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Optimal Investment, Growth Options, and Security Returns

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ABSTRACT

As a consequence of optimal investment choices, a firm's assets and growth options change in predictable ways. Using a dynamic model, we show that this imparts predictability to changes in a firm's systematic risk, and its expected return. Simulations show that the model simultaneously reproduces: (i) the time-series relation between the book-to-market ratio and asset returns; (ii) the cross-sectional relation between book-to-market, market value, and return; (iii) contrarian effects at short horizons; (iv) momentum effects at longer horizons; and (v) the inverse relation between interest rates and the market risk premium.

RECENT EMPIRICAL RESEARCH IN FINANCE has focused on regularities in the cross section of expected returns that appear anomalous relative to traditional models. Stock returns are related to book-to-market, and market value.[1] Past returns have also been shown to predict relative performance, through the documented success of contrarian and momentum strategies.[2] Existing explanations for these results are that they are due to behavioral biases or risk premia for omitted state variables.[3]

These competing explanations are difficult to evaluate without models that explicitly tie the characteristics of interest to risks and risk premia. For example, with respect to book-to-market, Lakonishok et al. (1994) argue: "The point here is simple: although the returns to the B/M strategy are impressive, B/M is not a 'clean' variable uniquely associated with eco-

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[1] See Fama and French (1992) for summary evidence.

[2] See Conrad and Kaul (1998) for a recent summary of evidence on this subject.

[3] See Lakonishok, Shleifer, and Vishny (1994) for arguments in favor of behavioral biases and Fama and French (1993) for an interpretation in terms of state variable risks.

nomically interpretable characteristics of the firms." A similar conclusion, though reached by a totally different line of argument, follows from Berk (1995), who argues that the finding of a cross-sectional relationship between average return and such variables as book-to-market is neither surprising nor informative in itself. Given expectations about security payoffs, market value must be correlated with systematic risk across securities. Thus, the size and book-to-market variables must be related to "economically interpretable characteristics of the firms." Additional empirical work can always enlarge our understanding of the relationships between the variables that explain expected returns and those that have more straightforward interpretations. Alternatively, by understanding the role variables such as book-to-market, size, or past returns play in a theoretical model, we can better understand the source of their explanatory power in the data.

In this paper we provide such a model and use it to develop an explanation for the above-mentioned empirical findings based on changes in firms' risks through time. Our model relates these changes in risk to firm-specific variables that empirically explain the cross-sectional variation in expected returns. In the model, firms that perform well tend to be those that have discovered particularly valuable investment opportunities. As they exploit those opportunities, their systematic risk changes. Through simulations we show that, in our model, time-series and cross-sectional regressions of returns on book-to-market and market value yield results that are quantitatively similar to those obtained by Fama and French (1992). Our simulations also reproduce qualitatively a number of other behaviors reported in the literature, such as an inverse relation between risk premia and interest rates, and the profitability of contrarian and momentum strategies at different horizons.

Changes in risk lead to a role for book-to-market because in our model good news is, on average, associated with lower risk and bad news is associated with higher risk. Firms in the model own two kinds of assets: assets that are in place and currently generating cash flows, and options to make positive net present value (NPV) investments in the future. During each period the cash flows from existing assets may die off, and new investment opportunities are presented to the firm. An investment that has very low systematic risk looks very attractive to the firm, other things equal, and investing in it leads to a large increase in value. As a result, however, the average systematic risk of the firm's cash flows in subsequent periods is lowered, which leads to lower returns on average. Similarly, when the firm loses a particularly low-risk asset, the result is both a drop in its current value and a rise in its average risk. In the model, book-to-market serves as a state variable summarizing the firm's risk relative to the scale of its asset base.

Asset turnover leads to an explanatory role for market value because it alters the relative importance of growth options versus existing assets within the firm. These two sources of value in the model have different levels of systematic risk, and firms with higher market values tend to be firms with larger current assets and cash flows. Thus, market value proxies for the state variable in the model that describes the relative importance of existing assets and growth options.

The changes in the firm's asset portfolio over its life cycle, and the interaction of these changes with interest rates lead to time-series behaviors that conform, at least qualitatively, to empirical findings in the literature. In the model, expected returns in a given period are positively related to lagged expected returns because the composition and systematic risk of the firm's assets are persistent, and they are negatively related to lagged realized returns because shocks to the composition of the firm's assets are negatively correlated with changes in systematic risk. This leads to momentum effects at longer horizons and contrarian effects at short horizons. At an aggregate level, the time series of portfolio expected returns shows positive correlation with book-to-market (as in Kothari and Shanken (1997) and Pontiff and Schall (1998)), and excess returns are negatively related to interest rates (as in, e.g., Breen, Glosten, and Jagannathan (1989)).

We use our model to explore the joint dynamics of the cross section of returns and firm characteristics in two ways. First, we derive analytical expressions for the firm's value and conditional expected return in terms of fundamental state variables in the model, and also in terms of book-to-market and market value. These expressions allow us to evaluate directly how, within the model, these variables proxy for the underlying economic characteristics of the firm which determine differences in expected return. In the model these underlying state variables summarize the systematic risk of existing assets and the relative importance of growth options for the firm. By making explicit the links between the underlying state variables and the observables used in empirical work, the model provides new insights into the role of market value and book-to-market. For example, we show that when the firm has no growth options, the conditional expected returns depend only on book-to-market. This may seem surprising given the traditional interpretation of book-to-market as summarizing growth potential. The dynamic turnover of firms' assets and the changes in firms' relative systematic risks also depend, through the investment decision, on the dynamic evolution of interest rates. In fact, we show that the present value of growth options can be written as a portfolio of options on pure discount bonds. Our solution for the value of these options allows us to examine the role of interest rates in determining the behavior of risk premia in cross section as well as time series.

Second, the closed-form expressions we obtain permit us to calibrate and simulate realized and conditional expected returns for large panels of assets. The simulated panels are similar in size to samples used in empirical studies based on CRSP and COMPUSTAT data.[4] We calibrate the model's pa-

[4] Such an experiment would be computationally infeasible without closed-form expressions for prices and expected returns because an unmanageably large number of conditional expectations would need to be evaluated numerically for thousands of firms at each point in time.

rameters to macroeconomic and aggregate stock portfolio moments, and then simulate large cross sections of returns. We then ask whether the model reproduces empirically documented regularities by simply replicating the empirical tests from past studies on the simulated data.

Our simulation results suggest that the model has the potential to explain several important features of the data simultaneously. The calibrated model captures most of the Fama and French (1992) regression results in magnitude and direction. It reproduces contrarian and momentum results in direction, although within the calibrated model momentum strategies continue to be profitable at longer horizons than in the data. Whether this model, or similar models, can quantitatively reproduce the full range of results reported in the literature will require additional research. For example, Lakonishok et al. (1994) and Haugen and Baker (1996) argue that the rewards to certain portfolio strategies are simply too large to be usefully understood as a rational response to risk. This paper does not definitively resolve all of these issues. To do so would require calibration, simulation, and then replication of complicated empirical strategies that involve conditioning on many different variables, some of which do not have direct analogues in our model. This paper does, however, suggest that dynamic evolution of systematic risks is both a promising source of explanatory power for understanding these types of phenomena in returns and a methodology for verifying whether this can be done in a quantitatively realistic setting.

This paper is related to two independent branches of the financial economics literature. First, it shares with a number of recent papers, such as Abel (1988), Epstein and Zin (1992), Telmer (1993), and Naik (1994), the goal of enriching the dynamics of the classic asset pricing theories of Merton (1973) and Lucas (1978) with the objective of explaining predictable variations in equity returns through time. However, these models focus less on how firms' relative risks change than on how aggregate risk premia evolve, and they do not explicitly analyze either the corporate investment decision or the importance of distinguishing between assets in place and growth options. Both of these features are central to our model.

A second line of research shares with our paper the recognition that the firm's investment policy involves the exercise of real options. (See Dixit and Pindyck (1993) for a comprehensive review.) The existing literature focuses on the implications this has for aggregate investment behavior. Our model develops its implications for the dynamics of returns and risk across firms.

A few papers in the literature share with ours an emphasis on the link between the dynamic evolution of firms' risks and the cross section of expected returns. Jagannathan and Wang (1996) show that a version of the conditional CAPM, in which betas depend on conditioning information, can empirically account for the explanatory power of variables such as book-to-market in the cross section. Chan (1988) and Ball and Kothari (1989) argue that changing systematic risk can explain the success of contrarian strategies. Bossaerts and Green (1989) show that shocks to the idiosyncratic portion of the firm's exogenous cash flow alter the systematic risk and expected return, and these changes persist. Kadiyala (1997) argues that the information contained in book-to-market should interact with both interest rates and changes in systematic risk, and analyzes the empirical evidence for these interactions. Our model produces expressions for conditional expected returns that involve all of these quantities, allowing us to evaluate their interactions explicitly. It illustrates how changes in systematic risk follow from value maximizing choices by firms. Finally, as in Berk (1995), market value and book-to-market in our model serve as proxies for more fundamental sources of differences in expected return.

Our paper is different in two respects, however. First, it provides independent roles for these two variables, both of which have been shown to be empirically important. Second, as mentioned earlier, our model reveals the more fundamental sources of variation in expected return. This makes it possible, for example, to simulate the model's outcomes and evaluate quantitatively its ability to match the empirical results. This would be impossible to do using Berk (1995), since no model of expected returns is specified, or required, to make the arguments that are the focus of that paper.

A number of papers in the literature offer explanations for various empirical results that our model explains, but the existing explanations involve behavioral biases or omitted variables.[5] The differences that distinguish our approach from previous explanations in the literature are twofold. First, rather than simply considering one effect in isolation, we use a single economic explanation—dynamic turnover of the firm's assets—to address a range of empirical regularities. Although these regularities do occur simultaneously in the data, it may not be obvious how they are related. Second, by undertaking a simulation study we can make *quantitative* assessments about how well the effects are explained by our theoretical analysis. This allows an objective assessment of the question of whether the effects identified in this paper are economically significant enough to explain the empirical results.

The paper is organized as follows. In the next section we set forth the notation, assumptions, and structure of the model. In Section II we discuss the valuation of the firm's cash flows. Section III presents our expressions for expected returns and a discussion of their properties. In Section IV we present results on the model's ability to capture important features of the cross-sectional and time-series behavior of stock returns. The paper concludes with Section V. Proofs of various lemmas used in our derivations are contained in Appendixes A and B. Details of our simulation study are in Appendix C.

I. The Model

The fundamental object of interest in our analysis is the individual firm. The random evolution of the firm's collection of projects determines how its risk and expected return change over time. Ours is a partial equilibrium

[5] As noted previously, examples are behavioral biases, as in Lakonishok et al. (1994), and omitted state variables, as in Fama and French (1993).

model. We take the process describing the pricing kernel to be exogenous, and we do not clear markets to ensure that the distribution of the pricing kernel is consistent with aggregated cash flows consequent to the investment decisions made at the firm level. This, however, gives us the tractability we need to focus on the dynamics for the relative risks of individual firms.

A. Technology

The firm operates with an infinite horizon in discrete time. At each date, $t \in \{0,1,2,\ldots\}$, a project becomes available. Undertaking the project requires an immediate investment that cannot be postponed or preserved. The cash flows the project generates in future periods have a distribution governed by several parameters. Some of these are constant across projects, and can be viewed as fixed features of the firm's technology. Others are specific to the project, and become known to the firm only at the date the project becomes available. The investment required to undertake a project is denoted I. At date t, the cash flows from a project that was undertaken at date $j < t$ are given by the process

$$C_j(t) = I \exp[\bar{C} - \tfrac{1}{2}\sigma_j^2 + \sigma_j \epsilon_j(t)], \tag{1}$$

where the innovations $\{\epsilon_j(t), t > j\}$ are serially independent standard normals. The parameter $\bar{C}$ controls the mean of the cash flow, and σ_j governs the variance.

Projects may become obsolete. To model this, we define a set of indicator variables, $\{\chi_j(t), t \geq j\}_{j=0}^{\infty}$, which equal 1 if the project that arrived at time j is alive at time t, and zero otherwise. We assume that

$$\chi_j(t+1) = \chi_j(t) Y_j(t+1), \tag{2}$$

where $Y_j(t+1)$ equals one with probability π and equals zero with probability $1-\pi$. The elements of $\{Y_j(t), t > j\}_{j=0}^{\infty}$ are independent of each other and of all other random variables in the model.[6] We start the process $\chi_j(t)$ by setting $\chi_j(j) = 1$ if the project that arrives at date j is actually taken on; $\chi_j(j) = 0$ otherwise.

B. Pricing Kernel and Interest Rates

To value the firm's future cash flows we employ a simple and tractable structure for the process governing the evolution of the pricing kernel. The

[6] In the expressions we derive for the firm's value and expected return, this structure is equivalent to assuming that projects' expected cash flows depreciate at rate Δ, with $\pi = e^{-\Delta}$, rather than randomly terminating. The other moments of the distribution of returns depend on this choice, however. For example, the variability of returns is higher if projects die suddenly than if they depreciate gradually.

price at date t of a random cash flow $C(T)$ delivered at date $T > t$ is assumed to be given by $E_t\{(z(T)/z(t))C(T)\}$, where

$$z(t+1) = z(t)\exp[-r(t) - \tfrac{1}{2}\sigma_z^2 - \sigma_z \nu(t+1)], \tag{3}$$

with $z(0) = 1$, and $\{\nu(t)\}_{t=0}^{\infty}$ serially independent standard normals. The dynamics of $r(t)$, the one-period, riskless, continuously compounded interest rate, are assumed to be those from the Vasicek (1977) model. That is,

$$r(t+1) = \kappa r(t) + (1-\kappa)\bar{r} + \sigma_r \xi(t+1), \tag{4}$$

where $\{\xi(t)\}_{t=0}^{\infty}$ are serially independent standard normal variates, and $0 < \kappa < 1$. The long-run mean of the short rate is given by $\bar{r}$, the rate with which it reverts to this mean is given by κ, and its volatility is given by σ_r. Let $\beta_{zr} \equiv \sigma_r \sigma_z \operatorname{cov}(\nu(t), \xi(t))$. This parameter influences the relative pricing of bonds in the model.

Lemma B.3 in Appendix B shows that the price at time t of a discount bond that matures at date s is given by

$$B[s-t, r(t)] = \exp[-\alpha_1(s-t)r(t) - \psi(s-t)], \tag{5}$$

where, for $n > 0$, $\alpha_1(n)$ and $\psi(n)$ are functions that are recursively defined in Appendix B (equations (B1) and (B3)).

C. Risk

The "systematic risk" of a project's cash flows, which we refer to as its "beta," is given by

$$\beta_j \equiv \sigma_j \sigma_z \operatorname{cov}(\epsilon_j(t), \nu(t)). \tag{6}$$

Note from equation (3) that a higher beta implies more negative correlation of cash flows with the pricing kernel, and thus, "more risk." This variable plays a role in our model analogous to that played by the market beta in the mean-variance CAPM, but they are not identical. In any asset pricing model that precludes arbitrage, there exists a positive process $\{m_{t+1}\}_{t=1}^{\infty}$ such that for any return, R_{t+1}, the following identity holds:

$$E_t(R_{t+1}) = e^{r(t)}[1 - \operatorname{cov}_t(m_{t+1}, R_{t+1})], \tag{7}$$

where $e^{r(t)}$ is the gross return on the riskless asset. Under the mean-variance CAPM, or conditional versions of it, m_{t+1} is a decreasing, linear function of the market return. Thus, $\operatorname{cov}_t(m_{t+1}, R_{t+1})$ is proportional to the market beta. In our model, $m_{t+1} = (z(t+1)/z(t))$. As in the CAPM, as the covariance between the pricing kernel and a given return is more negative the larger β_j is, but this dependence is nonlinear in our model. To see this explicitly, consider a one-period security that can be purchased at t and pays $C_j(t+1)$ at $t+1$, which is given by the process in equation (1). The value of this security is

$$E_t\left\{\frac{z(t+1)}{z(t)}C_j(t+1)\right\} = I\exp\{\bar{C} - \beta_j - r(t)\}. \tag{8}$$

It follows from the above equation that the covariance of the return on the above claim with the pricing kernel is

$$\operatorname{cov}_t(m_{t+1}, R_{t+1}) = \frac{Ie^{\bar{C}-r(t)}(e^{-\beta_j}-1)}{Ie^{\bar{C}-\beta_j-r(t)}} = 1 - e^{\beta_j}, \tag{9}$$

and the expected return is

$$\frac{Ie^{\bar{C}}}{Ie^{\bar{C}-\beta_j-r(t)}} = e^{r(t)+\beta_j}. \tag{10}$$

Thus, the risk premium of the above security is constant and is directly related to β_j which summarizes the systematic risk of the cash flows, just as the market beta does in the CAPM.

The beta of the project is only revealed to the firm on the date the project arrives, when the firm must decide whether to make the investment. Each β_j is drawn from a common distribution, denoted F_β, and is independent of all other variables in the model. This distribution is a fixed feature of the firm's technology. It is assumed that $E[\exp(-\beta_j)] < \infty$.

II. Valuation

In this section we derive the value of the firm. The development here focuses on those results that highlight the role of the firm's changing systematic risk, as well as the arguments that allow us to price explicitly the firm's growth options. In the interest of brevity, the details of the following derivations are provided in Appendixes A and B.[7]

[7] The propositions and lemmas in Appendixes A and B are proved for a more general version of the model where the required investment, I, follows a stochastic process. Most of the important cross-sectional features of the model can be understood without a stochastic I, so we focus on the simpler case here. Holding the required investment fixed limits the model in two ways. First, once firms reach a steady state, there is no expectation of growth. Second, the risk premium on the component of the firm's return associated with the present value of growth opportunities is relatively low because the only systematic risk it exhibits is interest rate risk. Modeling the required investment as a stochastic process allows the model to accommodate aggregate stochastic growth and different levels of risk for growth options relative to assets in place. This generality is important if the returns are interpreted in nominal terms because it allows for the possibility that some of this stochastic growth results from inflation.

