1\right)+\gamma_1}+\left(\frac{bp}{\upsilon}\right)^{\alpha(\gamma-1)+\gamma_1}\tilde{g}_2\left(x\right)\left[\left(1+b\right)^{(1-p+\nu p)(1+\beta_2)+p}-1\right]\right].
\end{aligned}$$

Furthermore, let the constants α_1 *and* m^* *be defined as*

$$\begin{aligned}
\alpha_1 &= \left[\frac{(\gamma-1)(1-\alpha)}{\alpha(\gamma-1)+\gamma_1}+\left(\frac{bp}{\upsilon}\right)^{\alpha(\gamma-1)+\gamma_1} g_2(0)\left[(1+b)^{(1-p+\nu p)(1+\beta_2)}-1\right]\right]\beta_1, \\
m^* &= \frac{\upsilon}{bp}(1+b)^{1-p+\nu p},
\end{aligned}$$

and the constants Δ_1 *and* Δ_2 *be given by*

$$\Delta_1 = -\frac{\alpha_1+\frac{\beta_1}{1-\lambda\beta_1}g_4(0)}{g_4(1-\gamma-\gamma_1^*)}, \tag{IA.10}$$

$$\Delta_2 = \frac{\beta_1}{1-\lambda\beta_1}. \tag{IA.11}$$

We assume throughout that $\Delta_1 > 0$*. Finally, let* χ_t *denote the following conditional expectation:*

$$\chi_t \equiv E_t\int_t^\infty e^{-\rho(s-t)}\left(\frac{C_s}{C_t}\right)^{-\nu}\left(\frac{\frac{M_s}{\theta_s}}{\frac{M_t}{\theta_t}}\right)^{(\gamma-1)(1-\alpha)}\frac{\theta_s}{\theta_t}ds. \tag{IA.12}$$

Then $\chi_t = \chi\left(\frac{\theta_t}{M_t}, m_t\right)$*, where*

$$\begin{aligned}
\chi\left(\tfrac{\theta_t}{M_t}, m_t\right) =\; & \Delta_2\left\{1+\left(\tfrac{\theta_t}{M_t}\right)^{\gamma-1+\gamma_1}\left[\tfrac{(\gamma-1)(1-\alpha)}{\alpha(\gamma-1)+\gamma_1}+\left(\tfrac{bpm_t}{\upsilon}\right)^{\alpha(\gamma-1)+\gamma_1} g_2(0)\left((1+b)^{(1-p+\nu p)(1+\beta_2)}-1\right)\right]\right\} \\
& +\Delta_1\left(\tfrac{\theta_t}{M_t}\right)^{\gamma+\gamma_1^*-1}\left\{1+\left(\tfrac{\theta_t}{M_t}\right)^{\gamma_1-\gamma_1^*}\left[\begin{array}{c} -\frac{\alpha(\gamma-1)+\gamma_1^*}{\alpha(\gamma-1)+\gamma_1}+\left(\frac{bpm_t}{\upsilon}\right)^{\alpha(\gamma-1)+\gamma_1}\times \\ g_2(1-\gamma-\gamma_1^*)\left[(1+b)^{(1-p+\nu p)(1+\beta_2)}-1\right]\end{array}\right]\right\}
\end{aligned}$$

when $m_t \leq \frac{\upsilon}{bp}$,

$$\begin{aligned}
\chi\left(\tfrac{\theta_t}{M_t}, m_t\right) =\; & \Delta_2\left\{1+\left(\tfrac{\theta_t}{M_t}\right)^{\gamma-1+\gamma_1}\left[-\tfrac{g_1(0)}{1+\beta_2}+\left(\tfrac{bpm_t}{\upsilon}\right)^{-(1+\beta_2)}(1+b)^{(1-p+\nu p)(1+\beta_2)} g_2(0)\right]\right\} \\
& +\Delta_1\left(\tfrac{\theta_t}{M_t}\right)^{\gamma+\gamma_1^*-1}\left\{1+\left(\tfrac{\theta_t}{M_t}\right)^{\gamma_1-\gamma_1^*}\left[\begin{array}{c} -\frac{g_1\left(1-\gamma-\gamma_1^*\right)}{1+\beta_2}+\left(\frac{bpm_t}{\upsilon}\right)^{-(1+\beta_2)}\times \\ (1+b)^{(1-p+\nu p)(1+\beta_2)} g_2(1-\gamma-\gamma_1^*)\end{array}\right]\right\}
\end{aligned}$$

when $\frac{\upsilon}{bp} \leq m_t \leq m^*$*, and finally*

$$\chi\left(\frac{\theta_t}{M_t}, m_t\right) = \Delta_2\left\{1+\left(\tfrac{\theta_t}{M_t}\right)^{\gamma-1+\gamma_1}\tfrac{(\gamma-1)(1-\alpha)}{\alpha(\gamma-1)+\gamma_1}\right\}+\Delta_1\left(\tfrac{\theta_t}{M_t}\right)^{-\left(1-\gamma-\gamma_1^*\right)}\left\{1-\tfrac{\alpha(\gamma-1)+\gamma_1^*}{\alpha(\gamma-1)+\gamma_1}\left(\tfrac{M_t}{\theta_t}\right)^{\gamma_1^*-\gamma_1}\right\}$$

when $m_t \geq m^*$.

Proof of Lemma 3. To save space we only give a sketch of the argument. As a first step, let $Z\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)$ be given as

$$Z\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right) \equiv E_{\tau_n}\int_{\tau_n}^\infty e^{-\rho(s-\tau_n)}\left(\frac{C_s}{C_{\tau_n}}\right)^{-\nu}\frac{\left(\frac{M_s}{\theta_s}\right)^{(\gamma-1)(1-\alpha)}}{\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{(\gamma-1)(1-\alpha)}}\frac{\theta_s}{\theta_{\tau_n}}ds.$$

$Z\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)$ satisfies the recursive relationship

$$
\begin{aligned}
Z\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right) &= E_{\tau_n}\int_{\tau_n}^{\tau_{n+1}} e^{-\rho(s-\tau_n)}\left(\frac{C_s}{C_{\tau_n}}\right)^{-\nu}\frac{\left(\frac{M_s}{\theta_s}\right)^{(\gamma-1)(1-\alpha)}}{\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{(\gamma-1)(1-\alpha)}}\frac{\theta_s}{\theta_{\tau_n}}ds \qquad \text{(IA.13)} \\
&+E_{\tau_n}\left[e^{-\rho(\tau_{n+1}-\tau_n)}\left(\frac{C_{\tau_{n+1}}}{C_{\tau_n}}\right)^{-\nu}\frac{\left(\frac{M_{\tau_{n+1}}}{\theta_{\tau_{n+1}}}\right)^{(\gamma-1)(1-\alpha)}}{\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{(\gamma-1)(1-\alpha)}}\left(\frac{\theta_{\tau_{n+1}}}{\theta_{\tau_n}}\right)Z\left(\frac{M_{\tau_{n+1}}}{\theta_{\tau_{n+1}}}\right)\right].
\end{aligned}
$$

Let $\omega_t \equiv \frac{\theta_t}{\theta_{\tau_n}}$ and let $\xi(\omega_t, m_t)$ be defined as

$$
\begin{aligned}
\xi(\omega_t, m_t) &\equiv E_t\int_t^{\tau_{n+1}} e^{-\rho(s-t)}\left(\frac{C_s}{C_{\tau_n}}\right)^{-\nu}\frac{\left(\frac{M_s}{\theta_s}\right)^{(\gamma-1)(1-\alpha)}}{\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{(\gamma-1)(1-\alpha)}}\frac{\theta_s}{\theta_{\tau_n}}ds \\
&= E_t\int_t^{\tau_{n+1}} e^{-\rho(s-t)}\left(1+F(K(m_t))\right)^{-\nu}\omega_t^{1-\gamma}m_t^{(\gamma-1)(1-\alpha)}ds, \qquad \text{(IA.14)}
\end{aligned}
$$

where the second line follows the definitions of ω_t and m_t and from $C_t = \theta_t X_{\tau_n}(1+F(K(m_t)))$. To provide a closed-form solution for $\xi(\omega_t, m_t)$, we solve the ordinary differential equation (ODE)

$$
\frac{\sigma^2}{2}\omega^2\xi_{\omega\omega} + \mu\omega\xi_\omega - (\rho+\lambda)\xi + (1+F(K(m_t)))^{-\nu}\omega_t^{1-\gamma}m_t^{(\gamma-1)(1-\alpha)} = 0 \qquad \text{(IA.15)}
$$

subject to the boundary conditions

$$
\xi_m\left(\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)m_t, m_t\right) = 0, \quad \lim_{\omega_t\to 0}\frac{\xi(\omega_t, m_t)}{(1+F(K(m_t)))^{-\nu}m_t^{-\alpha(\gamma-1)}\left(\frac{\omega_t}{m_t}\right)^{1-\gamma}} < \infty. \qquad \text{(IA.16)}
$$

By the results in Heinricher and Stockbridge (1991), a continuously differentiable function (in ω_t) that solves (IA.15) and (IA.16) is the solution to (IA.14).[2] The function that solves (IA.15) subject

[2] A sketch of the argument follows: Apply Ito's Lemma to $e^{-(\rho+\lambda)t}\xi(\omega_t m_t)$ to obtain

$$
\begin{aligned}
E\left(e^{-(\rho+\lambda)(T-t)}\xi(\omega_T, m_T)\right) - \xi(\omega_t.m_t) &= E\int_t^T e^{-(\rho+\lambda)(s-t)}\left(\frac{\sigma^2}{2}\omega^2\xi_{\omega\omega} + \mu\omega\xi_\omega - (\rho+\lambda)\xi\right)ds \\
&+E\int_t^T e^{-(\rho+\lambda)(s-t)}\xi_m\left(\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)m_s, m_s\right)dm_s,
\end{aligned}
$$

where the second line of the above display uses the fact that m_t increases whenever $\theta_t = M_t$, that is, whenever $\omega_t = \left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)m_t$. Now, let $T\to\infty$, and use (IA.15) together with (IA.16) to arrive at (IA.14).

to (IA.16) is given by

$$\xi\left(\omega_t, m_t\right)=\begin{cases}\beta_1 m_t^{-\alpha(\gamma-1)}\left(\frac{\omega_t}{m_t}\right)^{1-\gamma} & \\ \times\left\{1+\left(\frac{\theta_{\tau_n}\omega_t}{M_{\tau_n}m_t}\right)^{\gamma_1+\gamma-1}\left[\begin{array}{c}\frac{(\gamma-1)(1-\alpha)}{\alpha(\gamma-1)+\gamma_1}+\left(\frac{bpm_t}{\upsilon}\right)^{\alpha(\gamma-1)+\gamma_1} \\ \times g_2\left(0\right)\left[\left(1+b\right)^{(1-p+\nu p)(1+\beta_2]}-1\right)\end{array}\right]\right\}; & m_t\leq\frac{\upsilon}{bp} \\ \beta_1 m_t^{-\alpha(\gamma-1)}\left(\frac{bpm_t}{\upsilon}\right)^{\frac{-\nu p}{1-p+\nu p}}\left(\frac{\omega_t}{m_t}\right)^{1-\gamma} & \\ \times\left\{1+\left(\frac{\theta_{\tau_n}\omega_t}{M_{\tau_n}m_t}\right)^{\gamma_1+\gamma-1}\left[-\frac{g_1(0)}{1+\beta_2}+\left(\frac{bpm_t}{\upsilon}\right)^{-(1+\beta_2)}\left(1+b\right)^{(1-p+\nu p)(1+\beta_2)}g_2\left(0\right)\right]\right\}; & m_t\in\left[\frac{\upsilon}{bp}, m^*\right] \\ \beta_1 m_t^{-\alpha(\gamma-1)}\left(1+b\right)^{-\nu p}\left(\frac{\omega_t}{m_t}\right)^{1-\gamma}\left\{1+\left(\frac{\theta_{\tau_n}\omega_t}{M_{\tau_n}m_t}\right)^{\gamma_1+\gamma-1}\frac{(\gamma-1)(1-\alpha)}{\alpha(\gamma-1)+\gamma_1}\right\}; & m_t\geq m^*,\end{cases}\tag{IA.17}$$

which can be verified by direct substitution into (IA.15) and (IA.16).

We next take a number $\delta\geq 1-\gamma-\gamma_1$, and compute the function $\Phi\left(\omega_t, m_t;\delta\right)$, defined as

$$\begin{aligned}&\Phi\left(\omega_t, m_t;\delta\right) &&\text{(IA.18)}\\ &\quad\equiv\quad E_t\left[e^{-\rho(\tau_{n+1}-t)}\left(\frac{C_{\tau_{n+1}}}{C_{\tau_n}}\right)^{-\nu}\frac{\left(\frac{M_{\tau_{n+1}}}{\theta_{\tau_{n+1}}}\right)^{(\gamma-1)(1-\alpha)}}{\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{(\gamma-1)(1-\alpha)}}\left(\frac{\theta_{\tau_{n+1}}}{\theta_{\tau_n}}\right)\left(\frac{M_{\tau_{n+1}}}{\theta_{\tau_{n+1}}}\right)^{\delta}\right] &&\\ &\quad=\quad\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{\delta}\cdot B\left(\omega_t, m_t, \delta\right), &&\text{(IA.19)}\end{aligned}$$

where

$$B\left(\omega_t, m_t;\delta\right)\equiv E_t\left[e^{-\rho(\tau_{n+1}-t)}\left(\left(1+F(K(m_{\tau_{n+1}^-}))\right)\right)^{-\nu}m_{\tau_{n+1}^-}^{(\gamma-1)(1-\alpha)+\delta}\omega_{\tau_{n+1}^-}^{1-\gamma-\delta}\right].$$

The last line of equation (IA.18) follows from the definitions of ω_t and m_t and from $C_t=\theta_t X_{\tau_n}\left(1+F(K\left(m_t\right))\right)$. The expressions $m_{\tau_{n+1}^-}$ and $\omega_{\tau_{n+1}^-}$ denote the values of m_t and ω_t at the end of epoch n (i.e., an "instant" before the epoch changes).

To determine the expression for $B\left(\omega_t, m_t;\delta\right)$, we repeat the same argument as for $V\left(\omega_t, m_t\right)$. Specifically, $B\left(\omega_t, m_t\right)$ satisfies the ODE

$$\frac{\sigma^2}{2}\omega^2 B_{\omega\omega}+\mu\omega B_\omega-\left(\rho+\lambda\right)B+\left(1+F(K\left(m_t\right))\right)^{-\nu}m_t^{(\gamma-1)(1-\alpha)+\delta}\omega_t^{1-\gamma-\delta}=0\tag{IA.20}$$

subject to the boundary conditions

$$B_m\left(\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)m_t, m_t\right)=0,\quad\lim_{\omega_t\to 0}\frac{B\left(\omega_t, m_t;\delta\right)}{\omega_t^{1-\gamma-\delta}}<\infty.\tag{IA.21}$$

It can be verified by direct substitution that the solution to (IA.20) and (IA.21) is given by

$$B(\omega_t, m_t; \delta) = \begin{cases} g_3(\delta)\left(\frac{\omega_t}{m_t}\right)^{1-\gamma-\delta} m_t^{-\alpha(\gamma-1)} \\ \times\left\{1+\left(\frac{M_{\tau_n} m_t}{\theta_{\tau_n}\omega_t}\right)^{1-\gamma-\gamma_1-\delta}\left[\begin{array}{c} \frac{(\gamma-1)(1-\alpha)+\delta}{\alpha(\gamma-1)+\gamma_1}+\left(\frac{bpm_t}{\upsilon}\right)^{\alpha(\gamma-1)+\gamma_1} \\ \times g_2(\delta)\left[(1+b)^{(1-p+\nu p)(1+\beta_2)}-1\right]\end{array}\right]\right\}; m_t \leq \frac{\upsilon}{bp} \\ g_3(\delta)\left(\frac{bpm_t}{\upsilon}\right)^{\frac{-\nu p}{1-p+\nu p}}\left(\frac{\omega_t}{m_t}\right)^{1-\gamma-\delta} m_t^{-\alpha(\gamma-1)} \\ \times\left\{1+\left(\frac{M_{\tau_n} m_t}{\theta_{\tau_n}\omega_t}\right)^{1-\gamma-\gamma_1-\delta}\left[\begin{array}{c} -\frac{g_1(\delta)}{1+\beta_2}+\left(\frac{bpm_t}{\upsilon}\right)^{-(1+\beta_2)} \\ \times(1+b)^{(1-p+\nu p)(1+\beta_2)} g_2(\delta)\end{array}\right]\right\}; \quad m_t \in [\frac{\upsilon}{bp}, m^*] \\ g_3(\delta)(1+b)^{-\nu p}\left(\frac{\omega_t}{m_t}\right)^{1-\gamma-\delta} m_t^{-\alpha(\gamma-1)} \\ \times\left[1+\frac{(\gamma-1)(1-\alpha)+\delta}{\alpha(\gamma-1)+\gamma_1}\left(\frac{M_{\tau_n} m_t}{\theta_{\tau_n}\omega_t}\right)^{1-\gamma-\gamma_1-\delta}\right]; \quad m_t \geq m^*. \end{cases} \tag{IA.22}$$

Hence, at the beginning of epoch, $\omega_t = 1$ and $m_t = 1$, and therefore

$$B(1,1;\delta) = g_3(\delta) + g_4(\delta)\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{1-\gamma-\gamma_1-\delta}, \tag{IA.23}$$

where the function $g_3(\delta)$ and $g_4(\delta)$ are given in the statement of the lemma. Combining (IA.23) with (IA.19), it follows that

$$\Phi(1,1;\delta) = g_3(\delta)\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{\delta} + g_4(\delta)\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{1-\gamma-\gamma_1}. \tag{IA.24}$$

To complete the computation of $Z\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)$, we employ a "guess and verify" approach. We first guess that $Z\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)$ can be written as

$$Z\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right) = \Delta_1\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{\delta_1} + \Delta_2 \tag{IA.25}$$

for some appropriate constants δ_1, Δ_1, and Δ_2. Using (IA.25) inside the recursive equation (IA.14), recalling that at the beginning of the epoch $\omega_{\tau_n} = 1$ and $m_{\tau_n} = 1$, and using the definition of

$\Phi(\omega_{\tau_n}, m_{\tau_n}; \delta)$ (equation (IA.18)), we obtain

$$\begin{aligned}
& \Delta_1 \left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{\delta_1} + \Delta_2 \qquad \text{(IA.26)} \\
= \;& \xi(\omega_{\tau_n}, m_{\tau_n}) + \Delta_1 \Phi(\omega_{\tau_n}, m_{\tau_n}; \delta_1) + \Delta_2 \Phi(\omega_{\tau_n}, m_{\tau_n}; 0) \\
= \;& \beta_1 + \alpha_1 \left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{1-\gamma-\gamma_1} + \Delta_1 \left[g_3(\delta_1) \left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{\delta_1} + g_4(\delta_1) \left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{1-\gamma-\gamma_1} \right] \\
& + \Delta_2 \left[g_3(0) + g_4(0) \left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{1-\gamma-\gamma_1} \right] \\
= \;& [\beta_1 + \Delta_2 g_3(0)] + [\alpha_1 + \Delta_1 g_4(\delta_1) + \Delta_2 g_4(0)] \left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{1-\gamma-\gamma_1} + \Delta_1 g_3(\delta_1) \left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{\delta_1}.
\end{aligned}$$

Conjecture (IA.25) is true if the coefficients on the left- and the right-hand sides of (IA.26) match. Matching free-term coefficients, that is, setting

$$\Delta_2 = \beta_1 + \Delta_2 g_3(0) = \beta_1 + \Delta_2 \lambda \beta_1,$$

gives Δ_2 as in (IA.11). The value $\delta_1 = 1 - \gamma - \gamma_1^*$ follows from equating the coefficients of $\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{\delta_1}$:

$$\Delta_1 = \Delta_1 g_3(\delta_1).$$

Finally, the term that pre-multiplies $\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{1-\gamma-\gamma_1}$ needs to equal zero,

$$0 = \alpha_1 + \Delta_1 g_4(\delta_1) + \Delta_2 g_4(0),$$

which leads to Δ_1 as in (IA.10). This completes the computation of $Z\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)$.

Having determined $Z\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)$, we observe next that χ_t in equation (IA.12) can be written as

$$\begin{aligned}
\chi\left(\frac{\theta_t}{M_t}, m_t\right) = \;& E_t \left(\int_t^{\tau_{n+1}} e^{-\rho(s-t)} \left(\frac{C_s}{C_t}\right)^{-\nu} \left(\frac{\frac{M_s}{\theta_s}}{\frac{M_t}{\theta_t}}\right)^{(\gamma-1)(1-\alpha)} \frac{\theta_s}{\theta_t} ds \right) \\
& + E_t \left[e^{-\rho(\tau_{n+1}-t)} \left(\frac{C_{\tau_{n+1}}}{C_t}\right)^{-\nu} \left(\frac{\frac{M_{\tau_{n+1}}}{\theta_{\tau_{n+1}}}}{\frac{M_t}{\theta_t}}\right)^{(\gamma-1)(1-\alpha)} \frac{\theta_{\tau_{n+1}}}{\theta_t} \right] \times Z\left(\frac{M_{\tau_{n+1}}}{\theta_{\tau_{n+1}}}\right),
\end{aligned}$$

which implies that

$$
\begin{aligned}
&\chi\left(\frac{\theta_t}{M_t}, m_t\right) \\
&= \left(\frac{C_{\tau_n}}{C_t}\right)^{-\nu} \frac{\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{(\gamma-1)(1-\alpha)}}{\left(\frac{M_t}{\theta_t}\right)^{(\gamma-1)(1-\alpha)}} \frac{\theta_{\tau_n}}{\theta_t} \times E_t\left(\int_t^{\tau_{n+1}} e^{-\rho(s-t)}\left(\frac{C_s}{C_{\tau_n}}\right)^{-\nu}\left(\frac{\frac{M_s}{\theta_s}}{\frac{M_{\tau_n}}{\theta_{\tau_n}}}\right)^{(\gamma-1)(1-\alpha)} \frac{\theta_s}{\theta_{\tau_n}} ds\right) \\
&\quad + \left(\frac{C_{\tau_n}}{C_t}\right)^{-\nu} \frac{\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{(\gamma-1)(1-\alpha)}}{\left(\frac{M_t}{\theta_t}\right)^{(\gamma-1)(1-\alpha)}} \frac{\theta_{\tau_n}}{\theta_t} \\
&\quad \times E_t\left\{\left[e^{-\rho(\tau_{n+1}-t)}\left(\frac{C_{\tau_{n+1}}}{C_{\tau_n}}\right)^{-\nu}\left(\frac{\frac{M_{\tau_{n+1}}}{\theta_{\tau_{n+1}}}}{\frac{M_{\tau_n}}{\theta_{\tau_n}}}\right)^{(\gamma-1)(1-\alpha)} \frac{\theta_{\tau_{n+1}}}{\theta_{\tau_n}}\right] \times\left(\Delta_1\left(\frac{M_{\tau_{n+1}}}{\theta_{\tau_{n+1}}}\right)^{\delta_1}+\Delta_2\right)\right\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\chi\left(\frac{\theta_t}{M_t}, m_t\right) &= \left(\frac{C_{\tau_n}}{C_t}\right)^{-\nu} \frac{\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{(\gamma-1)(1-\alpha)}}{\left(\frac{M_t}{\theta_t}\right)^{(\gamma-1)(1-\alpha)}} \frac{\theta_{\tau_n}}{\theta_t} \cdot \xi\left(\omega_t, m_t\right) \\
&\quad + \left(\frac{C_{\tau_n}}{C_t}\right)^{-\nu} \frac{\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{(\gamma-1)(1-\alpha)}}{\left(\frac{M_t}{\theta_t}\right)^{(\gamma-1)(1-\alpha)}} \frac{\theta_{\tau_n}}{\theta_t}\left[\Delta_1 \Phi\left(\omega_t, m_t ; \delta_1\right)+\Delta_2 \Phi\left(\omega_t, m_t ; 0\right)\right].
\end{aligned} \tag{IA.27}
$$