Valuing the firm involves two steps. First, we provide an expression for the value of the cash flows from a given project, which we denote as

$$V_j(t) = E_t\left\{\sum_{s=t+1}^{\infty} \frac{z(s)}{z(t)} C_j(s)\chi_j(s)\right\}. \tag{11}$$

Note that if $\chi_j(t) = 0$, because the project has expired or was never undertaken, then $\chi_j(s) = 0$ for all $s > t$ and $V_j(t)$ must be zero. Next, we calculate the value of the options to invest in such projects in the future. Since the investment decision cannot be postponed, the value of each option on the investment date is just the maximum of the NPV of the project or zero. The value at date t is thus

$$V^*(t) = E_t\left[\sum_{s=t+1}^{\infty} \frac{z(s)}{z(t)} \max[V_s(s) - I, 0]\right]. \tag{12}$$

Finally, the value of the firm at time t, denoted $P(t)$, equals the value of future cash flows of all projects that are still "alive" plus the value of future investment opportunities:

$$P(t) = \sum_{j=0}^{t} V_j(t)\chi_j(t) + V^*(t). \tag{13}$$

A. The Valuation of Assets in Place

Consider, now, the value at time t of cash flows from the project that arrived at date j. Proposition A.1 in Appendix A shows that this can be written as

$$V_j(t) = I\exp[\bar{C} - \beta_j]D[r(t)], \tag{14}$$

where

$$D[r(t)] = \sum_{s=t+1}^{\infty} \pi^{(s-t)} B[s-t, r(t)] \tag{15}$$

is the value of a perpetual, riskless consol bond, where the payments on this bond depreciate at a constant rate $1 - \pi$.

Equation (14) gives the value of a project as the risk-adjusted discounted value of the perpetual stream of expected cash flows. The risk adjustment replaces the expected cash flow $\pi^{(s-t)}Ie^{\bar{C}}$, with its certainty equivalent $\pi^{(s-t)}Ie^{\bar{C}-\beta_j}$, which is then discounted at the riskless rate.

The value of all the firm's assets in place is given by summing across its projects that are currently "alive." This can be expressed using the "book value," the capital invested in projects that are still alive:[8]

$$b(t) \equiv I \sum_{j=0}^{t} \chi_j(t) \equiv I\, n(t), \tag{16}$$

where $n(t)$ is the number of projects alive. The value of the firm's assets in place is

$$\sum_{j=0}^{t} V_j(t)\chi_j(t) = b(t)e^{\bar{C}-\beta(t)}D[r(t)], \tag{17}$$

where

$$\beta(t) \equiv -\ln\left[\sum_{j=0}^{t} \frac{\chi_j(t)}{n(t)}\, e^{-\beta_j}\right] \tag{18}$$

summarizes the average systematic risk of the firm's existing assets. The beta of the assets-in-place of the firm is obtained by averaging across the betas of individual projects that are currently alive, and it varies through time depending on the optimal exercise of growth options and the obsolescence of existing projects.[9]

B. The Valuation of Growth Options

To value the firm's growth options, consider a particular term in the infinite sum, $V^*(t)$, given in equation (12):

$$E_t\left\{\frac{z(s)}{z(t)}\max[V_s(s) - I,0]\right\}. \tag{19}$$

This is the value at t of the option to invest at date $s > t$. The decision to take on the project at time s depends on its NPV, $V_s(s) - I$. From equation (14) we have

$$V_s(s) - I = I(\exp[\bar{C} - \beta_s]D[r(s)] - 1). \tag{20}$$

The NPV, therefore, depends on two variables that are not known at date t: $r(s)$ and β_s. By conditioning on β_s the problem of valuing the investment option can be reduced to valuing a portfolio of options on interest rates alone. In a single-factor model of interest rates, such as the Vasicek model, where

[8] The valuation formulas are unchanged if we view $1 - \pi$ as a depreciation rate. For that specification, $b(t)$ corresponds very closely to what accountants would call "book value."

[9] Under the CAPM, this averaging would be linear rather than geometric.

prices of all discount bonds are monotonic in the short rate, the above expectation can be evaluated using the procedure Jamshidian (1989) introduces to value an option on a portfolio of discount bonds.

For a given β_s, let $r^*_{\beta_s}$ be defined implicitly as the unique solution to the equation:

$$\exp[\bar{C} - \beta_s] \sum_{k=1}^{\infty} \pi^k B[k, r^*_{\beta_s}] = 1. \tag{21}$$

We can use this expression and equation (20) to compute the following:

$$E_t\left\{ \frac{z(s)}{z(t)} \max[V_s(s) - I, 0] \middle| \beta_s \right\}$$
$$= IE_t\left\{ \frac{z(s)}{z(t)} \max\left[\exp[\bar{C} - \beta_s] \sum_{k=1}^{\infty} \pi^k B[k, r(s)] - 1, 0 \right] \middle| \beta_s \right\} \tag{22}$$
$$= IE_t\left\{ \frac{z(s)}{z(t)} \max\left[\exp[\bar{C} - \beta_s] \sum_{k=1}^{\infty} \pi^k (B[k, r(s)] - B[k, r^*_{\beta_s}]), 0 \right] \middle| \beta_s \right\}. \tag{23}$$

By the definition of $r^*_{\beta_s}$, we know that

$$\exp[\bar{C} - \beta_s] \sum_{k=1}^{\infty} \pi^k (B[k, r(s)] - B[k, r^*_{\beta_s}]) \geq 0 \tag{24}$$

if and only if $r(s) \leq r^*_{\beta_s}$. But by monotonicity of the bond prices, $r(s) \leq r^*_{\beta_s}$ if and only if $B[k, r(s)] - B[k, r^*_{\beta_s}] \geq 0$ for every k. It follows that

$$\begin{aligned} &\max\left\{ \exp[\bar{C} - \beta_s] \sum_{k=1}^{\infty} \pi^k (B[k, r(s)] - B[k, r^*_{\beta_s}]), 0 \right\} \\ &\quad = \exp[\bar{C} - \beta_s] \sum_{k=1}^{\infty} \max\{B[k, r(s)] - B[k, r^*_{\beta_s}], 0\}, \end{aligned} \tag{25}$$

and so the option on the portfolio is expressed as a portfolio of European options. All the options have the same exercise date s. Each is a call on a pure discount bond maturing at a different date, $s + k$, with strike price $B[k, r^*_{\beta_s}]$. Substituting equation (25) into equation (23) provides

$$E_t\left\{ \frac{z(s)}{z(t)} \max[V_s(s) - I, 0] \middle| \beta_s \right\}$$
$$= IE_t\left\{ \frac{z(s)}{z(t)} \exp[\bar{C} - \beta_s] \sum_{k=1}^{\infty} \max\{B[k, r(s)] - B[k, r^*_{\beta_s}], 0\} \middle| \beta_s \right\}. \tag{26}$$

Let

$$J[r(t), s-t, k, r^*_{\beta_s}] \equiv E_t\left\{\frac{z(s)}{z(t)} \max[B[k, r(s)] - B[k, r^*_{\beta_s}], 0] \middle| \beta_s\right\} \tag{27}$$

denote the value of each option at time t. This value can be calculated in closed form under the Vasicek model. Lemma B.5 in Appendix B yields

$$J[r(t), s-t, k, r^*_{\beta_s}] = B[s-t+k, r(t)]N(d_1) - B[k, r^*_{\beta_s}]B[s-t, r(t)]N(d_2), \tag{28}$$

where $B[\cdot,\cdot]$ is given by equation (5), $N(\cdot)$ denotes the cumulative standard normal distribution function, and

$$d_1 = \frac{\ln B[s-t+k, r(t)] - \ln B[s-t, r(t)] - \ln B[k, r^*_{\beta_s}] + \frac{1}{2}\,\alpha_1^2(k)\sigma_2^2(s-t)}{\alpha_1(k)\sigma_2(s-t)}, \tag{29}$$

$$d_2 = d_1 - \alpha_1(k)\sigma_2(s-t), \tag{30}$$

with $\alpha_1(\cdot)$ and $\sigma_2(\cdot)$ as defined in equation (B1).

To obtain the unconditional present value, we integrate over β_s in equation (26). We can then sum across future investment dates to obtain the present value of all growth opportunities:

$$\begin{aligned} V^*(t) &= Ie^{\bar{C}} \sum_{s=t+1}^{\infty} \sum_{k=1}^{\infty} \pi^k \int_{\mathcal{B}} J[r(t), s-t, k, r^*_{\beta_s}] e^{-\beta_s}\, dF_\beta(\beta_s) \\ &\equiv Ie^{\bar{C}} J^*[r(t)]. \end{aligned} \tag{31}$$

Here $J^*[r(t)]$ is defined implicitly by the above identity as the per unit present value of all future growth options.

C. The Value of the Firm

Combining the value of assets in place and the value of growth options gives the value of the firm:

$$P(t) = b(t)e^{\bar{C}-\beta(t)}D[r(t)] + Ie^{\bar{C}}J^*[r(t)]. \tag{32}$$

Equation (32) is the analog, in an uncertain environment, of the pricing model under certainty that is often used as a standard introduction to corporate finance (see, e.g., Ross, Westerfield, and Jaffe (1988), p. 130). The price of a stock is described, in such treatments, as the value of future flows from ongoing projects, valued as a perpetuity, plus the value of growth opportunities. Under certainty, the value of growth opportunities is simply the present value of all future positive NPV projects. When there is no uncertainty, there is no option component.

III. Expected Returns

The objective of this section is to derive expressions for the expected return of the firm that show how empirically observed determinants of returns are related to underlying economic characteristics of the firm. The firm's (gross) return is given by the cash flow plus the future price, divided by the current price. To evaluate the conditional expected return we must obtain expressions for the conditional expectation of the next period's cash flow and the next period's price. Next period's price, in turn, consists of the value in the next period of existing projects, and the value of growth opportunities.

Appendix A derives analytical expressions for each of these conditional expectations in terms of the state variables, the parameters, and the following two quantities:

$$D_e[r(t)] \equiv E_t\{D[r(t+1)]\} \tag{33}$$

and

$$J_e^*[r(t)] \equiv E_t\{J^*[r(t+1)]\}, \tag{34}$$

where $D[r(t)]$ is defined in equation (15), and $J^*[r(t)]$ is the value of growth options defined in equation (31). The first of these is the expectation, given current interest rates, of the value of a depreciating perpetuity next period. It can be calculated in closed form and is given by equation (A20) in Appendix A. The second expression is the expectation, given the current interest rate, of the value next period of the portfolio of bond options embedded in the firm's growth options. It can be written as

$$J_e^*[r(t)] \equiv \sum_{s=t+1}^{\infty} \sum_{k=1}^{\infty} \pi^k \int_{\mathcal{B}} J_e[r(t), s-t, k, r_{\beta_s}^*] e^{-\beta_s} \, dF(\beta_s), \tag{35}$$

where $J_e[r(t), s-t, k, r_{\beta_s}^*]$ is, analogously, the expectation of the value of one of these options next period,

$$J_e[r(t), s-t, k, r_{\beta_s}^*] = E_t\{J[r(t+1), s-t, k, r_{\beta_s}^*] | \beta_s\}. \tag{36}$$

Under the Vasicek model, equation (36) can be calculated in closed form. From Proposition A.3 in Appendix A, it follows that

$$
\begin{aligned}
J_e[r(t), s-t, k, r^*_{\beta_s}] &= e^{r(t)}\Bigg(B[s-t+k, r(t)]\exp\left[-\frac{\beta_{zr}}{(1-\kappa)}\left(1-\kappa^{s-t+k-1}\right)\right]N(d_1^*) \\
&\quad - B[k, r^*_{\beta_s}]B[s-t, r(t)]\exp\left[-\frac{\beta_{zr}}{(1-\kappa)}\left(1-\kappa^{s-t-1}\right)\right]N(d_2^*)\Bigg), \qquad (37)
\end{aligned}
$$

where $B[\cdot,\cdot]$ is given by equation (5), $N(\cdot)$ denotes the cumulative standard normal distribution function, and

$$d_1^* = d_1 - \frac{\beta_{zr}\kappa^{s-t-1}}{\sigma_2(s-t)}, \qquad (38)$$

$$d_2^* = d_1^* - \alpha_1(k)\sigma_2(s-t), \qquad (39)$$

with d_1 as in equation (29) and $\alpha_1(\cdot)$ and $\sigma_2(\cdot)$ as defined in equation (B1).

We can now write, using Proposition A.4 in Appendix A, the conditional expected return on a proportional claim on the firm as

$$E_t[1 + R_{t+1}] = \frac{\pi n(t)[D_e[r(t)]e^{-\beta(t)} + 1] + J_e^*[r(t)]}{n(t)D[r(t)]e^{-\beta(t)} + J^*[r(t)]}, \qquad (40)$$

where $n(t) = b(t)/I$ is the number of ongoing projects. The first term in the numerator of equation (40) measures the expected (gross) return on the firm's assets in place and the second takes account of the firm's growth options. These two components of firm value respond differently to changes in the environment, which can be summarized through three state variables:

- $\beta(t)$, the average systematic risk of the firm's existing assets
- $r(t)$, the current interest rate
- $n(t)$, the number of active projects.

Each of these sources of variation in expected returns changes in predictable ways and imparts predictability to returns. The riskiness of the firm's assets, for instance, tends to revert toward the mean implicit in its investment opportunity set. The interest rate process we use obviously has a predictable component. The number of ongoing projects will be small early in the life of a firm, as well as after periods of high interest rates when the high discount rate will imply that relatively few attractive investments have been available.

Thus, our model captures the ideas that expected returns depend on the firm's life cycle and returns of mature firms behave differently from those of growth firms. Note that, in equation (40), $n(t)$ reflects the importance of existing assets relative to growth opportunities in the firm's value. It therefore determines how much of the expected return is attributable to each component of the value of the firm. When there are no ongoing projects and the value of the firm is all in its growth opportunities,

$$E_t[1 + R_{t+1}]|_{n(t)=0} = \frac{J_e^*[r(t)]}{J^*[r(t)]}. \tag{41}$$

In this situation, the expected return only depends on $r(t)$ and not on $\beta(t)$. As the number of ongoing projects goes to infinity, the growth component in expected return becomes negligible. In this case, the expected return of the firm is

$$\lim_{n(t)\to\infty} E_t[1 + R_{t+1}] = \frac{\pi[e^{\beta(t)} + D_e(r(t))]}{D(r(t))}. \tag{42}$$

This is increasing in $\beta(t)$, which summarizes the firm's systematic risk, relative to other firms with similar opportunities for future growth. Time variation in the firm's systematic risk thus plays an important role in determining expected returns of mature firms, but is less important for growth firms. Since $n(t)$ is also a measure of the physical size (as opposed to market value) of the firm, this model predicts that the expected returns of smaller firms differ from those of larger firms.

From equations (41) and (42), we can also conclude that the expected returns of high growth and low growth firms both depend on the level of interest rates in nonlinear and functionally different ways. For this reason we would expect the magnitude, and possibly the direction, of the premium to size to depend on the level of interest rates. This is consistent with the results in Kadiyala (1997).

If asset returns are consistent with our model and if an empiricist proceeds under the mistaken assumption that firms' expected returns are constant, then she will find that abnormally high returns are followed by abnormally low subsequent returns and vice versa. By inspecting equation (32) it is clear that the price of the firm is decreasing in $\beta(t)$. Consequently, an increase in the riskiness of the firm is associated with a current price decrease and subsequently higher returns. This is consistent with the "contrarian" evidence recently documented by empiricists. (See, e.g., Jegadeesh and Titman (1993).) Portfolios of recent winners underperform portfolios of recent losers. Informal appeals to "changing betas" as an explanation for contrarian strategies have been made in Chan (1988) and Ball and Kothari (1989). Our analysis makes this dependence explicit. Within our model, the returns to contrarian or momentum strategies are not "abnormal." They are compensation for bearing systematic risks that change in predictable ways.

Although equation (40) reveals the state variables driving expected returns in our model, to compute the average systematic risk of the firm's existing assets, $\beta(t)$, one would need to know a great deal about the firm.

This risk is a function of each ongoing project's covariance with the pricing kernel. To measure $\beta(t)$, it is not only necessary to know what the value of this covariance is for each project, it is also necessary to know which projects are ongoing at the time. A useful feature of our model, however, is that the expected returns equation can also be written in terms of observable variables. These variables are precisely those that have received so much attention in the empirical literature on the cross-sectional properties of expected returns: market value and book-to-market. This step allows us to make explicit the relationship between these variables and the underlying, economically interpretable firm characteristics that determine expected returns.

First, note that the risk-adjusted future expected cash flows are the source of any difference between the value, $P(t)$, and the value of the growth opportunities. That is, from equation (32):

$$e^{\bar{C}-\beta(t)}b(t) = \frac{P(t) - Ie^{\bar{C}}J^*[r(t)]}{D[r(t)]}. \tag{43}$$

Substituting this into our expression for expected returns, equation (40), allows us to eliminate the state-variable, $\beta(t)$. The following expression for conditional expected returns is obtained:

$$\begin{aligned} E_t[1 + R_{t+1}] = {} & \frac{\pi D_e(r(t))}{D(r(t))} + \pi e^{\bar{C}}\left[\frac{b(t)}{P(t)}\right] \\ & + Ie^{\bar{C}}\left[J_e^*[r(t)] - J^*[r(t)]\,\frac{\pi D_e(r(t))}{D(r(t))}\right]\left[\frac{1}{P(t)}\right]. \end{aligned} \tag{44}$$

Equation (44) decomposes the expected return into three parts. The first term is the expected rate of appreciation on the declining perpetuity, and it summarizes the effects of changing interest rates on the value of the cash flows produced by the firm's existing assets.