Plugging the expressions for $\xi\left(\omega_t, m_t\right)$ and $\Phi\left(\omega_t, m_t ; \delta_1\right)$ into equation (IA.27) and simplifying the resulting expression we arrive at the expression for $\chi\left(\frac{\theta_t}{M_t}, m_t\right)$ given in the statement of the lemma. ■

Corollary 1 *The value of assets in place for firm j is given by*

$$
P_{j,t}^A = X_{j,t}\theta_t \cdot \chi\left(\frac{\theta_t}{M_t}, m_t\right).
$$

Proof of Corollary 1. Combine the definitions of χ and $P_{j,t}^A$. ■

With this lemma we are now in a position to discuss the solution to the firm's optimization problem. The option to plant a tree in epoch N does not affect the option to plant a tree in any subsequent epoch.

The individual firm takes the processes for new trees ($K_{N,t}$) and consumption (C_t), and hence the stochastic discount factor H_t and the costs of planting a tree (equation (18)), as given. For the remainder of the proof we consider a firm that expects $K_{N,t}$ to behave as in (IA.9). Such a firm solves the problem

$$
J\left(\theta_t, M_t\right) = \sup_\tau E_t\left[1_{\{\tau<\tau_{N+1}\}} e^{-\rho(\tau-t)}\left(\zeta(i_{N,j}) G\left(\theta_\tau, M_\tau\right)-\eta M_{\tau_N} X_{\tau_N}^{-\nu} \theta_\tau^{-\nu}\left(\frac{M_\tau}{\theta_\tau}\right)^{(\gamma-1)(1-\alpha)}\right)\right],
$$

$$\text{(IA.28)}$$

with $G(\theta_t, M_t)$ defined as

$$\begin{aligned} G(\theta_t, M_t) &\equiv E_t \int_t^{\infty} e^{-\rho(s-\tau)} C_s^{-\gamma} \left(M_s^C\right)^{(\gamma-1)(1-\alpha)} \theta_s ds \\ &= \left[X_{\tau_N}\left(1+F\left(K(m_t)\right)\right)\right]^{-\nu} \theta_t^{\alpha(1-\gamma)} \left(\frac{M_t}{\theta_t}\right)^{(\gamma-1)(1-\alpha)} \cdot \chi\left(\frac{\theta_t}{M_t}, m_t\right). \end{aligned} \tag{IA.29}$$

Hence, the firm's optimization problem is

$$\begin{aligned} &J(\theta_t, M_t) \\ &= \sup_{\tau} E_t \left[1_{\{\tau<\tau_{N+1}\}} e^{-\rho(\tau-t)} \left(\begin{array}{c} \zeta(i_{N,j}) \left[X_{\tau_N}\left(1+F\left(K(m_t)\right)\right)\right]^{-\nu} \theta_\tau^{\alpha(1-\gamma)} \left(\frac{M_\tau}{\theta_\tau}\right)^{(\gamma-1)(1-\alpha)} \cdot \chi\left(\frac{\theta_\tau}{M_\tau}, m_\tau\right) \\ -\eta M_{\tau_N} X_{\tau_N}^{-\nu} \theta_\tau^{-\nu} \left(\frac{M_\tau}{\theta_\tau}\right)^{(\gamma-1)(1-\alpha)} \end{array} \right) \right]. \end{aligned} \tag{IA.30}$$

To solve the optimization problem inside the square brackets we proceed in two steps. First, we derive the optimal policy in a heuristic way by constraining attention to the class of "trigger strategies." Such strategies assume that the firm invests the first time that θ_t (and hence M_t) crosses an (optimally determined) threshold $\bar{\theta}$. Formally, the stopping times associated with these strategies are given by

$$\tau_{\bar{\theta}} = \inf\{s \geq t : \theta_s \geq \bar{\theta}\}. \tag{IA.31}$$

Additionally, we assume that the optimal $\bar{\theta}$ lies in the interval $\left[\frac{\upsilon}{bp} M_{\tau_N}, m^* M_{\tau_N}\right]$.[3] We let Θ denote the class of such trigger strategies. We do not attempt to justify ex-ante why the optimal strategy should lie in this class. Instead, in a second step, we verify the optimality of these strategies via a standard verification theorem for optimal stopping (Proposition 2).

To start, let $\tilde{V}(\theta_t, M_t)$ denote the value function for $\tau_{\bar{\theta}} \in \Theta$:

$$\begin{aligned} &\tilde{V}(\theta_t, M_t) \\ &\equiv \sup_{\tau_{\bar{\theta}} \in \Theta} E_t \left[1_{\{\tau<\tau_{N+1}\}} e^{-\rho(\tau_{\bar{\theta}}-t)} \left(\begin{array}{c} \zeta(i_{N,j}) \left[X_{\tau_N}\left(1+F\left(K(m_t)\right)\right)\right]^{-\nu} \theta_{\tau_{\bar{\theta}}}^{\alpha(1-\gamma)} \left(\frac{M_{\tau_{\bar{\theta}}}}{\theta_{\tau_{\bar{\theta}}}}\right)^{(\gamma-1)(1-\alpha)} \cdot \chi\left(\frac{\theta_{\tau_{\bar{\theta}}}}{M_{\tau_{\bar{\theta}}}}, m_{\tau_{\bar{\theta}}}\right) \\ -\eta M_{\tau_N} X_{\tau_N}^{-\nu} \theta_{\tau_{\bar{\theta}}}^{-\nu} \left(\frac{M_{\tau_{\bar{\theta}}}}{\theta_{\tau_{\bar{\theta}}}}\right)^{(\gamma-1)(1-\alpha)} \end{array} \right) \right]. \end{aligned} \tag{IA.32}$$

We first observe that $\tau_{\bar{\theta}} \in \Theta$ implies that $\theta_{\tau_{\bar{\theta}}} = M_{\tau_{\bar{\theta}}}$ and also $1 + F(K(m_t)) = [1 + bK(m_t)]^p = \left[\left(\frac{bp}{\upsilon}\right) m_t\right]^{\frac{p}{1-p+\nu p}}$. Furthermore, by Øksendal (2003), p. 210-211, we obtain

$$E_t\left[1_{\{\tau<\tau_{N+1}\}} e^{-\rho(\tau_{\bar{\theta}}-t)}\right] = \left(\frac{\theta_t}{\bar{\theta}}\right)^{\gamma_1}.$$

[3] This implies that equation (IA.9) simplifies to

$$K_{N,t} = \frac{\left[\left(\frac{bp}{\upsilon}\right) \frac{M_t}{M_{\tau_N}}\right]^{\frac{1}{1-p+\nu p}} - 1}{b}.$$

Accordingly, letting

$$\varphi\left(\frac{\bar{\theta}}{M_{\tau_N}}\right) \equiv \left(\frac{\bar{\theta}}{M_{\tau_N}}\right)^{\beta_2}\left[\frac{\bar{\theta}}{M_{\tau_N}}\zeta(i_{N,j})\chi\left(1,\frac{\bar{\theta}}{M_{\tau_N}}\right)-\eta\left(\frac{bp\bar{\theta}}{\upsilon M_{\tau_N}}\right)^{\frac{\nu p}{1-p+\nu p}}\right],$$

equation (IA.32) can be re-written as

$$\tilde{V}\left(\theta_t,M_t\right)=\left(\frac{bp}{\upsilon}\right)^{\frac{-\nu p}{1-p+\nu p}}M_{\tau_N}^{\alpha(1-\gamma)-\gamma_1}\theta_t^{\gamma_1}X_{\tau_N}^{-\nu}\times\sup_{\bar{\theta}\in\left[\frac{\upsilon}{bp}M_{\tau_N},m^*M_{\tau_N}\right]}\varphi\left(\frac{\bar{\theta}}{M_{\tau_N}}\right). \tag{IA.33}$$

By the assumption $\tau_{\bar{\theta}}\in\Theta$, $\frac{\upsilon}{bp}\leq m_{\tau_{\bar{\theta}}}\leq m^*$ and hence Lemma 3 implies that

$$\begin{aligned}\varphi\left(\frac{\bar{\theta}}{M_{\tau_N}}\right) &= \left(\frac{\bar{\theta}}{M_{\tau_N}}\right)^{\beta_2+1}\zeta(i_{N,j})\left\{\begin{array}{c}\Delta_2\left\{1-\frac{g_1(0)}{1+\beta_2}+\left(\frac{bp\bar{\theta}}{\upsilon M_{\tau_N}}\right)^{-(1+\beta_2)}(1+b)^{(1-p+\nu p)(1+\beta_2)}g_2\left(0\right)\right\}+\Delta_1\times\\ \left\{1-\frac{g_1(1-\gamma-\gamma_1^*)}{1+\beta_2}+\left(\frac{bp\bar{\theta}}{\upsilon M_{\tau_N}}\right)^{-(1+\beta_2)}(1+b)^{(1-p+\nu p)(1+\beta_2)}g_2\left(1-\gamma-\gamma_1^*\right)\right\}\end{array}\right\}\\ &\quad-\left(\frac{\bar{\theta}}{M_{\tau_N}}\right)^{\beta_2}\eta\left(\frac{bp\bar{\theta}}{\upsilon M_{\tau_N}}\right)^{\frac{\nu p}{1-p+\nu p}}.\end{aligned}$$

Assuming an interior solution and setting $\varphi'\left(\frac{\bar{\theta}}{M_{\tau_N}}\right)=0$, we obtain

$$\frac{\bar{\theta}}{M_{\tau_N}}=\frac{\left(\frac{\nu p}{1-p+\nu p}+\beta_2\right)\eta\left(\frac{bp\bar{\theta}}{\upsilon M_{\tau_N}}\right)^{\frac{\nu p}{1-p+\nu p}}}{\zeta(i_{N,j})\left[\left(1+\beta_2-g_1\left(0\right)\right)\Delta_2+\left(1+\beta_2-g_1\left(1-\gamma-\gamma_1^*\right)\right)\Delta_1\right]}. \tag{IA.34}$$

Notice that policy (IA.34) has the same form as policy (19), which is given by

$$\frac{\bar{\theta}}{M_{\tau_N}}=\frac{\upsilon\left(1+F\left(i_{N,j}\right)\right)^{\nu}}{\zeta(i_{N,j})}=\frac{\upsilon\left(\frac{bp\bar{\theta}}{\upsilon M_{\tau_N}}\right)^{\frac{\nu p}{1-p+\nu p}}}{\zeta(i_{N,j})}. \tag{IA.35}$$

Combining (IA.34) with (IA.35) implies that

$$\upsilon=\frac{\eta\left(\beta_2+\frac{\nu p}{1-p+\nu p}\right)}{\Delta_2\left(1+\beta_2-g_1\left(0\right)\right)+\Delta_1\left(1+\beta_2-g_1\left(1-\gamma-\gamma_1^*\right)\right)}. \tag{IA.36}$$

Notice that Δ_1 is a function of υ, although other parameters are independent of υ. Hence, equation (IA.36) is a non-linear equation in υ. We denote the solution to this equation by υ^* and assume that parameters are such that $\upsilon^*\geq bp$.[4]

Now note that if all other firms follow trigger strategies of the form (IA.34) with $\upsilon=\upsilon^*$, then the resulting process for K_t is given by (IA.9) with $\upsilon=\upsilon^*$, confirming the conjecture of firm j about the behavior of $K_{N,t}$. Assuming that the optimal stopping policy of any firm j lies in the interior of the "trigger" class Θ, firm j behaves optimally by following policy (IA.35) evaluated at $\upsilon=\upsilon^*$.

[4]This can be achieved by assuming a large enough value for η.

The next proposition shows that if all firms $j' \neq j$ follow policies of the form (IA.35) with $\upsilon = \upsilon^*$, then the optimal stopping strategy for firm j (across all possible stopping strategies) indeed takes the form (IA.35). We use the notation $x \wedge y$ for $\min(x, y)$ and $x \vee y$ for $\max(x, y)$.

Proposition 2 *Assume $\phi \equiv \rho + \lambda + \mu\gamma - (\gamma+1)\gamma\frac{\sigma^2}{2} > 0$ and $\gamma + \gamma_1 > 1$. Let υ^* denote the solution to (IA.36), $K(m_t)$ be given by (IA.9) with $\upsilon = \upsilon^*$, and $G(\theta_t, M_t)$ by (IA.29). Define $\bar{\theta}(M_t)$ as the solution to the equation*

$$\bar{\theta}(M_t) = \arg\max_{\bar{\theta}} \left(\frac{1}{\bar{\theta}}\right)^{\gamma_1} \left[\zeta(i_{N,j}) G\left(\bar{\theta}, M_t \vee \bar{\theta}\right) - \eta M_{\tau_N} \bar{\theta}^{-\nu} X_{\tau_N}^{-\nu} \left(\frac{\bar{\theta}}{M_t} \wedge 1\right)^{(1-\gamma)(1-\alpha)}\right]. \quad \text{(IA.37)}$$

Then it is optimal for firm j in epoch N to plant a tree the first time that $\theta_t \geq \bar{\theta}(M_t)$.

Proof of Proposition 2. The marginal firm solves the optimal stopping problem specified by (IA.28). For any C^1 function $f : \mathbb{R} \to \mathbb{R}$ that is twice-differentiable a.e. define the infinitesimal operator $\mathcal{A}(f) \equiv \frac{\sigma^2}{2}\theta^2 f_{\theta\theta} + \mu\theta f_\theta - (\rho + \lambda) f$. Next, note that Lemma 3 implies that the function $G(\theta_t, M_t)$ can be written as

$$\begin{aligned} G(\theta_t, M_t) \quad = \quad & [X_{\tau_N}(1 + F(K_t))]^{-\nu} M_t^{(\gamma-1)(1-\alpha)} \\ & \times \left[\Delta_2 \theta_t^{1-\gamma} + \Delta_1 \theta_t^{\gamma_1^*} \left(\frac{1}{M_t}\right)^{\gamma+\gamma_1^*-1} + Const(m_t) \cdot \theta_t^{\gamma_1}\right], \end{aligned} \quad \text{(IA.38)}$$

where $Const$ depends on m_t but is independent of θ_t. Since $\mathcal{A}(\theta_t^{\gamma_1}) = 0$, it is straightforward to check that

$$\mathcal{A}G(\theta_t, M_t) = -[X_{\tau_N}(1 + F(K_t))]^{-\nu} \left[\frac{1}{1 - \beta_1\lambda} \theta_t^{1-\gamma} (M_t)^{(\gamma-1)(1-\alpha)} + \lambda\Delta_1 \theta_t^{\gamma_1^*} M_t^{-\alpha(\gamma-1)-\gamma_1^*}\right]. \quad \text{(IA.39)}$$

Furthermore, by the construction of $G(\theta_t, M_t)$ (see Lemma 3) we also obtain

$$G_M(M_t, M_t) = 0. \quad \text{(IA.40)}$$

With these observations, let $\bar{\theta}(M_t)$ be defined as in equation (IA.37) and define the "candidate" value function $V(\theta_t, M_t)$ as

$$V(\theta_t, M_t) = \left(\frac{\theta_t}{\bar{\theta}}\right)^{\gamma_1} \left[\zeta(i_{N,j}) G\left(\bar{\theta}, M_t \vee \bar{\theta}\right) - \eta M_{\tau_N} \bar{\theta}^{-\nu} X_{\tau_N}^{-\nu} \left(\frac{\bar{\theta}}{M_t} \wedge 1\right)^{(1-\gamma)(1-\alpha)}\right] \quad \text{(IA.41)}$$

whenever $\theta_t \leq \bar{\theta}(M_t)$ and

$$V(\theta_t, M_t) = \zeta(i_{N,j}) G(\theta_t, M_t) - \eta M_{\tau_N} \theta_t^{-\nu} X_{\tau_N}^{-\nu} \left(\frac{\theta_t}{M_t}\right)^{(1-\gamma)(1-\alpha)} \quad \text{(IA.42)}$$

whenever $\theta_t > \bar{\theta}(M_t)$. In what follows we show the following four properties of the function $V(\theta_t, M_t)$:

$$V(\theta_t, M_t) \geq \zeta(i_{N,j})G(\theta_t, M_t) - \eta M_{\tau_N}\theta_t^{-\nu}X_{\tau_N}^{-\nu}\left(\frac{\theta_t}{M_t}\right)^{(1-\gamma)(1-\alpha)}, \tag{IA.43}$$

$$V(\theta_t, M_t) \text{ is continuously differentiable in } \theta_t, \tag{IA.44}$$

$$V_M(\theta_t, M_t) \leq 0 \text{ for } \theta_t = M_t, \tag{IA.45}$$

$$\mathcal{A}V(\theta_t, M_t) \leq 0\,. \tag{IA.46}$$

Property (IA.43) is immediate for $\theta_t \geq \bar{\theta}(M_t)$, and for $\theta_t \leq \bar{\theta}(M_t)$ it follows from

$$\begin{aligned}
&\zeta(i_{N,j})G(\theta_t, M_t) - \eta M_{\tau_N}\theta_t^{-\nu}X_{\tau_N}^{-\nu}\left(\frac{\theta_t}{M_t}\right)^{(1-\gamma)(1-\alpha)} = \\
&= \zeta(i_{N,j})G(\theta_t, M_t \vee \theta_t) - \eta M_{\tau_N}\theta_t^{-\nu}X_{\tau_N}^{-\nu}\left(\frac{\theta_t}{M_t}\wedge 1\right)^{(1-\gamma)(1-\alpha)} \\
&= \left(\frac{\theta_t}{\theta_t}\right)^{\gamma_1}\left[\zeta(i_{N,j})G(\theta_t, M_t \vee \theta_t) - \eta M_{\tau_N}\theta_t^{-\nu}X_{\tau_N}^{-\nu}\left(\frac{\theta_t}{M_t}\wedge 1\right)^{(1-\gamma)(1-\alpha)}\right] \\
&\leq \theta_t^{\gamma_1}\max_{\bar{\theta}}\left(\frac{1}{\bar{\theta}}\right)^{\gamma_1}\left[\zeta(i_{N,j})G\left(\bar{\theta}, M_t \vee \bar{\theta}\right) - \eta M_{\tau_N}\bar{\theta}^{-\nu}X_{\tau_N}^{-\nu}\left(\frac{\bar{\theta}}{M_t}\wedge 1\right)^{(1-\gamma)(1-\alpha)}\right] \\
&= V(\theta_t, M_t)\,.
\end{aligned}$$

To show property (IA.44) consider first the case $\theta_t \leq \bar{\theta}(M_t)$. Differentiating (IA.41) gives

$$\frac{\partial V}{\partial \theta} = \gamma_1 \frac{1}{\theta_t} V(\theta_t, M_t)\,, \tag{IA.47}$$

which is a continuous function. Furthermore, when $\theta_t \to \bar{\theta}(M_t)$, we obtain

$$\begin{aligned}
&\lim_{\theta_t \to \bar{\theta}(M_t)} \frac{\partial V(\theta_t, M_t)}{\partial \theta_t} \qquad \text{(IA.48)} \\
&= \frac{\gamma_1}{\bar{\theta}(M_t)}\left[\zeta(i_{N,j})G\left(\bar{\theta}(M_t), M_t \vee \bar{\theta}(M_t)\right) - X_{\tau_N}^{-\nu}\left[\bar{\theta}(M_t)\right]^{-\nu}\left(\frac{\bar{\theta}(M_t)}{M_t}\wedge 1\right)^{(1-\gamma)(1-\alpha)}\eta M_{\tau_N}\right].
\end{aligned}$$

Turning next to the case where $\theta_t > \bar{\theta}(M_t)$, direct differentiation of (IA.42) shows that the partial derivative of $V(\theta_t, M_t)$ with respect to θ_t is a continuous function, whose value at $\theta_t = \bar{\theta}(M_t)$ is given by

$$\begin{aligned}
&\lim_{\theta_t \to \bar{\theta}(M_t)} \frac{\partial V(\theta_t, M_t)}{\partial \theta} \qquad \text{(IA.49)} \\
&= \zeta(i_{N,j})G_\theta\left(\bar{\theta}(M_t), M_t\right) + \gamma\left[\bar{\theta}(M_t)\right]^{-\nu}\frac{1}{\bar{\theta}(M_t)}\left(\frac{M_t}{\bar{\theta}(M_t)}\right)^{(\gamma-1)(1-\alpha)}\eta X_{\tau_N}^{-\nu}M_{\tau_N}.
\end{aligned}$$

To establish (IA.44), we need to show that the "left" hand side derivative (equation (IA.48)) and the "right" hand side derivative (equation (IA.49)) coincide. Note that this statement is meaningful only when $\bar{\theta}(M_t) \leq M_t$, for otherwise $\theta_t \leq M_t < \bar{\theta}(M_t)$. Then the necessary condition for optimality (first order condition) of equation (IA.37) implies that

$$\begin{aligned} 0 &= \left(\frac{1}{\bar{\theta}}\right)^{\gamma_1}\left[\zeta(i_{N,j})G_\theta\left(\bar{\theta}, M_t\right) - (-\gamma)\,\bar{\theta}^{-\nu}\left(\frac{M_t}{\bar{\theta}}\right)^{(\gamma-1)(1-\alpha)}\frac{1}{\bar{\theta}}\eta X_{\tau_N}^{-\nu}M_{\tau_N}\right] \\ &\quad -\gamma_1\left(\frac{1}{\bar{\theta}}\right)^{\gamma_1}\frac{1}{\bar{\theta}}\left[\zeta(i_{N,j})G\left(\bar{\theta}, M_t\right) - \bar{\theta}^{-\nu}\left(\frac{M_t}{\bar{\theta}}\right)^{(\gamma-1)(1-\alpha)}\eta X_{\tau_N}^{-\nu}M_{\tau_N}\right]. \end{aligned} \tag{IA.50}$$

Dividing both sides of equation (IA.50) by $\left(\frac{1}{\bar{\theta}}\right)^{\gamma_1}$, we obtain that the right hand side of equation (IA.48) and the right hand side of equation (IA.49) are identical, so that $\frac{\partial V(\theta_t, M_t)}{\partial \theta}$ is continuous at $\theta_t = \bar{\theta}(M_t)$.