The second term is proportional to book-to-market. As new projects are adopted and old projects expire, this term varies in response to changes in the systematic risk of the firm. This systematic risk changes both because the firm's assets turn over and because the importance of the firm's existing assets relative to growth options varies. Since $\pi e^{\bar{C}}$ is a positive constant, the firm's expected return is positively correlated, in time series, with book-to-market. This is consistent with the empirical evidence reported in Kothari and Shanken (1997) and Pontiff and Schall (1998). Note that since $\pi e^{\bar{C}}b(t)$ is the expected cash flow of ongoing projects, this term can also be interpreted as the earnings-to-price ratio, or (in the case of no retained earnings) the dividend yield. The positive dependence on book-to-market also leads to cross-sectionally higher average returns for firms with high book-to-market ratios across firms that share common technologies. We explore these relations between book-to-market and expected returns further in Section IV through simulation.

The third term reflects the value of the firm's growth options. It changes in response to interest rate movements and the relative importance of growth options in the firm's value. The quantity $J^*[r(t)]$ is the value of a portfolio of bond options. The term $J_e^*[r(t)]$ is the expected value of this portfolio one period hence. Thus, the final term in equation (44) is proportional to the difference between the expected price of the portfolio of bond options and its current price accumulated at the appropriate risk-free rate—the expected return on the declining perpetuity. Hence it can be viewed as a certainty equivalent. This certainty equivalent is multiplied by the scale of investment opportunities, which quantifies the number of such bond options the firm has. Normalizing by the current price expresses this certainty equivalent as a risk premium.

The separate roles for book-to-market and market value in equation (44) derive directly from the dynamic evolution of the firm's assets. To illustrate, note that when a firm consists of just one long-lived project, the last term drops out leaving only the book-to-market ratio. In this case, the book-to-market ratio remains as a determinant of return despite the fact that the firm has no growth opportunities. Because of the existence of the third term in equation (44) it is quite possible for two firms to have identical book-to-market ratios and still have different growth potentials. In light of this, it might be worth questioning the current standard of using the book-to-market ratio as the de facto measure of a firm's growth potential.

Finally, note that the form of equation (44) does not depend on our appeal to the Vasicek (1977) model. A particular model is, of course, required to obtain the specific functional forms of J^*, J_e^*, D, and D_e. To express the relevant functions in terms of a single state variable, the short rate, a single factor process of the interest rate is required. Although a more general (e.g., multifactor) specification of the interest rate process would produce different functional forms for J^*, J_e^*, D, and D_e, many of the qualitative implications of equation (44) can be expected to hold more generally.

A. Constant Interest Rates

We can better evaluate the separate roles of the state variables that determine expected returns by considering the special case of the model that holds interest rates fixed. Any dynamics in the expected returns, in this case, are due entirely to the changing characteristics of the firm's assets.

When interest rates are constant, the value of growth is constant, so the functions J^* and D are constant. Accordingly, the intercept and the coefficients in equation (44) are all constant, which allows us to simplify it to

$$E_t[1 + R_{t+1}] = \pi + \pi e^{\bar{C}} \frac{b(t)}{P(t)} + K\left[\frac{1 - \pi e^{-r}}{1 - e^{-r}}\right]\frac{1}{P(t)}, \tag{45}$$

where $K \equiv IE_t\{\max[e^{\bar{C}-\beta_s}[1/(e^{-r} - 1)] - 1,0]\}$ does not depend on time because the β_s variables are independent and identically distributed.

In this case the expected return is a linear function of the two firm-specific variables that Fama and French (1993) argue effectively summarize the cross section of expected returns. By studying their role in a structural model we can relate these variables to the underlying state variables: $\beta(t)$, which summarizes the relative risk of the firm's existing assets, and $b(t)$, which summarizes their importance relative to the firm's growth opportunities. Further, we see that both book-to-market and market value are important. In our dynamic model, even with interest rates constant, there are two state variables that determine expected returns. This contrasts with the static model in Berk (1995), where one of these variables would be redundant.

B. The Impact of Interest Rates on Expected Returns

Interest rates affect the expected returns attributable to assets in place as well as those attributable to growth opportunities. First, with higher interest rates future cash flows are obviously discounted more heavily, leading to lower prices and higher expected returns. The second effect of higher interest rates is through their influence on current expectations about future values of $\beta(t)$. Fewer projects are undertaken when rates are high, and the rejected projects will be precisely the ones with relatively high systematic risks. This feedback effect represents an empirical prediction that might easily be overlooked in models where the firm's investment decision is simplified. Periods of high interest rates should be followed by lower risk premia in equities, because fewer high-risk investments will have been undertaken.

Interest rate movements also influence expected returns through the value of growth options. The risk premium of an option depends on the value of the underlying assets. Deep in-the-money options have lower risk premia than out-of-the-money options. Through this mechanism the expected excess return of the firm can change even if the asset base of the firm remains unchanged.

IV. Simulation Evidence

The purpose of this section is to evaluate the model's ability to reproduce quantitatively some important features of stock return data in simulation. Our model is nonlinear in the important state variables. As a result, some of the moments of interest, such as the unconditional expected return or the cross-sectional correlation between expected return and market value or book-to-market, are difficult to obtain analytically. We can, however, use the model to generate artificial data, and then calculate statistics similar to those reported by empirical researchers. Since conditional expected returns are observable in the simulation, and since we can generate very long artificial time series, we can determine whether the population moments of the returns generated by our model are, for appropriate parameter choices, similar in sign and magnitude to those measured by empirical researchers.

Some of the qualitative effects we discuss in previous sections involve dynamic effects that are too complex to be obvious from inspection or simple comparative statics. For example, conditional on a project being accepted, its systematic risk is likely to be lower in higher interest rate environments, leading to a complex, dynamic interaction between risk premia and interest rates. Similarly, the model's ability to produce contrarian effects in returns depends on the dynamic interaction between unexpected news about the systematic risk of new or expiring assets and subsequent expected returns. To determine whether these types of dynamic effects are quantitatively important for parameter values that lead to realistic behaviors on other dimensions, simulations are needed.

Our approach is in the spirit of the methodology recommended by Kydland and Prescott (1996). We first calibrate the model using the unconditional moments of interest rates and aggregate stock returns. Then, we compare the model's simulated outcomes to the features present in the data that are of particular interest to our study. These features include the cross-sectional relation between returns and firm-specific variables such as book-to-market and market value, the comovement of risk premium and interest rates in time series, and contrarian and momentum effects. We assume identical technology parameters across firms. All firms are therefore ex ante identical, but shocks to the cash flows, the death process, and β_j are drawn separately across firms. Thus, any cross-sectional effects are attributable to the endogenous evolution of their assets, which is our focus here.

The parameters for the pricing kernel and interest rate processes are calibrated to match sample yield volatilities and term premia from U.S. term structure data. Using the same simulated process for the pricing kernel and interest rate process, we generate 20,000 months of data for 50 individual firms, and examine the average risk premia, volatility, and the degree of predictability of an equally weighted portfolio of these firms' returns. These values are compared to actual empirical estimates of these moments, in order to calibrate those parameters for the firms' technology that do not have obvious relationships to macroeconomic variables.

Having thus calibrated the model, we confront it with the empirical results. First, we run time-series regressions of the equally weighted portfolio of the simulated firms' returns against interest rates and book-to-market. Then we turn to the cross-sectional properties of expected returns produced by the model. We repeatedly simulate panel data sets of returns similar in number of firms and number of months to the data sets used in existing empirical studies of cross sections of returns. On each simulated database, we perform cross-sectional tests similar to those in the literature. Across many simulations (1422), this produces a sampling distribution for the statistics of interest, under the null that our model holds with the calibrated parameter values. Using these sampling distributions, we can ask whether the estimates obtained by existing research could have been generated by our model with high probability. Finally, we evaluate the profitability of momentum and contrarian strategies at different horizons in our model.

A. Calibration of the Pricing Process

We interpret the model as applying to nominal returns, nominal interest rates, and the nominal pricing kernel, all at monthly frequencies,[10] and we use estimates from the term structure literature to obtain calibrated values for parameters governing the interest rate and pricing kernel process.

Backus, Foresi, and Zin (1994) provide monthly sample moments for the continuously compounded zero-coupon yields constructed by McCulloch and Kwon (1993) for 1982 to 1992. We set $\bar{r} = 0.07483/12$, the sample mean of the monthly rate over that period. (This choice is important in determining levels of average returns, but is not important to the behavior of risk premia in the model.) The first-order autocorrelation of the short rate is given by κ. Using the one-month yields from McCulloch and Kwan, this autocorrelation in the data is 0.906. Using the Fama Treasury bill data from CRSP, which covers the longer period from June 1952 to December 1995, the one-month T-bill rate has a sample autocorrelation of 0.969, and this autocorrelation increases with maturity. We set $\kappa = 0.95$, which lies within the range these estimates suggest. In the Vasicek (1977) model, the standard deviation of the yield on a bond with maturity T is given by:

$$\frac{1-\kappa^T}{(1-\kappa)T}\,\frac{\sigma_r}{\sqrt{1-\kappa^2}}. \tag{46}$$

A well-known deficiency of the Vasicek model, however, is its inability to reproduce the term structure of volatilities. Calibrating σ_r using longer maturity instruments produces higher values. Using $T = 1$, for example, and the Fama CRSP one-month T-bill yields with $\kappa = 0.95$ gives $\sigma_r = 0.0007$. Using the fitted 60-month yields for 1946 through 1987, Backus and Zin (1993) estimate the monthly standard deviation as 0.00275, which for our value of κ yields $\sigma_r = 0.0027$. We choose a value that reflects a longer rate, given the nature of the assets we value in the model, and use 0.002 for σ_r.[11]

What remains is the choice of β_{zr}. In our model the limiting yield (or the limiting forward rate) as maturity increases, is given by[12]

$$r_\infty \equiv \bar{r} + \frac{\sigma_r^2}{2}\left[\left(\frac{\beta_{zr}}{\sigma_r^2}\right)^2 - \left(\frac{\beta_{zr}}{\sigma_r^2} + \frac{1}{1-\kappa}\right)^2\right]. \tag{47}$$

The average yield differential between the 10-year zero coupon bond and the one-month T-bill reported by Backus et al. (1994) is 2.319 percent. We choose a value for β_{zr} of -0.00014, which gives a spread between the average short

[10] In the general form of our model, the cash flows from individual projects do not grow, but the investment scale process, $I(t)$, does. Thus, the general form of the model can accommodate growth in cash flows due to inflation at the firm level.

[11] Experiments with small values of σ_r produced returns with very little predictability.

[12] This computation is carried out explicitly, using different notation, in equation (10) of Backus et al. (1994).

Table I

Parameter Values for Simulations

Listed are the values chosen for all parameters required to simulate the model: the long-run mean of the short rate ($\bar{r}$), the rate of mean reversion (κ), the volatility of the short rate (σ_r), the covariance between the pricing kernel and the short rate (β_{zr}), the volatility of the pricing kernel (σ_r), the investment required to initiate a project (I), the log of the mean cash flow yield from a project ($\bar{C}$), the range of the distribution from which cash flow volatility is drawn (σ), and the probability a project's cash flows terminate ($1 - \pi$). The final two values are the probabilities that a new project will be acceptable ($\chi_t(t) = 1$), conditional on the interest rate. These probabilities determine the parameters of the distribution from which project risks are drawn.

Interest Rate Process		Firm's Technology	
Parameter	Value	Parameter	Value
$\bar{r}$	0.006236	I	1.0
κ	0.95	$\bar{C}$	-3.7
σ_r	0.002	σ	$0.3\lvert\bar{C}\rvert$
β_{zr}	-0.00014	π	0.99
σ_z	0.4	$\Pr(\chi_t(t) = 1 \mid r(t) = \bar{r})$	0.05
		$\Pr(\chi_t(t) = 1 \mid r(t) = 0)$	0.10

rate and the limiting yield of 2.4 percent. As a separate parameter, σ_z is not directly identified by the pricing relationships in our model.[13] Thus, the choice of σ_z is somewhat arbitrary, and is only needed to ensure that the parameters we do use are mutually consistent. We set σ_z to 0.4, which implies correlation between the pricing kernel and interest rate processes of -0.175. The first panel of Table I summarizes our choices of parameter values for the interest rate process.

B. Calibration of Technology

A few of the parameters characterizing the firm's technology can be related to simple macroeconomic statistics. The quantity $1 - \pi$ has a natural interpretation as the depreciation rate for the firm. Real business cycle models are generally calibrated with depreciation rates of 8 to 10 percent per year. (See, for example, Kydland and Prescott (1982), and for estimates of such a model see Christiano and Eichenbaum (1992).) We choose 0.99 for π. The model is calibrated to monthly intervals, which implies a depreciation rate slightly above 10 percent per year.

We experimented with the remaining parameters to bring the time-series behavior of portfolio returns into line with estimates of moments for aggregate stock portfolios. Once a project is undertaken, $\pi e^{\bar{C}}$ is an expected rate of cash return. Choosing this rate of return to be about 30 percent per year gives $\bar{C} = \log(0.3/12\pi) \approx -3.7$.

[13] This can be seen by noting that in the bond pricing formula in equation (5) all that is required is the function $\psi(\cdot)$, which, through recursive substitution from equation (B1), can be shown to depend on β_{zr} but not on σ_z. Neither do the formulas we derive for the value of cash flows or growth opportunities involve σ_z.

Table II

Descriptive Statistics for Aggregate Portfolios

The mean (Mean), standard deviation (Std), and excess kurtosis (Ex. Kurtosis) of the return on a one-month bond (Short bond return) and the expected (Portfolio expected return) and realized (Portfolio realized return) monthly returns of an equally weighted portfolio using 20,000 simulated monthly returns on 50 firms are reported under "Monthly Returns." The mean and standard deviation of the compounded monthly returns for each year are reported under "Annual Returns." Predictability of return is defined to be the R^2 of the regression of realized return onto expected return.

	Monthly Returns			Annual Returns	
Variable	Mean	Std.	Ex. Kurtosis	Mean	Std.
Short bond return	0.60%	0.65%	−0.01	7.40%	2.37%
Portfolio expected return	1.25%	0.66%	−0.05	–	–
Portfolio realized return	1.28%	4.12%	3.02	16.43%	16.35%
		R^2 coefficient			
Predictability of return		2.38%			

New projects differ in their systematic risk, which is given by β_j. Each β_j is assumed to have a translated exponential distribution with support $(-\infty, \beta^*]$ and density

$$f(\beta) = \frac{\exp\left(\dfrac{\beta - \beta^*}{\beta^* - \bar{\beta}}\right)}{\beta^* - \bar{\beta}}, \tag{48}$$

for $\bar{\beta} < \beta^*$. This distribution is tractable since truncation preserves its form. It also implies that for any set of parameters, risky, low value projects will be relatively more common. For given values of π and $\bar{C}$, $\bar{\beta}$ and β^* govern project turnover within the firm. We chose them by fixing the conditional probability that a new project has positive NPV at $r(t) = \bar{r}$ (1 in 20) and $r(t) = 0$ (1 in 10). These parameter values are summarized in the bottom panel of Table I. Together, these parameters give a project half-life of approximately $5\frac{2}{3}$ years.

In our simulations we draw σ_j, the standard deviation of the project's shock, from a distribution that is conditional on β_j. Since these two variables determine the correlation between the pricing kernel and the project's cash flow, their quotient must have a bounded support. We assume that the conditional distribution of σ_j is uniform with support $[|\beta_j|/\sigma_z, (|\beta_j|/\sigma_z) + \sigma]$. We set σ equal to 30 percent of the value of $\bar{C}$.

Using these parameters, we simulate 20,000 months of data for 50 firms. We restrict attention to firms that have reached a steady state distribution for the number of ongoing projects by dropping the first 200 observations. Table II contains descriptive statistics for the equally weighted portfolio of these 50 firms and the interest rate, which can be compared with empirical estimates. For example, Ibbotson Associates (1994) report an average annual risk premium of 8.6 percent for the period 1926–1994, and a standard deviation of 20.5 percent for the S&P 500 index.

In the postwar period, however, the standard deviation of annual returns tends to be somewhat lower. Using annual returns for the period 1951–1993 in Ibbotson Associates (1994), the standard deviation of annual returns is 16.76 percent, and Fama and French (1992) report monthly standard deviations of 4.47 percent for the value-weighted index for the period 1963–1990 and of 5.49 percent for the equally weighted index. Campbell, Lo, and MacKinley (1997, Table 1.1) report a monthly average return of the equally weighted portfolio of 1.25 percent with a standard deviation of 5.77 percent for the period 1962–1994. We searched for parameters that produced average risk premia for annual returns in the range of seven to ten percent, and a standard deviation of annual returns between 13 percent and 20 percent. We also sought parameters that produced a level of predictability in monthly returns consistent with empirical experience. For example, Pontiff and Schall (1998) report R-squared statistics of two to four percent in time-series regressions of monthly portfolio returns against book-to-market, dividend yield, and interest rate variables. Regressions of returns on their true conditional expectations provide an upper bound on the forecast ability of returns using such variables. Although we do not use excess kurtosis to calibrate the model, the parameterization nevertheless exhibits excess kurtosis relative to the normal distribution of approximately the order of magnitude observed (Campbell et al. (1997) report an excess kurtosis of 5.77 (4.33) for the equal-weighted (value-weighted) portfolio for the period 1962–1994).

Some other details of the simulations regarding the approximations used to compute the infinite sums and integrals appearing in our pricing equations are described in Appendix C.

C. Results

We begin our examination of the calibrated model by comparing the model numerically to valuations that ignore the dynamic evolution of the firm's assets. This provides some information about the economic significance of valuation errors that would come from treating the systematic risk as fixed, as do most applied valuation procedures. We then evaluate the model's dynamic and cross-sectional properties through simulation.

C.1. Changing Risk

To evaluate the quantitative importance of ignoring the dynamics of the firm's systematic risk, in Figure 1 we plot, for different values of beta and book-to-market, the difference between the expected return on the firm and the expected return that will be earned if all future projects have the same beta. Thus, the plot shows the error a researcher makes if he incorrectly assumes that the future beta of the firm is given by its current beta. The short interest rate is assumed to be seven percent.