To establish (IA.45), consider two cases. When $M_t \geq \bar{\theta}(M_t)$, then whenever $\theta_t = M_t$, equation (IA.42) along with (IA.40) leads to

$$\begin{aligned} V_M(M_t, M_t) &= \zeta(i_{N,j})G_M(M_t, M_t) - (\gamma-1)(1-\alpha)\,M_t^{(\gamma-1)(1-\alpha)-\gamma}\left(\frac{M_t}{M_t}\right)^{(\gamma-1)(1-\alpha)}\frac{M_{\tau_N}}{M_t}\eta \\ &= -\eta(\gamma-1)(1-\alpha)\,M_t^{-\nu-1}M_{\tau_N} \;\leq\; 0. \end{aligned}$$

When $M_t \leq \bar{\theta}(M_t)$, $M_t \vee \bar{\theta} = \bar{\theta}$ and hence whenever $\theta_t = M_t$, $V(\theta_t, M_t)$ is given by

$$\left(\frac{\theta_t}{\bar{\theta}}\right)^{\gamma_1}\cdot\left[\zeta(i_{N,j})G\left(\bar{\theta}, \bar{\theta}\right) - \bar{\theta}^{(\gamma-1)(1-\alpha)-\gamma}\eta X_{\tau_N}^{(\gamma-1)(1-\alpha)-\gamma}M_{\tau_N}\right],$$

which is independent of M_t. Hence, $V_M(M_t, M_t) = 0$.

To show property (IA.46), we start by noting that when $\theta_t < \bar{\theta}(M_t)$, $V(\theta_t, M_t)$ is given by (IA.41). Hence, $\mathcal{A}(V) = 0$, since $\mathcal{A}(\theta^{\gamma_1}) = 0$. When $\theta_t \geq \bar{\theta}(M_t)$, $V(\theta_t, M_t)$ is given by (IA.42). Using (IA.39) we obtain

$$\begin{aligned} \mathcal{A}V &= X_{\tau_N}^{(\gamma-1)(1-\alpha)-\gamma}\left(\frac{M_t}{\theta_t}\right)^{(\gamma-1)(1-\alpha)}\theta_t^{-\alpha(\gamma-1)-1} \\ &\quad\times\left[\phi\eta M_{\tau_N} - \zeta(i_{N,j})\left[1+F(K_t)\right]^{-\nu}\theta_t\left(\frac{1}{1-\lambda\beta_1} + \lambda\Delta_1\left(\frac{M_t}{\theta_t}\right)^{1-\gamma-\gamma_1^*}\right)\right]. \end{aligned} \tag{IA.51}$$

Hence, we only need to show that the term inside square brackets in (IA.51) is nonpositive for $\theta_t \geq \bar{\theta}(M_t)$. This amounts to showing that

$$\frac{\phi\eta M_{\tau_N}\left[(1+F(K_t))\right]^{\nu}}{\zeta(i_{N,j})} \leq \theta_t\cdot\left(\frac{1}{1-\lambda\beta_1} + \lambda\Delta_1\left(\frac{M_t}{\theta_t}\right)^{1-\gamma-\gamma_1^*}\right). \tag{IA.52}$$

Since the right hand side of (IA.52) is increasing in θ_t and $\theta_t \geq \bar{\theta}(M_t)$, it suffices to show that

$$\frac{\phi\eta M_{\tau_N}\left[1+F\left(K\left(M_t\right)\right)\right]^{\nu}}{\zeta(i_{N,j})} \leq \bar{\theta}\left(M_t\right)\left(\frac{1}{1-\lambda\beta_1}+\lambda\Delta_1\left(\frac{M_t}{\bar{\theta}\left(M_t\right)}\right)^{1-\gamma-\gamma_1^*}\right). \tag{IA.53}$$

Since $M_t \geq \theta_t \geq \bar{\theta}(M_t)$, equation (IA.50) can be re-written as

$$\begin{aligned} 0 &= \zeta(i_{N,j})\left[-\gamma_1\frac{1}{\bar{\theta}}G\left(\bar{\theta},M_t\right)+G_\theta\left(\bar{\theta},M_t\right)\right]+\gamma_1\frac{1}{\bar{\theta}}\bar{\theta}^{-\nu}\left(\frac{\bar{\theta}}{M_t}\right)^{(1-\gamma)(1-\alpha)}\eta X_{\tau_N}^{-\nu}M_{\tau_N} \\ &\quad +\gamma\frac{1}{\bar{\theta}}\bar{\theta}^{-\nu}\left(\frac{\bar{\theta}}{M_t}\right)^{(1-\gamma)(1-\alpha)}\eta X_{\tau_N}^{-\nu}M_{\tau_N}. \end{aligned} \tag{IA.54}$$

By (IA.38),

$$\begin{aligned} &-\gamma_1\frac{1}{\bar{\theta}}G\left(\bar{\theta},M_t\right)+G_\theta\left(\bar{\theta},M_t\right) \\ &= \left[X_{\tau_N}\left(1+F\left(K_t\right)\right)\right]^{-\nu}M_t^{(\gamma-1)(1-\alpha)}\times \\ &\left[-\gamma_1\Delta_2\bar{\theta}^{-\gamma}-\gamma_1\Delta_1\bar{\theta}^{\gamma_1^*-1}\left(\frac{1}{M_t}\right)^{\gamma+\gamma_1^*-1}+\Delta_2\left(1-\gamma\right)\bar{\theta}^{-\gamma}+\gamma_1^*\Delta_1\bar{\theta}^{\gamma_1^*-1}\left(\frac{1}{M_t}\right)^{\gamma+\gamma_1^*-1}\right]. \end{aligned} \tag{IA.55}$$

Combining equations (IA.54) and (IA.55) and simplifying yields

$$\bar{\theta}\left(M_t\right)=\frac{\phi\eta M_{\tau_N}\left(1+F\left(K\left(M_t\right)\right)\right)^{\nu}}{\zeta(i_{N,j})}\frac{\left(\gamma_1+\gamma\right)}{\phi\left(\frac{(\gamma+\gamma_1-1)}{1-\lambda\beta}\beta+\Delta_1\left(\gamma_1-\gamma_1^*\right)\left(\frac{M_t}{\bar{\theta}(M_t)}\right)^{1-\gamma-\gamma_1^*}\right)}.$$

Hence, to show equation (IA.53), we only need to verify that

$$\begin{aligned} &\frac{\phi\eta M_{\tau_N}\left(1+F\left(K_t\right)\right)^{\nu}}{\zeta(i_{N,j})}\frac{\left(\gamma_1+\gamma\right)\left(\frac{1}{1-\lambda\beta_1}+\lambda\Delta_1\left(\frac{M_t}{\bar{\theta}(M_t)}\right)^{1-\gamma-\gamma_1^*}\right)}{\phi\left(\frac{(\gamma+\gamma_1-1)}{1-\lambda\beta}\beta_1+\Delta_1\left(\gamma_1-\gamma_1^*\right)\left(\frac{M_t}{\bar{\theta}(M_t)}\right)^{1-\gamma-\gamma_1^*}\right)} \\ &\geq \quad \frac{\phi\eta M_{\tau_N}\left[1+F\left(K\left(M_t\right)\right)\right]^{\nu}}{\zeta(i_{N,j})}. \end{aligned}$$

To this end, we only need to show

$$\frac{\gamma_1+\gamma}{1-\lambda\beta_1}+\lambda\Delta_1\left(\gamma_1+\gamma\right)\left(\frac{M_t}{\bar{\theta}\left(M_t\right)}\right)^{1-\gamma-\gamma_1^*} \geq \phi\beta_1\frac{\gamma+\gamma_1-1}{1-\lambda\beta_1}+\phi\Delta_1\left(\gamma_1-\gamma_1^*\right)\left(\frac{M_t}{\bar{\theta}\left(M_t\right)}\right)^{1-\gamma-\gamma_1^*}. \tag{IA.56}$$

Define γ_2 as

$$\gamma_2 \equiv \frac{-\sqrt{\left(\mu-\frac{\sigma^2}{2}\right)^2+2\sigma^2\left(\rho+\lambda\right)}-\left(\mu-\frac{\sigma^2}{2}\right)}{\sigma^2} < 0.$$

Then,

$$\begin{aligned}\gamma_1\gamma_2 &= -\frac{2(\rho+\lambda)}{\sigma^2} \\ \gamma_1+\gamma_2 &= \frac{-2\left(\mu-\frac{\sigma^2}{2}\right)}{\sigma^2}.\end{aligned}$$

Hence,

$$\phi = -\left[(\gamma+1)\gamma\frac{\sigma^2}{2} - \mu\gamma - \rho - \lambda\right] = -\frac{\sigma^2}{2}(\gamma_1+\gamma)(\gamma+\gamma_2) > 0, \tag{IA.57}$$

which implies that $\gamma+\gamma_2<0$ (recall that, by assumption, $\phi>0$). Direct algebra gives

$$\beta_1\phi = \frac{\gamma+\gamma_2}{\gamma+\gamma_2-1}\times\frac{\gamma+\gamma_1}{\gamma+\gamma_1-1} \le \frac{\gamma_1+\gamma}{\gamma+\gamma_1-1}.$$

Consequently,

$$\frac{\gamma_1+\gamma}{1-\lambda\beta_1} \ge \phi\beta_1\frac{\gamma+\gamma_1-1}{1-\lambda\beta_1}. \tag{IA.58}$$

Therefore, to show inequality (IA.56), equation (IA.58) implies that we only need to show

$$\lambda\Delta_1(\gamma_1+\gamma)\left(\frac{M_t}{\bar\theta}\right)^{1-\gamma-\gamma_1^*} \ge \phi\Delta_1(\gamma_1-\gamma_1^*)\left(\frac{M_t}{\bar\theta}\right)^{1-\gamma-\gamma_1^*},$$

which is equivalent to showing that $\lambda(\gamma_1+\gamma)\ge\phi(\gamma_1-\gamma_1^*)$. Direct algebra shows that

$$\frac{2\lambda}{\sigma^2} = (\gamma_1^*-\gamma_2)(\gamma_1-\gamma_1^*). \tag{IA.59}$$

By (IA.57) and (IA.59),

$$\begin{aligned}\lambda(\gamma_1+\gamma) &= \frac{\sigma^2}{2}(\gamma_1+\gamma)(\gamma_1^*-\gamma_2)(\gamma_1-\gamma_1^*) \qquad \text{(IA.60)} \\ &= -\frac{\phi}{\gamma+\gamma_2}(\gamma_1^*-\gamma_2)(\gamma_1-\gamma_1^*).\end{aligned}$$

Furthermore,

$$(\gamma_1^*-\gamma_2) \ge -(\gamma+\gamma_2), \tag{IA.61}$$

since $\gamma_1^*+\gamma\ge0$. Given that $\gamma_1>\gamma_1^*$ and $\gamma+\gamma_2<0$, (IA.61) and (IA.60) yield the desired conclusion, namely, $\lambda(\gamma_1+\gamma)\ge\phi(\gamma_1-\gamma_1^*)$. This completes the proof of (IA.46).

The rest of the proof follows steps similar to Øksendal (2003), Chapter 9. For completeness we give a brief sketch omitting technical details. Take any stopping time τ and apply Ito's Lemma to $e^{-(\rho+\lambda)t}V(\theta_t,M_t)$ to obtain

$$\begin{aligned}Ee^{-(\rho+\lambda)(\tau-t)}V(\theta_\tau,M_\tau)-V(\theta_t,M_t) &= E_t\int_0^\tau e^{-(\rho+\lambda)(s-t)}\mathcal{A}V(\theta_s,M_s)\,ds + \qquad \text{(IA.62)} \\ &\quad +E_t\int_t^\tau e^{-(\rho+\lambda)(s-t)}V_M(M_s,M_s)\,dM_s.\end{aligned}$$

Re-arranging (IA.62) and using (IA.43)–(IA.46) yields

$$
\begin{aligned}
V\left(\theta_t, M_t\right) &\geq Ee^{-(\rho+\lambda)(\tau-t)}V\left(\theta_\tau, M_\tau\right) \\
&\geq Ee^{-(\rho+\lambda)(\tau-t)}\left[\zeta(i_{N,j})G\left(\theta_\tau, M_\tau\right) - \eta M_{\tau_N}\left[\theta_\tau\right]^{-\nu} X_{\tau_N}^{-\nu}\left(\frac{\theta_\tau}{M_\tau}\right)^{(1-\gamma)(1-\alpha)}\right].
\end{aligned}
$$

Since τ is arbitrary, $V\left(\theta_t, M_t\right)$ provides an upper bound to the value function for all feasible policies. Furthermore, this bound is attainable if the firm plants a tree the first time that $\theta_t = \bar{\theta}\left(M_t\right)$. Hence, $V\left(\theta_t, M_t\right)$ is the value function for firm j in round N and planting a tree once $\theta_t = \bar{\theta}\left(M_t\right)$ is optimal.

■

Proposition 2 shows that if firms perceive the equilibrium stochastic discount factor to be given by $H_t = e^{-\rho t}U_c(c_t, M_t^C)$, then it is optimal for them to plant a tree according to equation (19). Furthermore, Corollary 1 gives the equilibrium value of assets in place for firm j in round N at time t. To complete the determination of the value of a firm, the following proposition provides the equilibrium value of "current-epoch" growth options and "future-epoch" growth options.

Proposition 3 *Let $K(m_t)$ be given by (IA.9) with $\upsilon = \upsilon^*$. Then the price of firm j in technological epoch N is given by $P_{j,t}^A + P_{N,j,t}^o + P_{N,t}^f$, where the asset in place $P_{j,t}^A$ is given by*

$$
P_{j,t}^A = X_{j,t}\theta_t\chi\left(\frac{\theta_t}{M_t}, m_t\right) \tag{IA.63}
$$

and the current-epoch growth option at time t for firm j is

$$
\begin{aligned}
P_{N,j,t}^o &= X_{\tau_N}\theta_t\left(\frac{\theta_t}{M_t}\right)^{\gamma_1+\gamma-1}\left(\frac{M_t}{M_{\tau_N}}\right)^{\gamma_1+\alpha(\gamma-1)}\left(1+F\left(K\left(m_t\right)\right)\right)^{\nu} \\
&\quad \times\left(\frac{bp}{\upsilon^*}\right)^{\frac{-\nu p}{1-p+\nu p}-\beta_2}\upsilon^* C_{op}^{ind}\left(i_{N,j}\right)\left(1-1_{\{\widetilde{\chi}_{N,j}=1\}}\right),
\end{aligned} \tag{IA.64}
$$

where the constant $C_{op}^{ind}\left(i_{N,j}\right)$ is given by

$$
C_{op}^{ind}\left(i_{N,j}\right) = \left(1+bi_{N,j}\right)^{(1-p+\nu p)\beta_2+\nu p}\left(\begin{array}{c} -\frac{\eta}{\upsilon^*}\frac{1-\frac{\nu p}{1-p+\nu p}}{1+\beta_2} + \left(1+bi_{j,N}\right)^{-(1-p+\nu p)(1+\beta_2)} \\ \times\left(1+b\right)^{(1-p+\nu p)(1+\beta_2)}\left[\Delta_2 g_2\left(0\right)+\Delta_1 g_2\left(1-\gamma-\gamma_1^*\right)\right]\end{array}\right).
$$

Finally, define the constants C_{op} and $\tilde{\Delta}_1$ as

$$
\begin{aligned}
C_{op} &= \left[\frac{bp}{\upsilon^*}\right]^{\frac{-\nu p}{1-p+\nu p}-\beta_2}\upsilon^*\left(\begin{array}{c}\left(1+b\right)^{[1-p+\nu p](1+\beta_2)}\left[\Delta_2 g_2\left(0\right)+\Delta_1 g_2\left(1-\gamma-\gamma_1^*\right)\right]\frac{(1+b)^p-1}{bp} \\ -\frac{\eta}{\upsilon^*}\frac{\left(1-\frac{\nu p}{1-p+\nu p}\right)}{1+\beta_2}\frac{(1+b)^{[1-p+\nu p]\beta_2+\nu p+1}-1}{b([1-p+\nu p]\beta_2+\nu p+1)}\end{array}\right), \\
\tilde{\Delta}_1 &= -\frac{C_{op}}{\tilde{g}_4\left(1-\gamma-\gamma_1^*\right)}.
\end{aligned}
$$

Then the value of all "future epoch" growth options is given by

$$
\begin{aligned}
&P^f_{N,t} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(IA.65)}\\
&= \tilde{\Delta}_1 X_t \theta_t \begin{cases}
\left(\frac{\theta_t}{M_t}\right)^{\gamma+\gamma_1^*-1} \left\{ 1 + \left(\frac{\theta_t}{M_t}\right)^{\gamma_1-\gamma_1^*} \left[\begin{array}{c} -\frac{\alpha(\gamma-1)+\gamma_1^*}{\alpha(\gamma-1)+\gamma_1} + \left(\frac{bpm_t}{\upsilon^*}\right)^{\alpha(\gamma-1)+\gamma_1} \times \\ \tilde{g}_2 \left(1-\gamma-\gamma_1^*\right) \left[(1+b)^{(1-p+\nu p)(1+\beta_2)+p} - 1 \right] \end{array} \right] \right\}; & m_t \le \frac{\upsilon^*}{bp} \\
\left(\frac{\theta_t}{M_t}\right)^{\gamma+\gamma_1^*-1} \left\{ 1 + \left(\frac{\theta_t}{M_t}\right)^{\gamma_1-\gamma_1^*} \left[\begin{array}{c} \frac{-\tilde{g}_1(1-\gamma-\gamma_1^*)}{1+\beta_2+\frac{p}{1-p+\nu p}} + \left(\frac{bpm_t}{\upsilon^*}\right)^{-(1+\beta_2)-\frac{p}{1-p+\nu p}} \times \\ (1+b)^{(1-p+\nu p)(1+\beta_2)+p} \tilde{g}_2 \left(1-\gamma-\gamma_1^*\right) \end{array} \right] \right\}; & m_t \in \left[\frac{\upsilon^*}{bp}, m^*\right] \\
\left(\frac{\theta_t}{M_t}\right)^{\gamma+\gamma_1^*-1} \left[1 - \frac{\alpha-\alpha\gamma-\gamma_1^*}{\alpha-\alpha\gamma-\gamma_1} \left(\frac{\theta_t}{M_t}\right)^{\gamma_1-\gamma_1^*} \right]; & m_t \ge m^*.
\end{cases}
\end{aligned}
$$

Proof of Proposition 3. The proof of (IA.63) is given in Corollary 1. The proof of (IA.64) follows upon computing expression (IA.41) explicitly. The proof of equation (IA.65) follows similar steps to that of Lemma 3 and is omitted to save space. ∎

B. Some Remarks on the Model Setup and Extensions

B.1. The Representative-Agent Assumption

Throughout the paper we speak of a "representative" consumer-worker to expedite the presentation. As we also discuss in footnote 11 in the text, Rogerson (1988) shows how the assumption of a "representative" consumer-worker is consistent with the presence of indivisible labor supply, as long as one allows for labor-supply lotteries as one of the tradable contingent claims. Intuitively, even if firms randomly choose a worker to plant a tree, trading between workers allows them to share that risk, and hence there is no idiosyncratic endowment risk.

Even though the assumption of labor-supply lotteries introduced by Rogerson (1988) (and followed by the strand of the macroeconomics literature that builds on his paper) is sufficient to justify the existence of a representative agent, we note that it can be relaxed in our setup. Instead of introducing labor-supply lotteries, we can instead easily enrich the model and assume that the planting of each tree is a divisible task amongst workers. Specifically, if a) planting a single tree takes a continuum of tasks $z \in [0,1]$, b) each worker incurs a disutility of effort η_t per task performed, and c) any worker can perform any set of tasks in perfect competition, then there exists an equilibrium whereby tasks are divided equally across workers and the proceeds from planting a tree are allocated equally between them, even in the absence of labor-supply lotteries.

B.2. Market Completeness in the Context of the Full Model

In addition to markets in which agents can trade claims contingent on the realization of the diffusive shock θ_t, the assumption of market completeness in the context of the complete model also requires the existence of markets in which agents can trade securities (in zero net supply) that promise to pay one unit of the numeraire when technological round N arrives. An argument similar to Duffie and Huang (1985) implies, however, that these additional markets are redundant in general equilibrium, provided the existence of the usual traded assets, since agents are able to create dynamic portfolios of stocks and bonds that produce the same payoff as these claims. (For instance, the spread between short-maturity and long-maturity bonds jumps by a predictable magnitude when a new epoch arrives.) Hence, by trading in the stock market, a short-maturity bond, and a long-maturity bond, agents can synthetically "span" the assumed contingent-claims markets.

B.3. Trees and Previous Epochs

The assumption that a firm can plant a tree corresponding only to the current epoch can be relaxed (assuming that a firm can plant one tree each epoch), if we modify equation (12) to $A_{n+1} = A_n \bar{A} \left(1 + \int_0^{K_{n,\tau_{n+1}}} \zeta(i)\, di\right)$, where $\bar{A} \geq \frac{\zeta(0)}{\zeta(1)}$. Under this alternative assumption, for any firm j and any epoch n, we obtain $A_{n+1}\zeta(i_{j,n+1}) \geq A_{n+1}\zeta(1) \geq A_n\bar{A}\zeta(1) \geq A_n\zeta(0) \geq A_n\zeta(i_{j,n})$. Assuming that it costs the same to plant a tree of vintage $N+1$ or of vintage $n \in (-\infty, N]$, firm j would never find it optimal to plant a tree of a previous vintage. However, this model modification adds complexity but no extra insights, and hence we do not pursue it in the paper for parsimony.