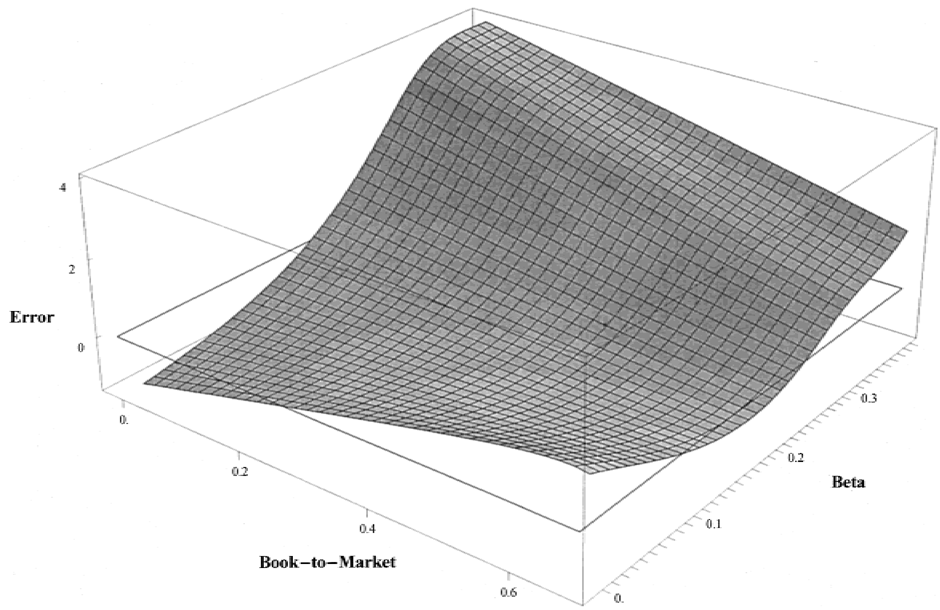


Figure 1. Error induced in expected returns by ignoring the dynamic evolution of the firm's risk. The (correct) annualized expected return (in percent) of a firm is calculated for a given current value of beta ($\beta(t)$). The annualized expected return is then recalculated holding the book value of the firm's assets fixed, and assuming the firm's beta remains at the current level forever (i.e., all future projects have the current beta). The correctly calculated expected return is subtracted from the incorrect one and the result is plotted as a function of the current beta and the correct book-to-market ratio. The short interest rate is assumed to be seven percent. The empty horizontal box marks a difference of zero.

It is quite evident from the figure that, when the firm is mostly growth options, the error in the expected return is increasing in the current beta. By ascribing the low current beta to future growth as well, the riskiness of the future assets the firm will acquire is understated—and the reverse holds at higher beta. With high current systematic risk, the error falls with book-to-market. Larger firms, where the growth component is a smaller fraction of value, will be misvalued less dramatically by standard methods. The fact that the error in the plot depends on book-to-market suggests that this variable will have additional explanatory power if a misspecified model that assumes a constant systematic risk is used to predict expected returns. The error in the misspecified model will also depend on interest rates, suggesting they too will show explanatory power.

C.2. Time-Series Simulation

Table III displays the results of several time-series regressions using the equally weighted portfolio of 50 firms generated in the simulation described in subsection B. These illustrate that conditional expected returns exhibit a

Table III

Time-Series Regressions Using Simulated Aggregate Stock Portfolio

Estimated coefficients are reported from regressions using simulated expected returns ($E_t(R_{t+1})$) and excess expected returns ($E_t(R_{t+1}) - r(t)$) of an equally weighted portfolio of 50 firms for 20,000 months as the dependent variable. The independent variables are book-to-market ($b(t)/P(t)$), the risk-free rate ($r(t)$), one-period lagged expected returns ($E_{t-1}(R_t)$), and the current realized return (R_t). The final line in the table uses the model's pricing kernel as the dependent variable. The last column in the table lists the R^2 coefficient of each regression.

Dep. Variable	Intercept	$b(t)/P(t)$	$r(t)$	$E_{t-1}(R_t)$	R_t	R^2
$E_t(R_{t+1})$	−0.0387	0.0638	—	—	—	0.6845
$E_t(R_{t+1}) - r(t)$	0.0011	0.0067	—	—	—	0.4523
$E_t(R_{t+1})$	−0.0044	0.0147	0.8601	—	—	0.9992
$E_t(R_{t+1}) - r(t)$	−0.0044	0.0147	−0.1399	—	—	0.9543
$E_t(R_{t+1})$	0.0007	—	—	0.9463	—	0.8956
$E_t(R_{t+1})$	0.0128	—	—	—	−0.0228	0.0206
$z(t)/z(t-1)$	1.084	—	—	—	−3.985	0.1454

number of behaviors that have been documented empirically. The t-statistics (not reported) in these regressions are extremely high, suggesting that we are close to the population moments.

The first two regressions in the table show that the conditional expected returns and excess expected returns are positively related in time series to book-to-market, as is suggested by equation (44), and as is shown empirically by Kothari and Shanken (1997) and Pontiff and Schall (1998). Pontiff and Schall report a coefficient of 0.0600 in a univariate regression of the equally weighted index return on book-to-market. This is quite similar in magnitude to our coefficient of 0.0638. The third regression shows that almost all of the movement in expected returns can be explained with a linear model using book-to-market and the interest rate. The fourth regression shows that risk premia are negatively related to interest rates, as has been shown empirically by, for example, Breen et al. (1989) and Ferson (1989). The univariate regression on interest rates yields a positive slope. This is because interest rates and book-to-market are correlated (correlation coefficient of 0.72): when rates rise, holding fixed the assets of the firm, market value must fall relative to book. In multivariate regressions, however, it is clear that interest rates are negatively related to risk premia. High interest rates in our model lead the firm to accept only low-risk projects. Further, interest rates are highly persistent. A firm facing high interest rates is unlikely to have undertaken high-risk projects in the recent past, and is unlikely to do so in the near future. Thus, both the cash flows from existing projects and those from anticipated future projects have lower expected risk and expected return. It is also the case in our model that lagged interest rates explain aggregate risk premia. Regressing the risk premium against lagged interest rates (not reported) produces negative coefficients beyond the first lag. This intuition seems to be robust to alternative specifications and parameterizations.

The fifth and sixth regressions in Table III show that returns are positively related to past expected returns and negatively related to past realized returns. This explains the model's ability to capture contrarian and momentum effects in returns. Jegadeesh and Titman (1993) and others show that past returns predict future returns, but that the direction of these predictions depends on the horizon. Over short horizons contrarian strategies generate returns superior to buy and hold. Over longer horizons, momentum strategies perform well. Our model suggests an economic rationale for this. In our model, conditional *expected* returns are positively autocorrelated because they depend on interest rates and on the systematic risk of the firm, both of which are persistent. Yet conditional expected returns depend *negatively* on realized past returns because the unanticipated component of past returns conveys information about changes in interest rates and the relative systematic risk of the firm's assets. We examine the implications of this for contrarian and momentum strategies in greater detail in Section C.4.

The final regression in Table III gives some indication of how much the standard CAPM explains in our model. Per se, nothing in our theory implies that the market portfolio is mean-variance efficient. From an empirical standpoint it is nevertheless interesting to know how well the static CAPM prices assets in our model. This question is addressed in detail in the next section, however the last regression does provide some preliminary evidence. It shows that about 15 percent of the variation in the equally weighted market portfolio can be explained by variation in the pricing kernel, which can be represented by a minimum-variance portfolio in this model. The correlation between the kernel and the market return is -0.38. These numbers indicate that the single beta market model might have at least some explanatory power in this context.

C.3. Cross-Sectional Simulations

Next we turn to whether the cross-sectional predictions of our model are consistent with the linear regression results widely reported in the literature on the cross section of expected returns. We perform these regression tests using data simulated with our model.

Each simulated data set consists of 2,000 firms for 600 months. As before, we then drop the first 200 observations leaving a sample period in each simulation of 400 months ($33\frac{1}{3}$ years). This is similar in size to the data sets used in a number of existing empirical studies.[14] For each simulation we then replicate four regression tests reported in Fama and French (1992, Table III). For each of the 400 months we run cross-sectional regressions of the monthly realized return against: (1) the firm's CAPM beta (as defined by Fama and French—hereafter, FF), (2) the log of market value of the firm, (3) the log of book-to-market and the log of market value, and (4) the CAPM beta and the log of market value, and then we calculate the time-series

[14] For example, Fama and French (1992) use an average of 2,267 firms over 318 months.

averages and t-statistics of the coefficients. The CAPM beta is calculated using the procedure outlined in FF.[15] Firms with zero book values are assigned a book value of 10^{-8}.[16] The only other point of departure between our simulations and the empirical procedure followed by FF is that they update their independent variables annually, but we update ours each month, since the full information set is observable in our model.[17]

We then repeat the simulation, and accompanying regression tests, 1,422 times. The relative frequencies estimate the distributions of the coefficients and t-statistics, given that our model holds with the parameters in Table I. We then investigate where the coefficients reported by FF fall within this distribution.

We begin with the first regression result reported in Fama and French (1992), Table III—the univariate regression of return on CAPM (or market) beta. Figure 2 shows two things. First, the population value of the regression coefficient in our model is clearly of the same sign as the empirical result. Second, the magnitude of the coefficient and t-statistic that FF obtained is well within the body of the frequency distribution from the simulations. This is confirmed by the summary statistics in Table IV. The last column reports the fraction of observations that deviate from the mean simulated outcome by more than the FF coefficient does. This corresponds to the p-value of the FF outcome when our model is taken to be the null. This value is large for both the coefficient (45.9 percent) and the t-statistic (35.8 percent). The other columns in the table report other characteristics of the distribution. Column 4 shows the frequency of positive coefficients (90 percent). The fifth column shows the frequency of deviations from zero of greater than the deviation of the FF observation from zero. In both cases this value is high (79.8 percent and 86.6 percent). Thus, the empirical outcomes could have occurred with high probability were the actual data generated by the calibrated model.

Figure 3 provides similar plots for the univariate regression of return on the logarithm of market value. Again, the model produces coefficients that reproduce the empirical results in terms of sign. Although the t-statistic is well within the body of the frequency distribution, the magnitude of the reward to market value is large relative to the coefficients typically produced by the parameterized model. Panel B of Table IV confirms this.

Figure 4 and Panel C of Table IV report on the multivariate regression of return on the logarithm of market value and book-to-market. Here the model's outcomes appear to conform closely to the empirical results. The table shows that the mean outcomes in the simulations are very close, in magni-

[15] The prebetas are calculated from the previous 60 monthly returns, so we use part of the first 200 observations to calculate the prebetas for the first 60 months of the sample. Details are in Appendix C.

[16] Simply removing the zero observations, the usual procedure in the cross-sectional literature, produces qualitatively similar results.

[17] Annual updating provides qualitatively similar results.

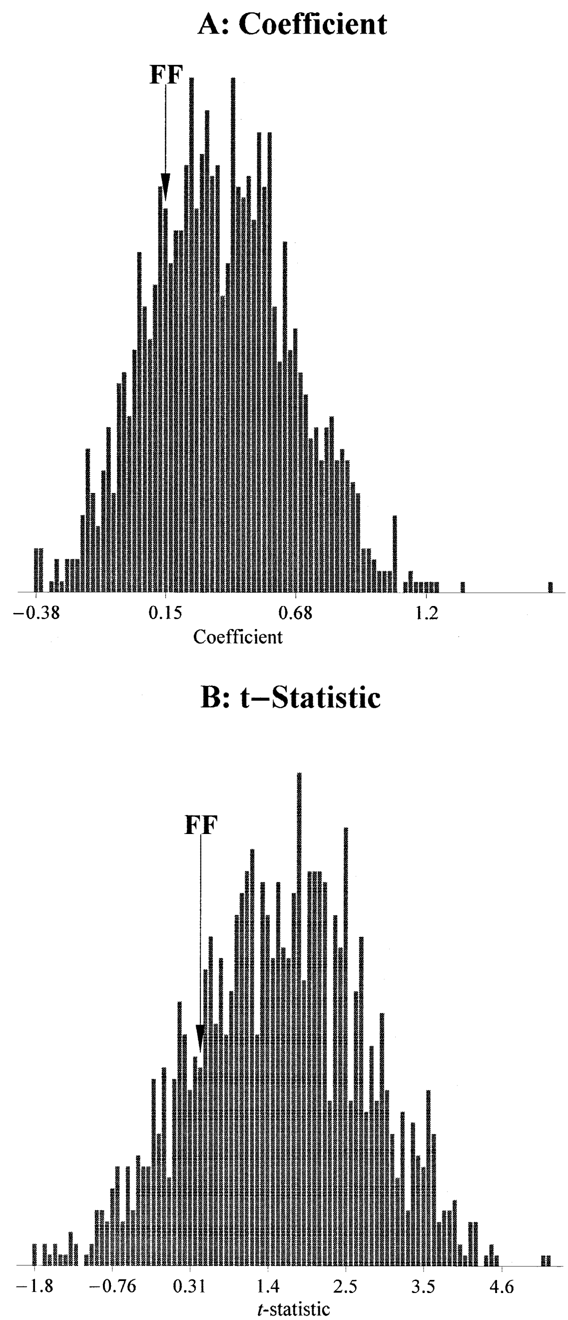


Figure 2. Frequency distribution of the coefficients and *t*-statistics of the Fama–Macbeth regression of return on beta. Panel A (B) is a histogram of the realized coefficients (*t*-statistics) over 1,422 simulations of univariate cross-sectional regressions of return on the CAPM beta. The coefficients (Panel A) are measured in percentage. The arrow marks the coefficient (*t*-statistic) obtained by Fama and French (1992, Table III).

Table IV

Summary Statistics of Cross-Sectional Regressions over 400 Months and 2,000 Firms

Summary statistics are listed for the coefficient (Coef.) and the t-statistic of Fama and French (FF) regressions across 1,422 independent simulations. The dependent variable is the realized return. Independent variables are market beta (β), log of market value ($P(t)$), and log of book to market ($b(t)/P(t)$). The FF column gives the empirical results obtained by Fama and French (1992, Table III), using the actual returns of 2,267 firms over 318 months. Columns 2 and 3 list means and standard deviations of the coefficient and the t-statistic over the simulations. The remaining columns give occurrence frequencies (in percentage) from the distribution of simulated outcomes. Column 4 gives the fraction of simulations for which the simulated coefficient (t-statistic) is greater than zero. Column 5 gives the fraction of simulations for which the coefficient (t-statistic) deviates from zero by more than the FF coefficient (t-statistic). Column 6 gives the fraction of simulations for which the simulated coefficient (t-statistic) deviates from the mean (column 2) by more than the FF coefficient (t-statistic) deviates from the mean. The rows labeled "Joint" provide these frequencies for both coefficients jointly. The measure of deviation in this case is the square root of the sum of the squared deviations, where coefficients are standardized by dividing by the standard deviations (column 3).

	(1)	(2)	(3)	(4)	(5)	(6)
	FF	Mean	Std. Dev.	$P[x > 0]$	$P[\lvert x\rvert > \lvert FF\rvert]$	$P[\lvert x - \bar{x}\rvert > \lvert FF - \bar{x}\rvert]$
Panel A: Univariate Regression on β						
β						
Coef.	0.15	0.377	0.295	90.01	79.82	44.94
t-stat.	0.46	1.542	1.166	90.01	86.64	35.79
Panel B: Univariate Regression on $\log[P(t)]$						
$\log[P(t)]$						
Coef.	−0.15	−0.035	0.048	20.96	0.77	1.82
t-stat.	−2.58	−0.956	1.172	20.96	7.95	15.68
Panel C: Multivariate Regression on $\log[P(t)]$ and $\log[b(t)/P(t)]$						
$\log[P(t)]$						
Coef.	−0.11	−0.093	0.071	5.63	36.85	82.77
t-stat.	−1.99	−2.237	1.449	5.63	54.92	87.69
$\log[b(t)/P(t)]$						
Coef.	0.35	0.393	0.422	98.95	41.49	93.46
t-stat.	4.44	2.641	1.050	98.95	4.92	8.93
Joint						
Coef.					39.24	94.51
t-stat.					12.73	22.22
Panel D: Multivariate Regression on β and $\log[P(t)]$						
β						
Coef.	−0.37	0.642	0.312	98.87	80.80	0.14
t-stat.	−1.21	2.273	0.942	98.87	87.34	0.00
$\log[P(t)]$						
Coef.	−0.17	0.053	0.057	83.12	2.39	0.21
t-stat.	−3.41	1.001	1.041	83.12	1.20	0.00
Joint						
Coef.					23.91	0.07
t-stat.					25.46	0.00

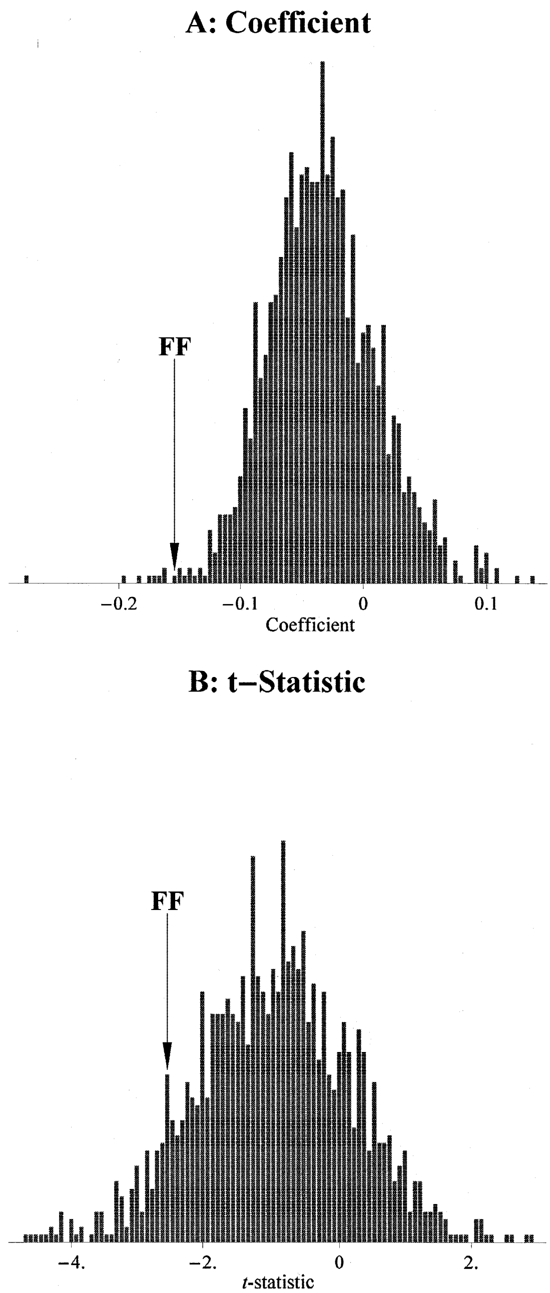
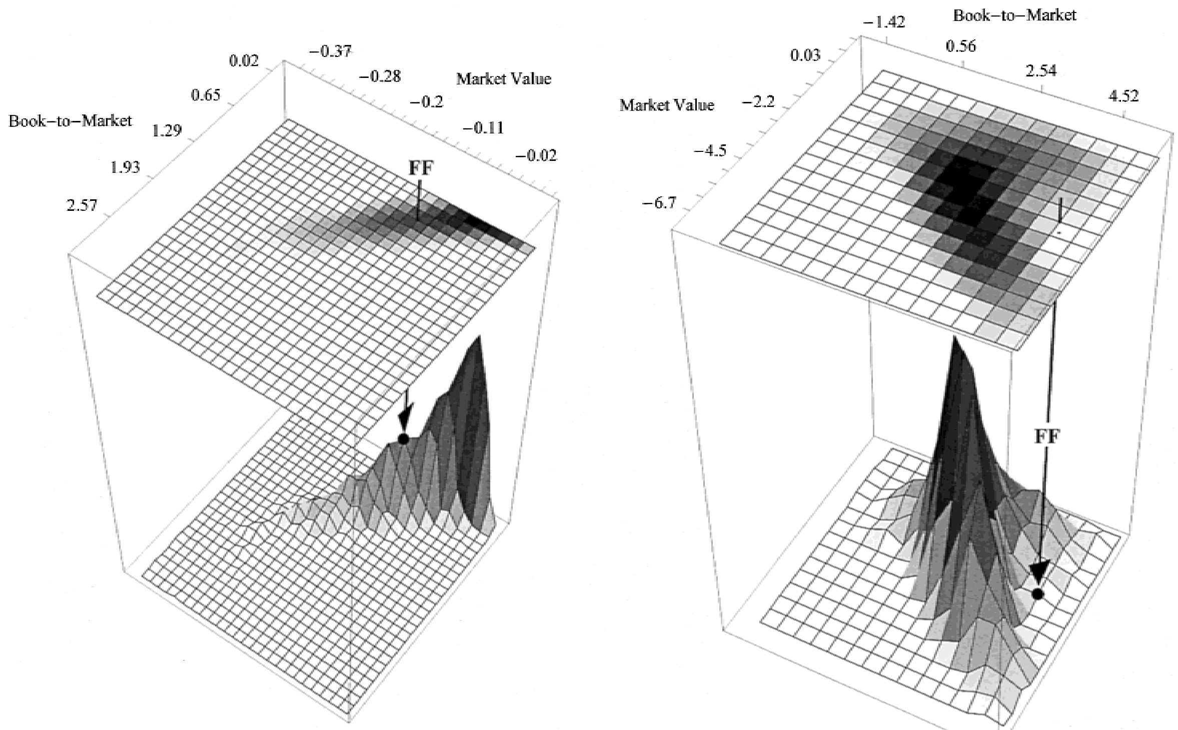


Figure 3. Frequency distribution of the coefficients and *t*-statistics of the Fama–Macbeth regression of return on market value. Panel A (B) is a histogram of the realized coefficients (*t*-statistics) over 1,422 simulations of univariate cross-sectional regressions of return on log market value. To enhance readability, one outlying coefficient at −0.71 percent in Panel A is not shown. The coefficients (Panel A) are measured in percentage. The arrow marks the coefficient (*t*-statistic) obtained by Fama and French (1992, Table III).



A: Coefficients **B: *t*-Statistics**

Figure 4. Frequency distribution of the coefficients and *t*-statistics of the Fama–Macbeth regression of return on market value and book-to-market. The left (right) panel is a histogram of the realized coefficients (*t*-statistics) over 1,422 simulations of multivariate cross-sectional regressions of return on log market value and log book-to-market. Both histograms are projected onto a plane to form the shadow plot above each histogram. The shadow plot is a contour plot where the darkness of a region indicates its height. To enhance readability, Panel A does not show one outlying observation which has a book-to-market coefficient of 1.88 percent and market value coefficient of −0.91 percent. The other outliers determine the plot ranges. The coefficients (Panel A) are measured in percentage. The arrow marks the coefficients (*t*-statistics) obtained by Fama and French (1992, Table III).

tude and in sign, to the coefficients Fama and French produce. Figure 4 shows that the coefficients are in the center of the distribution. The results for the t-statistics are less definitive, but based on the joint distribution from the model, the FF estimates have a p-value of 22.22 percent, and so are well within the range of probable outcomes.

The final regression, of return on both market value and market beta (see Figure 5 and Table IV), is one case where the model's outcomes seem clearly at odds with FF. When the log of market value is included in the regression along with market beta, the sign of the coefficient on market beta becomes negative (though insignificant) in the FF regressions. In our model, curiously, it is the reverse—the sign on market value becomes positive (though, based on the mean t-statistic, "insignificant"). Further, the FF outcomes are well outside the body of the frequency distribution generated by the simulations. Thus, although our model captures quite well the risk premia attributable to each variable independently, and also replicates the joint behavior of market value and book-to-market in the cross section, it has difficulty in matching the joint distribution of market beta and market value, at least for the particular parameterization that we use.

As a final point, it is worth noting the range of outcomes that may occur with significant probability in our model. The regression statistics resulting from the simulated data sets of length similar to historical experience are still quite volatile. Yet the model is calibrated with levels of volatility suggested by matching the moments of large portfolios. In light of this, it should not be surprising, perhaps, that such cross-sectional results often fail to prove empirically robust.

C.4. The Performance of Momentum and Contrarian Strategies

The role of asset turnover in the relationship between current returns and past returns is discussed in Section III. In Table III we see that in aggregate portfolios there is positive dependence on past expected returns and negative dependence on past realized returns. This suggests that contrarian and momentum strategies will be profitable at different horizons. In this section we evaluate this by simulation.

As Jegadeesh and Titman (1993) and Conrad and Kaul (1998) point out, in a world with no predictability in returns, but with cross-sectional differences in expected return due to risk, momentum strategies, which buy winners and sell losers, should perform well. Past returns estimate expected returns, so momentum strategies will tend to be long in higher expected return securities.

Consider, then, the effect of introducing asset turnover of the sort in our model. All firms are ex ante identical. All cross-sectional variation in expected returns arises endogenously as projects are accepted or lost. The expected return of the firm then evolves slowly as old projects die and are replaced with new ones. At time horizons that are comparable to the average life of a project, the momentum strategy is likely to be profitable for the

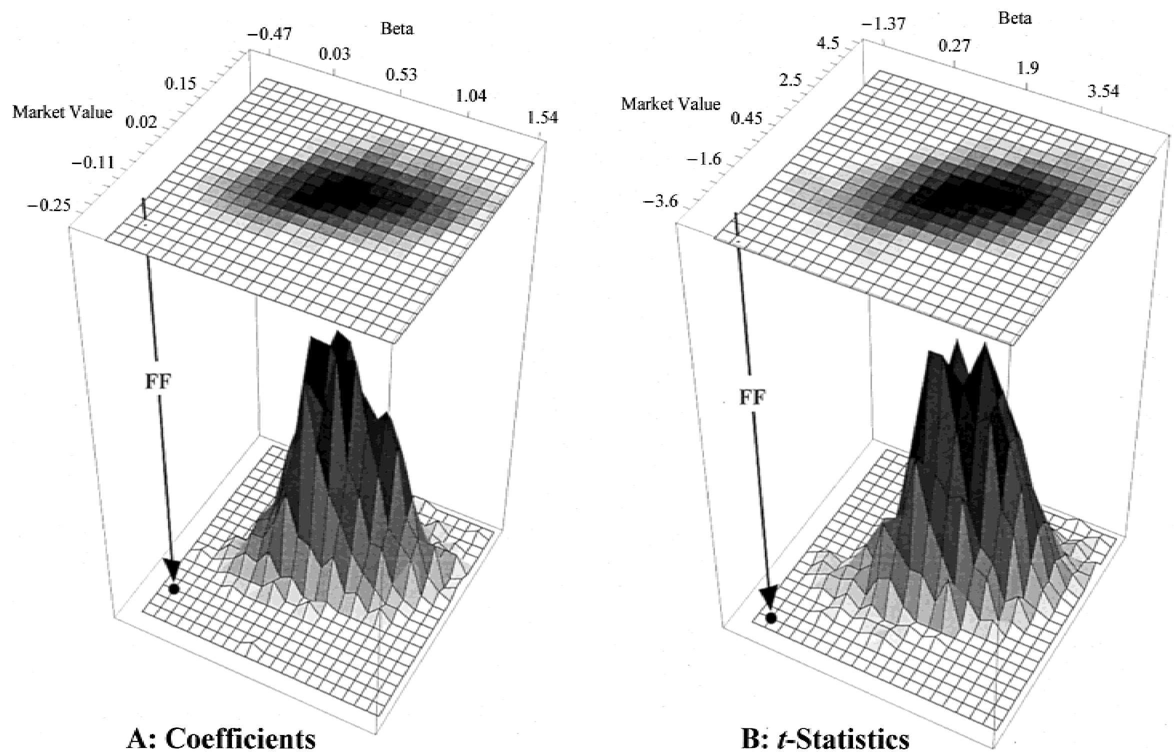


Figure 5. Frequency distribution of the coefficients and *t*-statistics of the Fama–Macbeth regression of return on market value and beta. The left (right) panel is a histogram of the realized coefficients (*t*-statistics) over 1,422 simulations of multivariate cross-sectional regressions of return on log market value and CAPM beta. Both histograms are projected into a plane to form the shadow plot above each histogram. The shadow plot is a contour plot where the darkness of a region indicates its height. To enhance readability, Panel A does not show an outlying observation that has a market value coefficient of 0.48 percent and beta coefficient of 3.37 percent. The other outliers determine the plot ranges. The coefficients (Panel A) are measured in percentage. The arrow marks the coefficients (*t*-statistics) obtained by Fama and French (1992, Table III).

same reason noted above—past returns are a measure of expected returns, which do not change much over such horizons. As is evident from Table III expected returns are highly persistent over short horizons. As the horizon lengthens, however, the probability that the projects in the earlier horizon will be around in the following horizons decreases, and this should lower the expected profits from momentum strategies.

At very short horizons the informational effect of a change in the firm's assets will be more important. In cross section, firms that have experienced the largest price increases are likely to be those firms that have found a low risk project. The net effect of taking on such projects is to reduce the overall riskiness of the firm, thus lowering the expected return. Similarly, the set of firms with the largest price decreases will contain a disproportionate number of firms that have lost low risk projects. In this case the net effect of losing these projects is to increase overall firm riskiness leading to higher returns. This reasoning suggests that contrarian strategies will be profitable, but only over shorter horizons. Since the information associated with asset evolution is reflected immediately in prices, the importance of this effect is diminished as the horizon lengthens and the return associated with the period in which the change occurs becomes a progressively smaller component of the overall horizon return.

To understand the relative importance of the above effects in our model we examine the performance, in simulation, of momentum strategies at 1-, 3-, 6-, and 9-month horizons and at 1-, 2-, 3-, 4-, 5-, 10-, 20-, and 30-year horizons. In each simulation we drop the first 200 months, leaving a data set consisting of 2,400 months and 500 firms. The realized return over each horizon is then calculated by multiplying together the gross realized return of each stock over the months that make up the horizon, denoted H. Firms are then sorted by this horizon return and equally weighted portfolios are formed out of the top 10 percent (*high* portfolio) and bottom 10 percent (*low* portfolio). The gross return of both of these portfolios over the following horizon is then calculated and expressed on an annual basis by taking the $12/H$th power. The annualized payoff of the zero cost momentum strategy of investing \$100 in the high portfolio and financing this by shorting the low portfolio is calculated by subtracting the annual gross return of the low portfolio from the high portfolio and multiplying by 100. For each horizon the average performance of the strategy over the whole simulation is calculated by averaging the annualized payoff over the $(2400/H) - 1$ investment periods in the simulation. This is then repeated over 284 independent simulations. The overall mean performance of the momentum strategy at each horizon interval is plotted in Figure 6.

As expected, the momentum strategy yields negative payoffs at very short horizons, namely one, three, and six months. The payoff is increasing in H, so by nine months it is essentially zero. It becomes positive by the one-year horizon and continues to increase until the five-year horizon, after which it steadily declines. The fact that the payoff from the strategy reaches a maximum at approximately five years accords well with the expected life of a

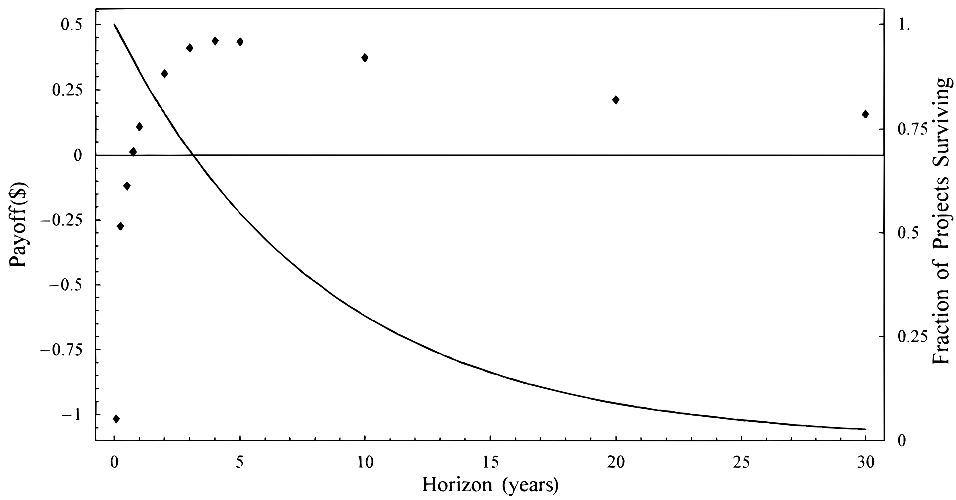


Figure 6. Mean payoffs of momentum strategy at different horizons. Each point plots the mean annualized payoff over 284 independent simulations of investing $100 in the portfolio consisting of the top 10 percent performers over the previous horizon and shorting the bottom 10 percent. Payoffs are annualized. The left vertical axes marks the annualized payoff. The curve plots the fraction of projects that are expected to survive over each horizon. The right vertical axes marks this survival fraction.

project. To see this, the project survival probability (as a function of the horizon) is plotted on the same figure. From this curve one can infer that about one-half of the projects last five years or longer.[18]

The figure confirms that our model can qualitatively reproduce the patterns widely reported in the literature of contrarian profits at shorter horizons and momentum profits at intermediate horizons. On a quantitative level, however, the model clearly has shortcomings. For one thing, the horizon with the maximum momentum strategy return is considerably shorter in the data—the horizon payouts usually reach a maximum between nine months and one year. (See Jegadeesh and Titman (1993), Table I.) Similarly, empirical studies report that the contrarian strategies are only profitable at horizons of about three months or less. (See Conrad and Kaul (1998), Table 1.) These horizons are determined in our model largely by the rate of turnover of the firms' assets. By decreasing the average life of a project (by increasing the depreciation rate), it would be possible to shorten them. However, even if further experimentation were to reveal that a higher depreci-

[18] The variability of these results across simulations is much larger for the one-month case. This appears to be due to the high volatility of the stocks this strategy selects (firms with few assets, where the information effects of acquiring or losing projects are most pronounced). This volatility is further exaggerated by our method of annualizing the payoff. To be sure that the outcome reported above was robust, we ran 2,004 additional simulations over a shorter time interval. The mean payoff obtained was very close, in sign and magnitude, to the one-month payoff in Figure 6.

ation rate could produce time horizons more in line with what is actually observed, this would still leave open the question of why the required depreciation rate is higher than that suggested by the macroeconomic data used to calibrate the model initially.

On the positive side, the model does seem to generate profits of the same order of magnitude as that previously observed. For example, Conrad and Kaul (1998, Table 1), find that the payoff to the momentum strategy reaches a maximum at about one year and ranges between 0.2 and 0.7, depending on the sample period. Although our maximum payoff occurs at the five-year horizon, the size of the (annualized) profits of 0.4 are within the range that Conrad and Kaul report.[19]

V. Conclusions

The model developed in this paper describes investment decision making by individual firms. The valuation of the cash flows that result from these decisions, along with the firm's options to grow in the future, leads to dynamics for conditional expected returns. Our model of expected returns helps explain a number of the important features of the cross-sectional and time-series behavior of stock returns, and the biases that might be induced by a model that ignores these dynamics. Within the model, firms' assets turn over as new investment opportunities with differing risk characteristics arrive, existing assets expire, interest rates change, and firms respond optimally through their investment choices. In the expression for conditional expected returns that we derive, book-to-market appears explicitly, and changes in this variable summarize the effects of movements in the systematic risk of the firm due to asset turnover. Our simulation results show that the model can reproduce simultaneously several important cross-sectional and time-series behaviors that researchers have documented for stock returns, including the explanatory power of book-to-market, market value, and interest rates, and the success of contrarian and momentum strategies at different horizons.