B.4. The Intertemporal Elasticity of Substitution Implied by the Preference Specification (14)

We start by defining the intertemporal elasticity of substitution for two arbitrary times $t_1 < t_2$ as

$$IES \equiv \frac{d\log(c_2/c_1)}{d\log\left(U_{c_1}/U_{c_2}\right)},$$

where c_1 and c_2 denote consumption at times t_1 and t_2, respectively. For the purpose of building intuition, suppose that the representative agent has preferences of the form (14), and consider a deterministically growing consumption path at the rate g (so that $d\log(c_t) = d\log(M_t^C)$). Then a simple computation yields $IES = \frac{1}{\gamma+(\gamma-1)(\alpha-1)}$, and the equilibrium interest rate is given by $\rho + \frac{g}{IES}$. Accordingly, for levels of γ above one, the equilibrium interest rate in an economy where the agent has no external habit formation ($\alpha = 1$) is higher than in an economy where $\alpha < 1$. Similarly, the higher values of IES associated with external habit formation imply that changes in g have a smaller impact on the interest rates.

B.5. Tree-Specific Shocks

As we note in Section III.C of the text, the cross-sectional simulations in Table V assume the presence of idiosyncratic (disembodied) tree-specific shocks.

Besides allowing us to better match the cross-sectional distributions of firm size and valuation ratios, such shocks seem plausible on first principles. Specifically, in the paper we make the assumption that technology is fully embodied in the new trees. However, in reality new technological paradigms also affect the internal organization of firms, their marketing practices, and potentially the way existing technologies are used in the production process. Hence, the arrival of a new epoch may affect the profitability of *existing* trees. To account for this possibility, we allow for the presence of tree-specific shocks $Z(i,t)$, so that the time-t output of tree $i \in [0,1]$ that is planted at time s in epoch N is given by $A_N \zeta(i) Z(i,t) \theta_t$. The shock $Z(i,t)$ is equal to one at the time s that the tree is planted (i.e., $Z(i,s) = 1$), stays constant within each epoch (i.e., $Z(i,t) = Z(i,\tau_N),\ t \in [\tau_N, \tau_{N+1})$), and jumps between epochs so that $Z(i,\tau_{N+1}) = Z(i,\tau_N) u(i,\tau_{N+1})$, where $u(i,\tau_{N+1})$ is i.i.d. across trees and epochs, distributed lognormally with mean one and variance $\sigma^2_{u(\tau_{N+1})}$, and independent of all other shocks in the model.

As we explain in the text, by their construction the idiosyncratic shocks $Z(i,t)$ do not affect a firm's optimal stopping problem, the stochastic discount factor, or any other aggregate quantity. Hence, they do not affect any of the conclusions of the paper. They simply add more variability to the stationary cross-sectional distribution of the size and book-to-market ratios, so as to allow us to match these distributions more accurately. With this goal in mind, we choose $\sigma_{u(\tau_{N+1})} = 2$, thus approximately matching the deciles of each of the two distributions.[5]

B.6. Investment-Related Statistics

In this section of the Internet Appendix, we report additional investment-related statistics. To match model-implied statistics to National Income and Products Accounting (NIPA), we start by defining Gross Domestic Product (GDP) in the sense of NIPA within our model. Over the period of a year, aggregate output (in units of the numeraire good, namely, consumption) is simply given by the total added value in the consumption sector and the investment sector of the economy. Given that investment goods have a price of q_s in units of the numeraire, GDP is thus given by

$$GDP_{t+1} = \underbrace{\int_t^{t+1} C_s ds}_{\text{Consumption}} + \underbrace{\int_t^{t+1} q_s dK_{N,s}}_{\text{Investment}}. \tag{IA.66}$$

Equation (IA.66) states the familiar fact that in the absence of a government and an external sector, GDP is equal to consumption plus investment. Table IA.I tabulates statistics of the investment-to-GDP ratio in both the model and the data. Since the model abstracts from both government expenditure and net exports, we compute the investment-to-GDP ratio (in the data) as private investment divided by the sum of the consumption of nondurables and services plus private investment.

Table IA.I shows that the model implies an investment-to-GDP ratio that is on average lower and has a wider range of values than in the data. (Since investment in the model is skewed,

[5]A technical condition to ensure stationarity of the cross-sectional size distribution is $lim_{N\to\infty} \sigma^2_{u(\tau_{N+1})} = 0$. In the simulations we enforce this condition by simply assuming that the idiosyncratic shocks have constant variance σ_u^2 for M epochs after the tree is planted and zero variance thereafter. We choose σ_u^2 and M as free parameters to match as closely as possible the 20 cross-sectional moments of the size and book-to-market distributions. Specifically, we choose $\sigma_u = 2$ and $M = 2$.

Table IA.I: Investment-Related Statistics

The data are annual and time integrated. Source: Bureau of Economic Analysis. Years: 1929-2010.

	Data	Model
Mean of $\frac{\text{Investment}}{\text{Consumption+Investment}}$	19.9%	12.4%
First decile of $\frac{\text{Investment}}{\text{Consumption+Investment}}$	10.8%	0%
Ninth decile of $\frac{\text{Investment}}{\text{Consumption+Investment}}$	25.0%	40.8%

we choose to report the 10th and 90th percentiles, so as to give a better picture of both the variability and the skewness of the investment-to-GDP ratio). Given our highly stylized modeling of investment, we consider the model's ability to reproduce the properties of the investment-to-GDP ratio as satisfactory.

In conclusion, it is useful to note that our modeling of investment is intentionally stylized, since we want to isolate and highlight only one motivation for investing, namely, the adoption of new technological vintages. If one were to set the additional goal to exactly match all business cycle statistics, the performance of the model could be further improved by introducing additional motivations for investment (replacing depreciated capital, adjusting the scale of existing investments, etc.), while attenuating the motivation we isolate and highlight in the paper. However, such an extension is outside the scope of the current paper. For our purposes, it suffices that our general-equilibrium framework reproduces (qualitatively and quantitatively) the joint time-series properties of consumption and returns, which form the focus of our analysis.

C. The Covariance between Current Excess Returns and Long-Run Consumption Growth: Some Further Details

In this section we provide additional details on the covariance between excess returns and subsequent consumption growth from the perspective of alternative models. We structure this section as follows.

First, we postulate a decomposition of log-consumption into a stochastic trend and a stochastic cycle. This decomposition encompasses both our model and the model by Bansal and Yaron (2004) as special cases, and it serves two purposes: a) it facilitates some of the derivations that follow in Sections C.2 and C.3, and b) it helps us relate our (endogeneous) consumption process to the one assumed by Bansal and Yaron (2004).

Second,we use the decomposition postulated in Section C.1 to derive the covariance between current excess returns and subsequent consumption growth in the models by Campbell and Cochrane (1999) (Section C.2) and Bansal and Yaron (2004) (Section C.3).

C.1. Preliminaries: A Trend-Cycle Decomposition of Log Consumption

For the derivations that follow, we use the following Beveridge and Nelson (1981) decomposition of log-consumption:

$$\log c_t = T_t + \widetilde{x}_t \tag{IA.67}$$

$$T_{t+1} = T_t + \mu + \xi_{t+1} \tag{IA.68}$$

$$g_{t+1} = \log c_{t+1} - \log c_t = \mu + \xi_{t+1} + \widetilde{x}_{t+1} - \widetilde{x}_t, \tag{IA.69}$$

where T_t is the stochastic trend component of consumption, $\widetilde{x}_t$ is a stationary process capturing the economic cycle, and $E_t(\xi_{t+1}) = 0$. The decomposition (IA.67)–(IA.69) is useful because it is general enough to encompass many models in the literature. For instance, the basic random walk specification for log-consumption as assumed by Campbell and Cochrane (1999) is a special case of (IA.67)–(IA.69) with $\widetilde{x}_t = 0$. Our own consumption process also allows such a trend-cycle decomposition. Finally, the model by Bansal and Yaron (2004) can be easily written in the form (IA.67)–(IA.69). To see how, assume that $\widetilde{x}_t$ follows an AR(1) process

$$\widetilde{x}_{t+1} \approx \rho_x \widetilde{x}_t + \varepsilon_{t+1}, \tag{IA.70}$$

where $\rho_x \in (0,1)$ and $E_t(\varepsilon_{t+1}) = 0$. Defining $x^*_{t+1} \equiv (\rho_x - 1)\widetilde{x}_{t+1}$, $\sigma\eta_{t+1} \equiv \xi_{t+1} + \varepsilon_{t+1}$, and $\psi_e \sigma e_{t+1} \equiv (\rho_x - 1)\varepsilon_{t+1}$, and using these definitions inside (IA.67)–(IA.69) results exactly in the Bansal-Yaron specification

$$\begin{aligned} g_{t+1} &= \mu + x^*_t + \sigma\eta_{t+1}, \qquad \text{(IA.71)} \\ x^*_{t+1} &= \rho_x x^*_t + \psi_e \sigma e_{t+1}. \end{aligned}$$

The constants σ and ψ_e refer to notation used in Bansal and Yaron (2004) and the shocks η_{t+1} and e_{t+1} are normalized to have unitary variance. Bansal and Yaron (2004) additionally assume that $cov(\eta_t, e_t) = 0$.

For future reference, we next prove that the model of Bansal and Yaron (2004) implies that the correlation between shocks to the stochastic trend (ξ_t) and shocks to the cycle (ε_t) are negative.

LEMMA 4 *If $\widetilde{x}_t$ follows an AR(1) process of the form (IA.70) and $cov(\eta_t, e_t) = 0$, then the correlation $\rho_{\xi,\varepsilon} \equiv corr(\xi_t, \varepsilon_t)$ is negative. More precisely,*

$$\rho_{\xi,\varepsilon} = -\frac{\sigma_\varepsilon}{\sigma_\xi}, \tag{IA.72}$$

where σ_ξ and σ_ε denote the standard deviations of the shocks ξ_t and ε_t, respectively.

Proof of Lemma 4. By the definitions of η_t and e_t we obtain

$$\begin{aligned} 0 \quad = \quad cov(\eta_t, e_t) \quad &= \quad \frac{1}{\sigma\psi_e\sigma} cov(\xi_t + \varepsilon_t, (\rho_x - 1)\varepsilon_t) \\ &= \quad \frac{(\rho_x - 1)}{\sigma^2\psi_e}\left[\rho_{\xi,\varepsilon}\sigma_\varepsilon\sigma_\xi + \sigma_\varepsilon^2\right], \end{aligned}$$

which gives (IA.72). ■

Similar to Bansal and Yaron (2004), our process for the stochastic cycle is persistent. However, in our model the dynamics of the stochastic cycle arise endogenously and follow more complicated dynamics than the AR(1) dynamics postulated by Bansal and Yaron (2004).

To summarize, several models in the literature can be thought of as special cases of the decomposition (IA.67)–(IA.69). Besides allowing us to relate different models, our primary motivation for using the decomposition (IA.67)–(IA.69) is that it simplifies the computations that follow in the next two subsections.