Appendix A. Derivation of the Model with Stochastic Investment

Here we derive closed-form expressions for the value and expected return on the firm for a generalized version of the model presented in the text. We allow the investment and scale of the cash flows to evolve as a stochastic process:

$$I(t+1) = I(t)\exp\left[\mu_I - \tfrac{1}{2}\sigma_I^2 + \sigma_I \zeta(t+1)\right]. \tag{A1}$$

The members of the sequence $\{\zeta(t)\}_{t=0}^{\infty}$ are serially independent and normally distributed with mean zero and variance one.

[19] Direct comparisons to other numbers in their table as well as to other studies are uninformative because, since the portfolios require zero net investment, the profits cannot be interpreted as returns. This implies that a fair comparison of magnitudes must ensure that such features as how the returns are compounded and the time period over which the profits are standardized are consistent.

The value of $I(t)$ determines the cost of the project arriving at t. The cash flows from this project at some future date s also depend on $I(t)$, as in equation (1) in the text where $I(t) = I$. We allow the investment process to have systematic risk. Let $\beta_{zi} \equiv \sigma_z \sigma_I \operatorname{cov}(\zeta(t), \nu(t))$, and $\beta_{ri} \equiv \sigma_r \sigma_I \operatorname{cov}(\zeta(t), \xi(t))$.

Finally, to ensure that the value of future growth opportunities is finite, we must require that

$$r_\infty > \mu_I - \beta_{zi} - \frac{\beta_{ri}}{1-\kappa} \text{ and } \sigma_\infty > 0. \tag{A2}$$

This inequality is the equivalent of the assumption in the Gordon growth model that the firm's growth rate not exceed its discount rate. Here, the limiting, risk-adjusted discount rate must exceed the growth rate of the investment process, $I(t)$.

A. Valuation

PROPOSITION A.1: *The time-t value of the cash flows of a project that is alive and arrived at date $j \leq t$ is*

$$\begin{aligned} V_j(t) &= I(j)\exp[\bar{C} - \beta_j] \sum_{s=t+1}^{\infty} \pi^{(s-t)} B[s-t, r(t)] \\ &= I(j)\exp[\bar{C} - \beta_j] D[r(t)], \end{aligned} \tag{A3}$$

where $D[r(t)]$ is defined in equation (15) and $B[\cdot,\cdot]$ is defined in equation (5).

Proof: For a project that is currently alive, consider a particular term $E_t[(z(s)/(z(t))C_j(s)\chi_j(s)]$ in the infinite sum in equation (11) defining $V_j(t)$. This equals

$$\begin{aligned} &E_t\left[\left\{\frac{z(s-1)}{z(t)}\right\}\left\{\frac{z(s)}{z(s-1)}\,C_j(s)\chi_j(s)\right\}\right] \\ &\quad = \pi^{(s-t)} E_t\left[\left\{\frac{z(s-1)}{z(t)}\right\} E_{s-1}\left\{\frac{z(s)}{z(s-1)}\,C_j(s)\right\}\right] \\ &\quad = \pi^{(s-t)} I(j)\exp[\bar{C} - \beta_j] E_t\left[\frac{z(s-1)}{z(t)}\,e^{-r(s-1)}\right] \\ &\quad = \pi^{(s-t)} I(j)\exp[\bar{C} - \beta_j] B[s-t, r(t)], \end{aligned} \tag{A4}$$

where

$$B[s-t, r(t)] \equiv E_t\left[\frac{z(s)}{z(t)}\right] = E_t\left[\frac{z(s-1)}{z(t)}\,e^{-r(s-1)}\right]$$

is the price at time t of a riskless pure discount bond (with face value 1) that matures in $s - t$ periods. In equation (A4), the first equality follows from the law of iterated expectations and from two other facts: (i) $\chi_j(s)$ is independent of all other random variables in the model, and (ii) $E_t[\chi_j(s)|\chi_j(t) = 1] = \pi^{(s-t)}$. The second equality follows from the lognormality of the cash flows and the pricing kernel, which yields

$$E_{s-1}\left\{\frac{z(s)}{z(s-1)}C_j(s)\right\} = \exp[-r(s-1)]I(j)\exp[\bar{C}-\beta_j]. \tag{A5}$$

Substituting each term in equation (A4) back into equation (11) gives equation (A3). ■

The value of assets in place is then the sum of the values of the individual projects times the indicator function for whether they are currently alive:

$$\sum_{j=0}^{t} V_j(t)\chi_j(t) = b(t)e^{\bar{C}-\beta(t)}D[r(t)], \tag{A6}$$

where

$$\beta(t) \equiv -\ln\left[\sum_{j=0}^{t}\frac{I(j)\chi_j(t)}{b(t)}e^{-\beta_j}\right], \tag{A7}$$

and

$$b(t) \equiv \sum_{j=0}^{t} I(j)\chi_j(t). \tag{A8}$$

Next we must compute the value of growth options. As in Section II.B of the text, conditioning on β_s, take a particular term in the infinite sum, $V^*(t)$, given in equation (12), and using equation (A3) we get

$$\begin{aligned}
E_t&\left\{\frac{z(s)}{z(t)}\max[V_s(s)-I(s),0]\middle|\beta_s\right\}\\
&= E_t\left\{\frac{z(s)}{z(t)}I(s)\max\left[\exp[\bar{C}-\beta_s]\sum_{k=1}^{\infty}\pi^k B[k,r(s)]-1,0\right]\middle|\beta_s\right\}\\
&= E_t\left\{\frac{z(s)}{z(t)}I(s)\max\left[\exp[\bar{C}-\beta_s]\sum_{k=1}^{\infty}\pi^k(B[k,r(s)]-B[k,r^*_{\beta_s}]),0\right]\middle|\beta_s\right\}\\
&= \exp[\bar{C}-\beta_s]\sum_{k=1}^{\infty}\pi^k E_t\left\{\frac{z(s)}{z(t)}\max[I(s)B[k,r(s)]-I(s)B[k,r^*_{\beta_s}],0]\middle|\beta_s\right\}\\
&= I(t)\exp[\mu_I(s-t)]\exp[\bar{C}-\beta_s]\\
&\quad\times\sum_{k=1}^{\infty}\pi^k E_t\left\{\frac{z(s)}{z(t)}\max\left[\frac{I(s)}{E_tI(s)}B[k,r(s)]-\frac{I(s)}{E_tI(s)}B[k,r^*_{\beta_s}],0\right]\middle|\beta_s\right\}\\
&= I(t)\exp[\mu_I(s-t)]\exp[\bar{C}-\beta_s]\sum_{k=1}^{\infty}\pi^k J[r(t),s-t,k,r^*_{\beta_s}].
\end{aligned} \tag{A9}$$

The value of the growth opportunity at s, conditional on β_s, is given by a portfolio of European bond options which expire at s. The value of each option is implicitly defined in equation (A9) to be $J[r(t), s-t, k, r^*_{\beta_s}]$. Each option delivers $I(s)/E_t[I(s)]$ pure discount bonds with remaining maturity k and each has a strike price (for each bond) of $B[k, r^*_{\beta_s}]$. Note that because the value of $I(s)$ only becomes known at time s, the number of bonds that the option delivers is stochastic.

PROPOSITION A.2: *The value at t, given the spot interest rate $r(t)$, of a European option on $I(s)/E_t[I(s)]$ pure discount bonds with k periods to maturity, that expires at date s and has strike price $(I(s)/E_t[I(s)])B(k, r^*_{\beta_s})$ is*

$$J[r(t), s-t, k, r^*_{\beta_s}] = B[s-t+k, r(t)]G_1(s-t, k)N\big(\hat{d}_1(r(t), s-t, k, \beta_s)\big)$$
$$- B[k, r^*_{\beta_s}]B[s-t, r(t)]G_2(s-t)N\big(\hat{d}_2(r(t), s-t, k, \beta_s)\big), \tag{A10}$$

where $B[\cdot,\cdot]$ is as in equation (5), $N(\cdot)$ is the standard normal distribution function, and where $G_1(\cdot)$, $G_2(\cdot)$, $\hat{d}_1(\cdot)$, and $\hat{d}_2(\cdot)$ are functions defined in Lemma B.8 in Appendix B.

Proof: From equation (A9) we have that

$$J[r(t), s-t, k, r^*_{\beta_s}] = E_t\left\{\frac{z(s)}{z(t)} \max\left[\frac{I(s)}{E_t I(s)} B[k, r(s)] - \frac{I(s)}{E_t I(s)} B[k, r^*_{\beta_s}], 0\right] \Bigg| \beta_s\right\}, \tag{A11}$$

so the result follows directly from Lemma B.8 in Appendix B. ■

The present value of all growth opportunities is then the sum of these option values across all future dates s integrated over all values of β_s:

$$V^*(t) = I(t)e^{\bar{C}} \sum_{s=t+1}^{\infty} \exp[\mu_I(s-t)] \sum_{k=1}^{\infty} \pi^k \int_{\mathcal{B}} J[r(t), s-t, k, r^*_{\beta_s}] e^{-\beta_s} dF_\beta(\beta_s)$$
$$\equiv I(t)e^{\bar{C}} J^*[r(t)]. \tag{A12}$$

The following lemma shows that the infinite sums in this expression converge under our assumptions, which in turn justifies reversing the order of integration and summation.

LEMMA A.1: *The infinite sum in equation (A12) converges.*

Proof: The following proof shows that under equation (A2) the sum in equation (A12) converges. First, note from Proposition A.2 that

$$J[r(t), s-t, k, r^*_{\beta_s}] \leq K \exp\left[-\left(\beta_{zi} + \frac{\beta_{ri}}{1-\kappa}\right)(s-t)\right] B[s-t+k, r(t)] \quad \text{(A13)}$$

for $K = \exp[\beta_{ri}/(1-\kappa)^2]$.

For some $m > 0$, let $\hat{\mu}_I = \mu_I - \beta_{zi} - [\beta_{ri}/(1-\kappa)]$, $\epsilon = (r_\infty - \hat{\mu}_I)/m$ and $y_\epsilon = r_\infty - \epsilon$. Equation (A2) implies that $\epsilon > 0$ and $y_\epsilon > \hat{\mu}_I$. Pick m large enough the ensure that $y_\epsilon \geq 0$. Since, under the Vasicek model, yields converge to r_∞ at long maturities, there exists an N_ϵ such that for all $n \geq N_\epsilon$, $B[n, r(t)] \leq \exp[-y_\epsilon n]$. Using (A13), this means that equation (A12) is bounded by

$$\begin{aligned}
&K \sum_{s=t+1}^{N_\epsilon} \exp[\hat{\mu}_I(s-t)] \sum_{k=1}^{\infty} \pi^k \int_{\mathcal{B}} B[s-t+k, r(t)] e^{-\beta_s} dF(\beta_s) \\
&\quad + K \sum_{s=N_\epsilon+1}^{\infty} \exp[\hat{\mu}_I(s-t)] \sum_{k=1}^{\infty} \pi^k \int_{\mathcal{B}} \exp[-y_\epsilon(s-t+k)] e^{-\beta_s} dF(\beta_s) \quad \text{(A14)} \\
&= K \sum_{s=t+1}^{N_\epsilon} \exp[\hat{\mu}_I(s-t)] \sum_{k=1}^{\infty} \pi^k B[s-t+k, r(t)] E[e^{-\beta_s}] \\
&\quad + K \sum_{s=N_\epsilon+1}^{\infty} \exp[\hat{\mu}_I(s-t)] \sum_{k=1}^{\infty} \pi^k \exp[-y_\epsilon(s-t+k)] E[e^{-\beta_s}] \quad \text{(A15)} \\
&\leq K \sum_{s=t+1}^{N_\epsilon} \exp[\hat{\mu}_I(s-t)] E[e^{-\beta_s}] \sum_{k=1}^{\infty} B[s-t+k, r(t)] \\
&\quad + K \sum_{s=N_\epsilon+1}^{\infty} \exp[\hat{\mu}_I(s-t) - y_\epsilon(s-t)] E[e^{-\beta_s}] \sum_{k=1}^{\infty} (\pi \exp[-y_\epsilon])^k. \quad \text{(A16)}
\end{aligned}$$

In the final expression above, the first term is finite because the consol price $(\sum_{k=1}^{\infty} B[s-t+k, r(t)])$ is finite in the Vasicek model. The second term above is finite because $y_\epsilon > \hat{\mu}_I$ by equation (A2), and because $(\pi \exp[-y_\epsilon]) < 1$. Similar arguments imply that J^*_e is finite as well. ■

Combining the value of assets in place, equation (A6), and the value of growth options, equation (A12), gives the value of the firm:

$$P(t) = b(t) e^{\bar{C} - \beta(t)} D[r(t)] + I(t) e^{\bar{C}} J^*[r(t)]. \quad \text{(A17)}$$

B. Derivation of the Expected Return of the Firm

To evaluate the conditional expected return, we must obtain expressions for the conditional expectations of next period's cash flow, of the value next period of existing projects, and of the value of growth opportunities.

We begin with the cash flow.

$$E_t\left[\sum_{j=0}^{t} C_j(t+1)\chi_j(t+1)\right] = \pi\sum_{j=0}^{t} I(j)\exp[\bar{C}]\chi_j(t)$$
$$= \pi\exp(\bar{C})b(t). \tag{A18}$$

We can compute the expectation at t of the value of ongoing projects next period as

$$E_t\left[\sum_{j=1}^{t} V_j(t+1)\chi_j(t)Y_j(t+1)\right] = \pi\left[\sum_{j=0}^{t} I(j)\chi_j(t)\exp[\bar{C}-\beta_j]\right]D_e[r(t)]$$
$$= \pi e^{\bar{C}-\beta(t)}b(t)D_e[r(t)], \tag{A19}$$

where $D_e[r(t)]$ is defined in equation (33). Under the Vasicek model,

$$D_e[r(t)] = \sum_{k=1}^{\infty}\pi^k\exp\left[-\alpha_1(k)[\kappa r(t)+(1-\kappa)\bar{r}]-\psi(k)+\frac{1}{2}\alpha_1^2(k)\sigma_r^2\right]. \tag{A20}$$

Finally, we calculate the expectation, at time t, of the value of growth opportunities at time $t+1$:

$$E_t[V^*(t+1)+\max[V_{t+1}(t+1)-I(t+1),0]]$$
$$= \sum_{s=t+1}^{\infty} E_t\left[\frac{z(s)}{z(t+1)}\max[V_s(s)-I(s),0]\right]. \tag{A21}$$

Note that in the summation on the right-hand side, at $s = t+1$, $z(s)/z(t+1) = 1$, so that it includes the expected NPV of the potential investment at time $t+1$. Now consider a single term in this summation. Conditioning on β_s, and substituting from equation (A3) for $V_s(s)$, we have

$$E_t\left\{\frac{z(s)}{z(t+1)}\max[I(s)e^{\bar{C}-\beta_s}\sum_{k=1}^{\infty}\pi^k B[k,r(s)]-I(s),0]\middle|\beta_s\right\}$$
$$= \sum_{k=1}^{\infty}\pi^k e^{\bar{C}-\beta_s}E_t\left\{\frac{z(s)}{z(t+1)}\max[I(s)B[k,r(s)]-I(s)B[k,r^*_{\beta_s}],0]\middle|\beta_s\right\}$$
$$= I(t)e^{\bar{C}-\beta_s}e^{\mu_I(s-t)}$$
$$\times\sum_{k=1}^{\infty}\pi^k E_t\left\{\frac{z(s)}{z(t+1)}\frac{I(s)}{E_t[I(s)]}\max[B[k,r(s)]-B[k,r^*_{\beta_s}],0]\middle|\beta_s\right\}$$
$$= I(t)e^{\bar{C}-\beta_s}e^{\mu_I(s-t)}\sum_{k=1}^{\infty}\pi^k J_e[r(t),s-t,k,r^*_{\beta_s}]. \tag{A22}$$

The first equality exploits the arguments that lead to equation (25). The second relies on the relation between $I(s)$ and $E_t[I(s)]$. The final equality implicitly defines $J_e[r(t), s-t, k, r^*_{\beta_s}]$. Taking expectations across possible values of β_s and summing across future dates, s, gives

$$
\begin{aligned}
&E_t\big[V^*(t+1) + \max[V_{t+1}(t+1) - I(t+1), 0]\big] \\
&\quad = I(t)e^{\bar{C}} \sum_{s=t+1}^{\infty} \exp[\mu_I(s-t)] \sum_{k=1}^{\infty} \pi^k \int_{\mathcal{B}} J_e[r(t), s-t, k, \beta_s] e^{-\beta_s}\, dF(\beta_s) \\
&\quad = I(t)e^{\bar{C}} J_e^*[r(t)]. \qquad \text{(A23)}
\end{aligned}
$$

Under the Vasicek model we can compute $J_e[\cdot]$ explicitly. Lemma B.10 in Appendix B can be applied directly to yield the following proposition.

PROPOSITION A.3:

$$
\begin{aligned}
J_e[r(t), s-t, k, r^*_{\beta_s}] = e^{r(t)}\big[&B[s-t+k, r(t)]H_1(s-t,k)N\big(d_1^*(r(t), s-t, k, \beta_s)\big) \\
&- B[k, r^*_{\beta_s}]B[s-t, r(t)]H_2(s-t) \\
&\times N\big(d_2^*(r(t), s-t, k, \beta_s)\big)\big] \qquad \text{(A24)}
\end{aligned}
$$

where $B[\cdot,\cdot]$ is defined in equation (5), $N(\cdot)$ is the cumulative standard normal distribution function, and where $H_1(\cdot)$, $H_2(\cdot)$, $d_1^*(\cdot)$, and $d_2^*(\cdot)$ are functions defined in Lemma B.10. By setting $\beta_{zi} = \beta_{zr} = 0$ in these functions, $J_e[r(t), s-t, k, r^*_{\beta_s}]$ for the special case when $I(t)$ is constant can be obtained.

Proof: Equation (A22) implicitly defines $J_e[r(t), s-t, k, r^*_{\beta_s}]$ to be

$$
J_e[r(t), s-t, k, r^*_{\beta_s}] \equiv E_t\left\{ \frac{z(s)}{z(t+1)} \frac{I(s)}{E_t[I(s)]} \max[B[k, r(s)] - B[k, r^*_{\beta_s}], 0] \,\middle|\, \beta_s \right\}. \qquad \text{(A25)}
$$

So $J_e[r(t), s-t, k, r^*_{\beta_s}]$ corresponds to the time t expectation of the price at $t+1$ of a call option that matures at s, which, upon exercise, delivers $I(s)/(E_t[I(s)])$ riskless discount bonds maturing at time $s+k$, and has strike price $(I(s)/(E_t[I(s)]))B[k, r^*_{\beta_s}]$ (i.e., you have to pay $B[k, r^*_{\beta_s}]$ for each bond). The result then follows by applying Lemma B.10 in Appendix B. ■

Combining equations (A23), (A19), and (A18) yields the following expression for the current expectation of the cum-dividend value of the firm at $t+1$:

$$
\pi e^{\bar{C}-\beta(t)} b(t) D_e[r(t)] + \pi e^{\bar{C}} b(t) + I(t)e^{\bar{C}} J_e^*[r(t)]. \qquad \text{(A26)}
$$

PROPOSITION A.4: *The following two expressions describe the expected return earned by holding a proportional claim on the firm's assets from t to $t+1$:*

$$E_t[1+R_{t+1}] = \frac{\pi \dfrac{b(t)}{I(t)}[D_e[r(t)]e^{-\beta(t)} + 1] + J_e^*[r(t)]}{\dfrac{b(t)}{I(t)} D[r(t)]e^{-\beta(t)} + J^*[r(t)]} \tag{A27}$$

and

$$E_t[1+R_{t+1}] = \frac{\pi D_e(r(t))}{D(r(t))} + \pi e^{\bar{C}}\left[\frac{b(t)}{P(t)}\right] + \left[J_e^*[r(t)] - J^*[r(t)]\frac{\pi D_e(r(t))}{D(r(t))}\right]e^{\bar{C}}\left[\frac{I(t)}{P(t)}\right]. \tag{A28}$$

The comparable expressions for the special case when $I(t) = I\ \forall t$ are obtained by setting $I(t) = I$ in the above expressions.

Proof: Dividing equation (A26) by the current price at t as given in equation (A17) yields

$$E_t[1+R_{t+1}] = \frac{\pi e^{\bar{C}-\beta(t)} b(t) D_e[r(t)] + \pi e^{\bar{C}} b(t) + I(t)e^{\bar{C}} J_e^*[r(t)]}{b(t)e^{\bar{C}-\beta(t)} D[r(t)] + I(t)e^{\bar{C}} J^*[r(t)]}. \tag{A29}$$

The above equation can be rewritten as equation (A27). The expression for the value of the firm in equation (A17) implies that

$$e^{\bar{C}-\beta(t)} b(t) = \frac{P(t) - I(t)e^{\bar{C}} J^*[r(t)]}{D[r(t)]}. \tag{A30}$$

Substituting this in equation (A29), gives equation (A28). ∎

Appendix B. Bond and Bond Option Prices in the Vasicek Model

The prices of riskless pure discount bonds in the Vasicek model can be expressed using the following recursively defined functions, which depend on maturity but are constant across interest rates. For $n > 0$, define

$$\alpha_1(n+1) = \alpha_1(n) + \alpha_2(n);$$

$$\phi_1(n+1) = \phi_1(n) + \phi_2(n) + \tfrac{1}{2}\sigma_z^2;$$

$$\alpha_2(n+1) = \kappa\alpha_2(n);$$

$$\phi_2(n+1) = \kappa\phi_2(n) + (1-\kappa)\bar{r};$$

$$\sigma_1^2(n+1) = \sigma_1^2(n) + \sigma_2^2(n) + 2\sigma_{12}(n) + \sigma_z^2;$$

$$\sigma_2^2(n+1) = \kappa^2\sigma_2^2(n) + \sigma_r^2;$$

$$\sigma_{12}(n+1) = \kappa\sigma_{12}(n) + \kappa\sigma_2^2(n) + \beta_{zr}. \tag{B1}$$

Additionally, the following useful random variables can be recursively defined:

$$\delta_r(t,n+1) = \kappa\delta_r(t,n) + \sigma_r\xi(t+n+1);$$

$$\delta_z(t,n+1) = \delta_z(t,n) + \delta_r(t,n) + \sigma_z\nu(t+n+1), \tag{B2}$$

where $\{\nu(t), t > 0\}$ and $\{\xi(t), t > 0\}$ are the shocks to the pricing kernel and the interest rate processes respectively (as defined in equations (3) and (4)).

To complete the preceding definitions, impose the boundary conditions that $\alpha_2(0) = 1$ and $\alpha_1(0) = \phi_1(0) = \phi_2(0) = \sigma_1^2(0) = \sigma_2^2(0) = \sigma_{12}(0) = \delta_r(t,0) = \delta_z(t,0) = 0$. We denote, for $n > 0$, the function

$$\psi(n) \equiv \phi_1(n) - \tfrac{1}{2}\sigma_1^2(n). \tag{B3}$$

This function is used in the bond pricing formula, equation (B5).

Using these definitions, the following series of lemmas derives bond and bond option prices in the Vasicek model.

LEMMA B.1: *Let x and y be jointly normally distributed with $Ex = \mu_x$, $Ey = \mu_y$, $\mathrm{var}(x) = \sigma_x^2$, $\mathrm{var}(y) = \sigma_y^2$, and $\mathrm{cov}(x,y) = \sigma_{xy}$. Then, for any constants A and K,*

$$\begin{aligned} E\big[\exp(y)\max[A\exp(x) - K,0]\big] &= A\exp\left[\mu_x + \mu_y + \frac{1}{2}(\sigma_x^2 + 2\sigma_{xy} + \sigma_y^2)\right]N\left[\frac{\ln A - \ln K + (\mu_x + \sigma_x^2 + \sigma_{xy})}{\sigma_x}\right] \\ &\quad - K\exp\left[\mu_y + \frac{1}{2}\sigma_y^2\right]N\left[\frac{\ln A - \ln K + (\mu_x + \sigma_{xy})}{\sigma_x}\right], \end{aligned} \tag{B4}$$

where $N(\cdot)$ is the cumulative standard normal distribution function.

Proof: First write $y = \mu_y + (\sigma_{xy}/\sigma_x^2)(x - \mu_x) + \epsilon$, where ϵ is normally distributed with $E\epsilon = 0$, $\mathrm{var}(\epsilon) = \sigma_y^2 - (\sigma_{xy}^2/\sigma_x^2)$, and $\mathrm{cov}(x,\epsilon) = 0$ (implying, by normality, independence). Substituting this into the left-hand side of equation (B4) and simplifying provides

$$\begin{aligned}
&E\left[\exp\left[\mu_y + \frac{\sigma_{xy}}{\sigma_x^2}(x - \mu_x) + \epsilon\right]\max[A\exp[x] - K,0]\right]\\
&\quad = \exp\left[\mu_y + \frac{\sigma_y^2}{2} - \frac{\sigma_{xy}^2}{2\sigma_x^2}\right]E\left[\exp\left[\frac{\sigma_{xy}}{\sigma_x^2}(x - \mu_x)\right]\max[A\exp(x) - K,0]\right]\\
&\quad = \exp\left[\mu_y + \frac{\sigma_y^2}{2} - \frac{\sigma_{xy}^2}{2\sigma_x^2} - \frac{\sigma_{xy}}{\sigma_x^2}\mu_x\right]E\left[\exp\left[\frac{\sigma_{xy}}{\sigma_x^2}x\right]\max[A\exp(x) - K,0]\right] \qquad \text{(B5)}\\
&\quad = \exp\left[\mu_y + \frac{\sigma_y^2}{2}\right]\left(A\exp\left[\mu_x + \frac{\sigma_x^2 + 2\sigma_{xy}}{2}\right]N\left[\frac{\ln A - \ln K + (\mu_x + \sigma_x^2 + \sigma_{xy})}{\sigma_x}\right]\right.\\
&\qquad\qquad \left. - KN\left[\frac{\ln A - \ln K + (\mu_x + \sigma_{xy})}{\sigma_x}\right]\right),
\end{aligned}$$

which is the right-hand side of equation (B4). The last step follows from a standard result on truncated means of lognormal distributions. (See, e.g., Ingersoll (1987), pp. 14–15.) ■

LEMMA B.2. *Let $\{z(t), t \geq 0\}$ and $\{r(t), t \geq 0\}$ evolve as in equations (3) and (4). Then, for $n > 0$,*

$$\ln z(t+n) = \ln z(t) - \alpha_1(n)r(t) - \phi_1(n) - \delta_z(t,n) \qquad \text{(B6)}$$

and

$$r(t+n) = \alpha_2(n)r(t) + \phi_2(n) + \delta_r(t,n), \qquad \text{(B7)}$$

where $\delta_z(t,n)$ and $\delta_r(t,n)$ are random variables that are jointly normally distributed and that depend only on information that arrives from date $t+1$ to date $t+n$. Moreover, $E\delta_z(t,n) = E\delta_r(t,n) = 0$, $\mathrm{var}(\delta_z(t,n)) = \sigma_1^2(n)$, $\mathrm{var}(\delta_r(t,n)) = \sigma_2^2(n)$, and $\mathrm{cov}(\delta_z(t,n),\delta_r(t,n)) = \sigma_{12}(n)$. The functions $\alpha_i(\cdot)$, $\phi_i(\cdot)$, $\sigma_i^2(\cdot)$ for $i = 1,2$, and $\sigma_{12}(\cdot)$ are recursively defined in equation (B1).

Proof: The proof is by induction. First write $z(t+n+1)$ in terms of $z(t+n)$ using equation (3). Do the same for $r(t+n+1)$ and $r(t+n)$ using equation (4). Then use the induction hypothesis to reexpress $z(t+n)$ and $r(t+n)$. The lemma follows from applying equation (B1) to the resulting expressions. ■

LEMMA B.3 (BOND PRICES IN THE VASICEK MODEL): *Suppose that the processes $\{z(t)\}$ and $\{r(t)\}$ evolve as in equations (3) and (4). Then the price at time t of a pure discount bond maturing on date $t + n$ is given by $B[n,r(t)] = \exp(-\alpha_1(n)r(t) - \psi(n))$ where $\psi(n) = \phi_1(n) - \frac{1}{2}\sigma_1^2(n)$ and other functions are as defined in equation (B1).*

Proof: The price at time t of a riskless discount bond maturing at time n is given by $E_t[z(t+n)/z(t)]$. Using Lemma B.2, this equals $\exp[-\alpha_1(n)r(t) - \phi_1(n)]E_t \exp(-\delta_z(t,n))$. Since $\delta_z(t,n)$ is normally distributed with mean zero and variance $\sigma_1^2(n)$, the result follows from the expression for the moment generating function of the normal distribution. ■

LEMMA B.4 (MORE VASICEK TRIVIA): *The following relations hold in the Vasicek model. For $n > 0$ and $k > 0$,*

$$\alpha_1(n) - \alpha_1(n+k) = -\alpha_1(k)\alpha_2(n) \tag{B8}$$

and

$$\psi(n) - \psi(n+k) = -\alpha_1(k)\phi_2(n) + \tfrac{1}{2}\alpha_1^2(k)\sigma_2^2(n) + \alpha_1(k)\sigma_{12}(n) - \psi(k). \tag{B9}$$

Proof: Note that

$$E_t\left[\frac{z(t+n+k)}{z(t)}\right] = E_t\left[\left(\frac{z(t+n)}{z(t)}\right)E_{t+n}\left(\frac{z(t+n+k)}{z(t+n)}\right)\right] \tag{B10}$$

and that $E_{t+n}[z(t+n+k)/z(t+n)]$ equals $\exp(-\alpha_1(k)r(t+n) - \psi(k))$ from Lemma B.3. Lemma 4 then follows from using expressions (B6) and (B7) for $r(t+n)$ and $z(t+n)$, taking expectations and equating coefficients. ■

LEMMA B.5 (BOND OPTION PRICES): *Suppose that $z(t)$ and $r(t)$ evolve as in equations (3) and (4). Then, the price at t of a call option that matures at $t + n$, which has an exercise price of K and which, upon exercise, delivers a riskless discount bond maturing at time $t + n + k$ is given by*

$$B[n+k,r(t)]N(d_1(n,k)) - KB[n,r(t)]N(d_2(n,k)), \tag{B11}$$

where $B[\cdot,\cdot]$ is given in Lemma B.3, $N(\cdot)$ is the cumulative standard normal distribution function, and where

$$d_1(n,k) = \frac{\ln B[n+k,r(t)] - \ln B[n,r(t)] - \ln K + \frac{1}{2}\alpha_1^2(k)\sigma_2^2(n)}{\alpha_1(k)\sigma_2(n)}, \tag{B12}$$

$$d_2(n,k) = d_1(n,k) - \alpha_1(k)\sigma_2(n), \tag{B13}$$

and $\alpha_1(\cdot)$ and $\sigma_2(\cdot)$ are as defined in equation (B1).

Proof: The price of the option mentioned in Lemma B.5 equals

$$E_t\left\{\frac{z(t+n)}{z(t)}\max[\exp(-\alpha_1(k)r(t+n)-\psi(k))-K,0]\right\}. \tag{B14}$$

Using the expressions for $z(t+n)$ and $r(t+n)$ given in equations (B6) and (B7), the above expectation can be written as

$$\begin{aligned}&\exp[-\alpha_1(n)r(t)-\phi_1(n)]\\ &\quad\times E_t\big[\exp[-\delta_z(t,n)]\max\big[A[r(t),n,k]\exp[-\alpha_1(k)\delta_r(t,n)]-K,0\big]\big],\end{aligned} \tag{B15}$$

where $A[r(t),n,k] = \exp[-\alpha_1(k)\alpha_2(n)r(t) - \alpha_1(k)\phi_2(n) - \psi(k)]$. Lemma B.1 implies that equation (B15) equals

$$\begin{aligned}&\exp[-\alpha_1(n)r(t)-\phi_1(n)]A[r(t),n,k]\\ &\quad\times\exp[\tfrac{1}{2}(\sigma_1^2(n)+2\alpha_1(k)\sigma_{12}(n)+\alpha_1^2(k)\sigma_2^2(n))]N(d_1)\\ &\quad-\exp[-\alpha_1(n)r(t)-\phi_1(n)]K\exp[\tfrac{1}{2}\sigma_1^2(n)]N(d_2),\end{aligned} \tag{B16}$$

with

$$d_1 = \frac{[\ln A[r(t),n,k]-\ln K+\alpha_1^2(k)\sigma_2^2(n)+\alpha_1(k)\sigma_{12}(n)]}{\alpha_1(k)\sigma_2(n)} \tag{B17}$$

and

$$d_2 = d_1 - \alpha_1(k)\sigma_2(n). \tag{B18}$$

The term multiplying $N(d_2)$ in equation (B16) is $B[n,r(t)]$. Lemma B.4 implies that the term multiplying $N(d_1)$ in equation (B16) equals $B[n+k,r(t)]$, and that d_1 equals

$$\frac{[\alpha_1(n)-\alpha_1(n+k)]r(t)+[\psi(n)-\psi(n+k)]-\ln K+\frac{1}{2}\alpha_1^2(k)\sigma_2^2(n)}{\alpha_1(k)\sigma_2(n)}. \tag{B19}$$

Using the expression for bond prices in the Vasicek model, the above term is seen to equal the equivalent term in the expression given in Lemma B.5. This completes the proof. ■

LEMMA B.6: *Let $I(t)$, $z(t)$, and $r(t)$ evolve as in equations (A1), (3), and (4). Let $\hat{z}(t) \equiv I(t)z(t)$, $\operatorname{cov}(\zeta(t),\xi(t)) = \rho_{ri}$, and $\operatorname{cov}(\zeta(t),\nu(t)) = \rho_{zi}$. Then, for $n > 0$,*

$$\ln\hat{z}(t+n) = \ln\hat{z}(t) - \alpha_1(n)r(t) - \hat{\phi}_1(n) - \delta_{\hat{z}}(t,n), \tag{B20}$$

where $\delta_{\hat{z}}(t,n)$ *and* $\delta_r(t,n)$*, as in Lemma B.2, are jointly normally distributed and depend only on information that arrives from date* $t+1$ *to date* $t+n$*. Moreover,* $E\delta_{\hat{z}}(t,n) = E\delta_r(t,n) = 0$*,* $\text{var}(\delta_{\hat{z}}(t,n)) = \hat{\sigma}_1^2(n)$*,* $\text{var}(\delta_r(t,n)) = \sigma_2^2(n)$*, and* $\text{cov}(\delta_{\hat{z}}(t,n),\delta_r(t,n)) = \hat{\sigma}_{12}(n)$*. The functions* $\alpha_1(\cdot)$*,* $\alpha_2(\cdot)$*,* $\phi_2(\cdot)$*, and* $\sigma_2(\cdot)$ *are defined as before. The remaining functions and random variables are defined recursively as follows:*

$$\hat{\phi}_1(n+1) = \hat{\phi}_1(n) + \phi_2(n) + \tfrac{1}{2}\sigma_z^2 - (\mu_I - \tfrac{1}{2}\sigma_I^2), \tag{B21}$$

$$\delta_{\hat{z}}(t,n+1) = \delta_{\hat{z}}(t,n) + \delta_r(t,n) + \sigma_z\nu(t+n+1) - \sigma_I\zeta(t+n+1), \tag{B22}$$

$$\hat{\sigma}_1^2(n+1) = \hat{\sigma}_1^2(n) + \sigma_2^2(n) + 2\hat{\sigma}_{12}(n) + \sigma_z^2 + \sigma_I^2 - 2\beta_{zi}, \tag{B23}$$

$$\hat{\sigma}_{12}(n+1) = \kappa\hat{\sigma}_{12}(n) + \kappa\sigma_2^2(n) + \beta_{zr} - \beta_{ri}, \tag{B24}$$

where $\beta_{zi} = \rho_{zi}\sigma_z\sigma_I$ *and* $\beta_{ri} = \rho_{ri}\sigma_r\sigma_I$ *and where* $\{\nu(t), t > 0\}$ *and* $\{\zeta(t), t > 0\}$ *are shocks to the pricing kernel and the investment outlay processes respectively (as defined in equations (3) and (A1)). Moreover, the following additional boundary conditions apply:* $\hat{\phi}_1(0) = \hat{\sigma}_1^2(0) = \hat{\sigma}_{12}(0) = \delta_{\hat{z}}(t,0) = 0$.