C.2. The Covariance between Excess Returns and Subsequent Consumption Growth in Campbell and Cochrane (1999)

The key mechanism in Campbell and Cochrane (1999) is that current excess returns are inversely related to the "surplus ratio," that is, the ratio of current consumption over a moving, smooth average of past consumption. Specifically, the excess return r_{t+1}^e can be expressed as

$$r_{t+1}^e = -\beta s_t + u_{t+1}, \tag{IA.73}$$

where $E_t(u_{t+1}) = 0$ and $\beta \geq 0$. The process for the surplus s_t is a carefully designed process that reflects exclusively past consumption-growth shocks, so that excess returns are time varying, but interest rates are not. For our purposes, we can consider more general specifications of the surplus ratio by allowing it to take any form

$$s_t = \sum_{j=0}^{\infty} \phi_j g_{t-j}. \tag{IA.74}$$

As Wachter (2006) and Yu (2007) discuss, the special case $\phi_j = \phi^j$ for some positive $\phi < 1$ provides a good approximation to the model by Campbell and Cochrane (1999). To allow for more generality, while capturing the idea that s_t is persistent and "countercyclical" (in the sense of Campbell and Cochrane (1999)), we simply require that $\phi_j \geq 0$ for all j and $\sum_{j=0}^{\infty} \phi_j < \infty$.

Clearly, since Campbell and Cochrane (1999) assume that log-consumption growth is i.i.d., their model implies that $cov\left(r_t^e, g_{t+k}\right) = 0$ for any $k \geq 1$. Therefore, the baseline model of Campbell and Cochrane (1999) implies that $cov\left(r_t^e, \sum_{k=1}^{T} g_{t+k}\right) = 0$ for any $T > 1$ and cannot explain the upward-sloping of covariances in Figure 3 of the paper.

However, even with a stochastic cycle of the form (IA.70), we obtain the following result.

LEMMA 5 *If the excess return r_{t+1}^e is given by (IA.73), s_t is given by (IA.74), $cov\left(u_t, \varepsilon_t\right) > 0$, and $1 + \rho_{\xi,\varepsilon} \frac{\sigma_\xi}{\sigma_\varepsilon}\left(1 + \rho_x\right) \leq 0$, then*

$$cov\left(r_t^e, g_{t+k}\right) \leq 0 \tag{IA.75}$$

for any $k \geq 1$.

Proof of Lemma 5. Combining (IA.73) and (IA.70) gives, for any $k \geq 1$,

$$\begin{aligned}
cov\left(r_{t,}^e g_{t+k}\right) &= cov\left(-\beta s_{t-1} + u_t, \xi_{t+k} + \widetilde{x}_{t+k} - \widetilde{x}_{t+k-1}\right) \\
&= cov\left(-\beta s_{t-1} + u_t, \left(\rho_x - 1\right) \widetilde{x}_{t+k-1}\right) \\
&= cov\left(-\beta s_{t-1} + u_t, \left(\rho_x - 1\right) \rho_x^{k-1}\left(\rho_x \widetilde{x}_{t-1} + \varepsilon_t\right)\right) \\
&= \left(\rho_x - 1\right) \rho_x^{k-1} cov\left(u_t, \varepsilon_t\right) + \left(1 - \rho_x\right) \rho_x^k \beta cov\left(s_{t-1}, \widetilde{x}_{t-1}\right).
\end{aligned} \tag{IA.76}$$

Clearly, the assumption that $cov\left(u_t, \varepsilon_t\right) > 0$ implies that the first term in (IA.76) is negative since $\rho_x < 1$. To show that the second term in (IA.76) is negative, we let $var\left(x_t\right) = \sigma_x^2$ and we compute

$$\begin{aligned}
cov\left(s_{t-1}, \widetilde{x}_{t-1}\right) &= \sum_{j=1}^{\infty} \phi_{j-1} cov\left(\xi_{t-j} + \left(\rho_x - 1\right) \widetilde{x}_{t-j-1} + \varepsilon_{t-j}, \widetilde{x}_{t-1}\right) \\
&= \sum_{j=1}^{\infty} \phi_{j-1}\left[cov\left(\xi_{t-j} + \left(\rho_x - 1\right) x_{t-j-1} + \varepsilon_{t-j}, \sum_{i=1}^{j} \rho_x^{i-1} \varepsilon_{t-i} + \rho_x^j x_{t-j-1}\right)\right] \\
&= \sum_{j=1}^{\infty} \phi_{j-1}\left[\left(\rho_x - 1\right) \rho_x^j \sigma_x^2 + \rho_x^{j-1} cov\left(\xi_{t-j} + \varepsilon_{t-j}, \varepsilon_{t-j}\right)\right] \\
&= \sum_{j=1}^{\infty}\left(\rho_x^{j-1} \phi_{j-1}\right)\left[\left(\rho_x^2 - \rho_x\right) \sigma_x^2 + \sigma_x^2\left(1 - \rho_x^2\right) + cov\left(\xi_{t-j}, \varepsilon_{t-j}\right)\right] \\
&= \sum_{j=1}^{\infty}\left(\rho_x^{j-1} \phi_{j-1}\right)\left[\sigma_x^2\left(1 - \rho_x\right) + cov\left(\xi_{t-j}, \varepsilon_{t-j}\right)\right] \\
&= \sigma_x^2\left(1 - \rho_x\right) \sum_{j=1}^{\infty}\left(\rho_x^{j-1} \phi_{j-1}\right)\left[1 + \rho_{\xi,\varepsilon} \frac{\sigma_\xi}{\sigma_\varepsilon}\left(1 + \rho_x\right)\right].
\end{aligned}$$

Given $1+\rho_{\xi,\varepsilon}\frac{\sigma_\xi}{\sigma_\varepsilon}(1+\rho_x) \leq 0$, it follows that $cov\left(s_{t-1}, \widetilde{x}_{t-1}\right) \leq 0$, and hence (IA.76) implies (IA.75).
■

Lemma 5 asserts that the covariance between the current excess return and consumption growth at some future point is negative, as long as two conditions are met. The first condition is that innovations to excess returns are positively correlated with innovations in the cycle. This requirement is empirically desirable,[6] so it is plausible to work out the implications of the model under the assumption that it holds. The second condition states that $1+\rho_{\xi,\varepsilon}\frac{\sigma_\xi}{\sigma_\varepsilon}(1+\rho_x) \leq 0$. This condition is easy to satisfy if the standard deviation of trend-shocks (ξ_t) is larger than the standard deviation of cycle-shocks (ε_t), or the correlation between ξ_t-shocks and ε_t-shocks is sufficiently negative. For instance, $1+\rho_{\xi,\varepsilon}\frac{\sigma_\xi}{\sigma_\varepsilon}(1+\rho_x)$ is automatically negative in the model specification of Bansal and Yaron (2004), in light of Lemma 4.

To summarize, in this section we consider an external habit formation model in the spirit of Campbell and Cochrane (1999) appropriately extended to allow for predictable consumption growth as in Bansal and Yaron (2004). We show that any external-habit specification associated with positively correlated innovations between cycle and returns would also lead to negative covariances between current excess returns and future consumption growth. Accordingly, such a model would not be able to reproduce the pattern of covariances in Figure 3 of the paper.

C.3. The Covariance between Excess Returns and Subsequent Consumption Growth in Bansal and Yaron (2004)

In this section, we show that the Bansal and Yaron (2004) model implies a positive covariance between the excess return at time t and consumption growth at some $t+k$ for $k \geq 1$.

To save notation, we simply consider a Bansal and Yaron (2004) model without any shocks to volatility. None of our conclusions depends on this simplifying assumption, as long as we retain the assumption of Bansal and Yaron (2004) that shocks to the cycle and shocks to stochastic volatility are independent.

Specifically, up to a first-order approximation, the excess return on a dividend claim in the model of Bansal and Yaron (without stochastic voalility) is given by

$$r_t^e = \bar{r} + \kappa_{1m} A_{1m} \psi_e \sigma e_t + \psi_d \sigma u_t^d, \tag{IA.77}$$

where u_t^d is not correlated with any other shocks in the model, and the constants $\bar{r}, \kappa_{1m}, A_{1m}, \psi_e, \psi_d$, and σ refer to the notation used in their paper. Equation (IA.77) and the assumed dynamics (IA.71)

[6] In the data the price-to-dividend ratio is low when consumption is below its stochastic trend (low values of $\widetilde{x}_t$), and high when consumption is above its stochastic trend (high values of $\widetilde{x}_t$). This implies that innovations to returns (which mirror movements in the price-to-dividend ratio over short frequencies of the data, such as a quarter) are positively related to innovations in $\widetilde{x}_t$.

imply

$$
\begin{aligned}
cov\left(r_t^e, g_{t+k}\right) &= \kappa_{1m} A_{1m} \psi_e \sigma cov\left(e_t, x_{t+k}^*\right) \\
&= \kappa_{1m} A_{1m} \psi_e \sigma cov\left(e_t, \rho_x^k x_t^*\right) \\
&= \rho_x^k \kappa_{1m} A_{1m} \left(\psi_e \sigma\right)^2 > 0.
\end{aligned} \tag{IA.78}
$$

Even though the Bansal and Yaron (2004) model leads to a positive covariance between current excess returns and subsequent consumption growth, and hence can explain the pattern in Figure 3 of the paper, inspection of (IA.78) reveals that this covariance between current excess returns and subsequent consumption growth in Bansal and Yaron (2004) is driven exclusively by the innovations (e_t) to the excess return, rather than the expected component of the excess return.

D. Covariance Decomposition along the Lines of Equation (24) in the Paper

Table IA.II gives the model-implied covariance of expected excess returns with subsequent consumption growth inside the model (the first term on the right-hand side of equation (24)). In computing the expected excess returns inside the model, we use the two underlying state variables as instruments (i.e., $\frac{\theta_t}{M_t}$ and $\frac{M_t}{M_{\tau_N}}$). We note that, since all endogenous valuation ratios, interest rates, investment, the economic cycle, etc, are combinations of these two variables, the results remain virtually identical when we use combinations of two of the aforementioned endogenous variables as predictive instruments. To take into account small-sample estimation issues, we draw 1,000 samples of 240-quarter-long paths, and report in the first row of the table the average value of the resulting covariance (by horizon T).

We also report in the second and third rows the respective quantities in the data using the instruments in Panels A and B of Table IV in the paper. The bottom part of the table (fourth through sixth rows) reports the results of the top three rows as fractions of the overall covariance between excess returns and subsequent consumption growth (the left-hand side of equation (24)). Specifically, the fourth row expresses the covariance of expected excess returns with subsequent consumption growth as a fraction of the overall model-implied covariance between excess returns and subsequent consumption growth, as depicted in Figure 3 in the paper. Similarly, rows five and six report the results in rows two and three as fractions of the overall covariance between excess returns and subsequent consumption growth in the data. These last two rows correspond to the respective rows denoted "Exp. component" in Table IV in the paper.

Table IA.II: Covariance Decompositions: Data and Model

Horizon (T)	1	2	3	7	11	15	19
Cov. - Model ($\times 10^{-4}$)	0.113	0.222	0.326	0.698	1.017	1.246	1.409
Cov. - Data - IV of Panel A ($\times 10^{-4}$)	0.291	0.622	0.964	2.092	2.457	2.503	2.925
Cov. - Data - IV of Panel B ($\times 10^{-4}$)	0.196	0.450	0.722	1.523	2.140	2.495	3.107
Cov. - Model (fraction)	0.215	0.240	0.294	0.380	0.411	0.471	0.513
Cov. - Data - IV of Panel A (fraction)	0.296	0.367	0.379	0.878	0.864	0.974	0.853
Cov. - Data - IV of Panel B (fraction)	0.199	0.266	0.284	0.639	0.752	0.971	0.906

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