Proof: The proof of Lemma B.6 follows the same steps as those used in the proof of Lemma B.2. ∎

LEMMA B.7: *The functions defining the evolution of* $z(t+n)$ *in equation (B6) and those defining the evolution of* $\hat{z}(t+n)$ *in equation (B20) are related as follows:*

$$\hat{\phi}_1(n) = \phi_1(n) - (\mu_I - \tfrac{1}{2}\sigma_I^2)n, \tag{B25}$$

$$\hat{\sigma}_{12}(n) = \sigma_{12}(n) - \beta_{ri}\frac{(1-\kappa^n)}{1-\kappa}, \tag{B26}$$

and

$$\hat{\sigma}_1^2(n) = \sigma_1^2(n) + \left[\sigma_I^2 - 2\beta_{zi} - \frac{2\beta_{ri}}{1-\kappa}\right]n + \left[\frac{2\beta_{ri}}{1-\kappa}\right]\left[\frac{(1-\kappa^n)}{1-\kappa}\right]. \tag{B27}$$

Proof: The proof is by induction. Equation (B21) holds by definition. Suppose equation (B25) holds for n. Then we must show these imply that equation (B25) holds for $n+1$. Substituting equation (B25) into equation (B21) gives

$$\begin{aligned}\hat{\phi}_1(n+1) &= \phi_1(n) + \phi_2(n) + \tfrac{1}{2}\sigma_z^2 - (\mu_I - \tfrac{1}{2}\sigma_I^2)(n+1)\\ &= \phi_1(n+1) - (\mu_I - \tfrac{1}{2}\sigma_I^2)(n+1),\end{aligned} \tag{B28}$$

where the last step simply applies the definition of $\phi_1(n+1)$ from equation (B1). Exactly analogous steps verify the other two equations. ■

LEMMA B.8 (VALUE OF AN OPTION ON A STOCHASTIC NUMBER OF DISCOUNT BONDS): *Suppose that $I(t)$, $z(t)$, and $r(t)$ evolve as in equations (A1), (3), and (4). Then, the price at t of a call option that matures at $t+n$, which, upon exercise, delivers $I(t+n)$ riskless discount bonds maturing at time $t+n+k$ and has strike price $KI(t+n)$,*

$$E_t\left[\frac{z(t+n)}{z(t)}\max\big[I(t+n)B[k,r(t+n)]-I(t+n)K,0\big]\right], \tag{B29}$$

is given by

$$I(t)\exp[\mu_I n]\big[B[n+k,r(t)]G_1(n,k)N\big(\hat{d}_1(n,k)\big) - KB[n,r(t)]G_2(n)N\big(\hat{d}_2(n,k)\big)\big], \tag{B30}$$

where $B[\cdot,\cdot]$ is defined in Lemma B.3, $N(\cdot)$ is the cumulative standard normal distribution function, and

$$G_1(n,k) = \exp\left[-\beta_{zi}n - \frac{\beta_{ri}}{1-\kappa}n + \frac{\beta_{ri}}{(1-\kappa)^2}\kappa^k(1-\kappa^n)\right], \tag{B31}$$

$$G_2(n) = \exp\left[-\beta_{zi}n - \frac{\beta_{ri}}{1-\kappa}n + \frac{\beta_{ri}}{(1-\kappa)^2}(1-\kappa^n)\right], \tag{B32}$$

$$\hat{d}_1(n,k) = d_1(n,k) - \frac{\beta_{ri}(1-\kappa^n)}{(1-\kappa)\sigma_2(n)}, \tag{B33}$$

$$\hat{d}_2(n,k) = \hat{d}_1(n,k) - \alpha_1(k)\sigma_2(n), \tag{B34}$$

with $d_1(\cdot,\cdot)$ as defined in Lemma B.5 and $\alpha_1(\cdot)$ and $\sigma_2(\cdot)$ as defined in equation (B1).

Proof: Lemma B.5 proves this result for the case where the exercise price on the bond is fixed. We must now generalize this argument to account for the systematic risk associated with the $I(t)$ process. Using the definition $\hat{z}(t+n)$ as well as the expression for $\hat{z}(t+n)$ and $r(t+n)$ given in equations (B20) and (B7), we get

$$\begin{aligned} E_t&\left[\frac{z(t+n)}{z(t)}\max\big[I(t+n)B[k,r(t+n)]-I(t+n)K,0\big]\right] \\ &= I(t)E_t\left(\frac{\hat{z}(t+n)}{\hat{z}(t)}\max[B[k,r(t+n)]-K,0]\right) \\ &= I(t)\exp[-\alpha_1(n)r(t)-\hat{\phi}_1(n)] \\ &\quad\times E_t\big[\exp(-\delta_{\hat{z}}(t,n))\max\big[A[r(t),n,k]\exp[-\alpha_1(k)\delta_r(t,n)]-K,0\big]\big]. \end{aligned} \tag{B35}$$

The expression in equation (B35) compares with that in equation (B15). Thus, Lemma B.1, as before, implies that equation (B35) equals

$$
\begin{aligned}
& I(t)\exp[-\alpha_1(n)r(t)-\hat{\phi}_1(n)]A[r(t),n,k] \\
& \quad \times \exp[\tfrac{1}{2}(\hat{\sigma}_1^2(n)+2\alpha_1(k)\hat{\sigma}_{12}(n)+\alpha_1^2(k)\sigma_2^2(n))]N(\hat{d}_1) \\
& \quad - I(t)\exp[-\alpha_1(n)r(t)-\hat{\phi}_1(n)]K\exp[\tfrac{1}{2}\hat{\sigma}_1^2(n)]N(\hat{d}_2),
\end{aligned} \tag{B36}
$$

with

$$
\hat{d}_1 = \frac{[\ln A[r(t),n,k]-\ln K+\alpha_1^2(k)\sigma_2^2(n)+\alpha_1(k)\hat{\sigma}_{12}(n)]}{\alpha_1(k)\sigma_2(n)} \tag{B37}
$$

and $\hat{d}_2 = \hat{d}_1 - \alpha_1(k)\sigma_2(n)$. Expression (B36) can be rewritten using the relations derived in Lemma B.7. Simplifying the result using the fact that $\alpha_1(k) \equiv [(1-\kappa^k)/(1-\kappa)]$ gives equation (B30). ■

LEMMA B.9 (EVEN MORE VASICEK TRIVIA): *The following relations hold. For* $n \geq 0$,

$$
\hat{\sigma}_{12}(n+1) = \hat{\sigma}_{12}(n) + \alpha_1(n)\kappa^n\sigma_r^2 - \kappa^n(\beta_{ri}-\beta_{zr}), \tag{B38}
$$

$$
\hat{\sigma}_1^2(n+1) = \hat{\sigma}_1^2(n) + \sigma_I^2 - 2\alpha_1(n)(\beta_{ri}-\beta_{zr}) - 2\beta_{zi} + \alpha_1^2(n)\sigma_r^2 + \sigma_z^2, \tag{B39}
$$

$$
\phi_2(n) = \phi_2(1)\alpha_1(n). \tag{B40}
$$

Proof: Write

$$
\ln\hat{z}(t+n) - \ln\hat{z}(t) = (\ln\hat{z}(t+n) - \ln\hat{z}(t+1)) + (\ln\hat{z}(t+1) - \ln\hat{z}(t)), \tag{B41}
$$

and use Lemma B.6 to express each difference. Matching random variables provides

$$
\delta_{\hat{z}}(t,n+1) = \delta_{\hat{z}}(t+1,n) + \alpha_1(n)\sigma_r\xi(t+1) + \sigma_z\nu(t+1) - \sigma_I\zeta(t+1). \tag{B42}
$$

Computing the variance provides equation (B39). Doing the same thing for $r(t)$ (using Lemma B.2) provides

$$
\delta_r(t,n+1) = \delta_r(t+1,n) + \alpha_2(n)\sigma_r\xi(t+1). \tag{B43}
$$

Equation (B38) follows by computing $\hat{\sigma}_{12}(n+1) = \mathrm{cov}(\delta_{\hat{z}}(t,n+1), \delta_r(t,n+1))$ using the above equation and equation (B42). Finally, match the constant term in the expression for $r(t+n)$:

$$\phi_2(n) - \phi_2(n-1) = \alpha_2(n-1)(1-\kappa)\bar{r} = \kappa^{n-1}\phi_2(1), \tag{B44}$$

where the second equality follows from manipulating equation (B1). Summing both sides of the above equation provides equation (B40). ■

LEMMA B.10 (EXPECTED VALUE OF AN OPTION ON A STOCHASTIC NUMBER OF BONDS): *Suppose that $I(t)$, $z(t)$, and $r(t)$ evolve as in equations (A1), (3), and (4). Then, the time t expectation of the price at $t+1$ of a call option that matures at $t+n$, which, upon exercise, delivers $I(t+n)$ riskless discount bonds maturing at time $t+n+k$ and has strike price $KI(t+n)$,*

$$E_t\left[\left\{\frac{z(t+n)}{z(t+1)}\right\}\max\left[I(t+n)B[k,r(t+n)] - I(t+n)K,0\right]\right], \tag{B45}$$

is given by

$$\begin{aligned} I(t)\exp[\mu_I n + r(t)]\big[&B[n+k,r(t)]H_1(n,k)N(d_1^*(n,k)) \\ &- KB[n,r(t)]H_2(n)N(d_2^*(n,k))\big], \end{aligned} \tag{B46}$$

where $B[\cdot,\cdot]$ is defined in Lemma B.3, $N(\cdot)$ is the cumulative standard normal distribution function, and

$$H_1(n,k) = G_1(n,k)\exp\left[\beta_{zi} - \frac{\beta_{zr}}{(1-\kappa)}(1-\kappa^{n+k-1})\right], \tag{B47}$$

$$H_2(n) = G_2(n)\exp\left[\beta_{zi} - \frac{\beta_{zr}}{(1-\kappa)}(1-\kappa^{n-1})\right], \tag{B48}$$

$$d_1^*(n,k) = \hat{d}_1(n,k) - \frac{\beta_{zr}\kappa^{n-1}}{\sigma_2(n)}, \tag{B49}$$

$$d_2^*(n,k) = d_1^*(n,k) - \alpha_1(k)\sigma_2(n), \tag{B50}$$

with $G_1(\cdot,\cdot)$, $G_2(\cdot)$, and $\hat{d}_1(\cdot,\cdot)$ as defined in Lemma B.8, and with $\alpha_1(\cdot)$ and $\sigma_2(\cdot)$ as defined in equation (B1).

Proof: The proof follows the same lines as the proof of Lemma B.8. Using the definition $\hat{z}(t+n)$ as well as the expression for $\hat{z}(t+n)$ and $r(t+n)$ given in equations (B20) and (B7), we get

$$
\begin{aligned}
E_t&\left[\left\{\frac{z(t+n)}{z(t+1)}\right\}\max\big[I(t+n)B[k,r(t+n)]-I(t+n)K,0\big]\right]\\
&= E_t\left(I(t+1)\left\{\frac{\hat{z}(t+n)}{\hat{z}(t+1)}\right\}\max\big[B[k,r(t+n)]-K,0\big]\right)\\
&= I(t)\exp[\mu_I-\tfrac{1}{2}\sigma_I^2]E_t\big\{\exp[-\alpha_1(n-1)r(t+1)-\hat{\phi}_1(n-1)]\\
&\qquad\qquad\qquad\qquad\times\exp[\sigma_I\zeta(t+1)-\delta_{\hat{z}}(t+1,n-1)]\\
&\qquad\qquad\qquad\qquad\times\max\big[A[r(t),n,k]\exp[-\alpha_1(k)\delta_r(t,n)]-K,0\big]\big\}\\
&= I(t)\exp[\mu_I-\tfrac{1}{2}\sigma_I^2-\alpha_1(n-1)\alpha_2(1)r(t)-\alpha_1(n-1)\phi_2(1)-\hat{\phi}_1(n-1)]\\
&\quad\times E_t\big\{\exp[\sigma_I\zeta(t+1)-\delta_{\hat{z}}(t+1,n-1)-\alpha_1(n-1)\sigma_r\xi(t+1)]\\
&\qquad\quad\times\max\big[A[r(t),n,k]\exp[-\alpha_1(k)\delta_r(t,n)]-K,0\big]\big\}\\
&= I(t)\exp[-\alpha_1(n)r(t)-\hat{\phi}_1(n)+r(t)+\tfrac{1}{2}\sigma_z^2]\\
&\quad\times E_t\big\{\exp[\sigma_I\zeta(t+1)-\delta_{\hat{z}}(t+1,n-1)-\alpha_1(n-1)\sigma_r\xi(t+1)]\\
&\qquad\quad\times\max\big[A[r(t),n,k]\\
&\qquad\quad\times\exp[-\alpha_1(k)\kappa^{n-1}\sigma_r\xi(t+1)-\alpha_1(k)\delta_r(t+1,n-1)]-K,0\big]\big\},
\end{aligned}
\tag{B51}
$$

where the last step follows by using equations (B8), (B21), and (B40). Expression (B52) compares with that in equation (B35). Thus, Lemmas B.1 and B.9 imply that the expectation in equation (B52) equals

$$
\begin{aligned}
A[r(t),n,k]&\exp\big[\tfrac{1}{2}\big(\hat{\sigma}_1^2(n)+2\alpha_1(k)\hat{\sigma}_{12}(n)+\alpha_1^2(k)\sigma_2^2(n)-\sigma_z^2-2\beta_{zr}\alpha_1(n-1)\\
&\qquad+2\beta_{zi}-2\alpha_1(k)\kappa^{n-1}\beta_{zr}\big)\big]N(d_1^*)\\
&-K\exp\big[\tfrac{1}{2}\big(\hat{\sigma}_1^2(n)-\sigma_z^2-2\beta_{zr}\alpha_1(n-1)+2\beta_{zi}\big)\big]N(d_2^*)
\end{aligned}
\tag{B52}
$$

with

$$
d_1^* = \frac{\ln A[r(t),n,k]-\ln K+\alpha_1^2(k)\sigma_2^2(n)+\alpha_1(k)\hat{\sigma}_{12}(n)-\alpha_1(k)\kappa^{n-1}\beta_{zr}}{\alpha_1(k)\sigma_2(n)}
\tag{B53}
$$

and $d_2^* = d_1^* - \alpha_1(k)\sigma_2(n)$. The expression in equation (B53) can be substituted back into equation (B52), which can then be rewritten using the relations derived in Lemma B.7. Simplifying the result gives equation (B46). ■

Appendix C. Simulation Details

The most computationally demanding feature of the model is calculating the option value of future investment opportunities, $J^*[r(t)]$, in equation (31). This involves an infinite double summation and a numerical integration. With the model calibrated to monthly data, the effects of discounting and mean-reversion act slowly on the values in these summations. The inner summation (over k), involves J, which goes to zero, times π^k. Thus, the summand decreases reasonably quickly. It decreases monotonically in $s - t$. At typical parameters and $s - t = 1$, the summand falls below 0.01 percent of the value of the summation up to that point by $k = 400$, our choice of a truncation point. The summand in the outer summation goes to zero more slowly. Experimentation at representative parameters shows it falls below 0.01 percent of the value of the summation at that point by $s - t = 950$ and so this is used as the truncation point in this case. The small residual effect of these truncations on the absolute level of prices is further minimized in the calculation of expected returns where the relative rather than the absolute levels of prices are important.

To avoid the recomputation of $J^*[r(t)]$ and $J_e^*[r(t)]$ at every date, we discretize the domain of these functions and calculate the function at just those points. We then linearly interpolate to determine a value for the function during the simulation. The number of grid points is set to ensure the accuracy of prices meets a prespecified tolerance, which is set to 0.01 percent of the price level.

At the beginning of the simulation all firms consist of no assets and $I(0) = 1$. It takes about 200 months before the number of projects stabilizes. We therefore drop the first 200 months of the simulated data in the time-series simulations. We also begin the cross-sectional regressions in month 200; however, because the beta estimates depend on estimates of prebetas that are taken on the previous 60 months of data, the cross-sectional regressions actually use data beginning in month 140. The details on how this is calculated are as follows. Recall that each simulated data set consists of 600 months. For each of the last 400 months, stocks are sorted into 10 portfolios by market value. Within each market value portfolio, stocks are sorted into 10 more portfolios by the their prebetas—their beta calculated using the previous 60 months of data. The beta of each of these 100 portfolios over the last 400 months is then calculated. In each of the last 400 months, we then allocate the portfolio beta to each of the stocks that makes up the portfolio. The cross-sectional regressions are then run monthly over the last 400 months using these beta estimates as well as the current values for $\log(P(t))$ and $\log(b(t)/P(t))$. All the coefficient estimates of each simulation are the average coefficients over the 400 cross-sectional regressions. The t-statistics are these averages divided by the standard error. All portfolios are formed using equal weights and all betas are calculated by summing the slopes of a regression of return on the (equally weighted) market portfolio return in the current and prior month.

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