Equilibrium in a DeFi Lending Market

Thomas J. Rivera* Fahad Saleh† Quentin Vandeweyer‡

Abstract

We study lending in decentralized finance facilitated by a programmable interest rate rule set by a Protocol for Loanable Funds (PLF). PLFs suffer a disadvantage when compared to traditional lending platforms, given their inability to incorporate off-chain information into the borrowing and lending rates that they set. For this reason, for a pre-determined PLF interest rate function, the DeFi equilibrium is sub-optimal when compared to a competitive lending market equilibrium. We nonetheless show that an optimally designed PLF interest rate function is able to generate equilibrium interest rates, and therefore welfare, that is arbitrarily close to a competitive lending market equilibrium when there are no frictions in the DeFi lending market.

Keywords: Decentralized Finance, DeFi, Protocol for Loanable Funds, PLF

*McGill University. Email: thomas.rivera@mcgill.ca
†Wake Forest University. Email: salehf@wfu.edu
‡University of Chicago, Booth School of Business. Email: quentin.vandeweyer@chicagobooth.edu

1 Introduction

Decentralized finance (DeFi) is a new technology that allows users to access traditional financial services without relying on a trusted intermediary. In this paper, we study a new type of DeFi protocol that allows agents to borrow and lend funds in a peer-to-peer fashion through smart contracts on a blockchain (e.g., Ethereum). These smart contracts, generally referred to as a protocol for loanable funds (PLF), allow agents to freely supply loanable funds from which other agents may borrow. Agents who borrow pay interest on their loan, and the PLF passes along the borrower interest payments to the suppliers of loanable funds.

This paper derives a model of DeFi lending in order to understand the welfare implications of the PLF interest rate-setting mechanism. A defining feature of DeFi lending is that technical constraints limit the ability of blockchain applications to incorporate external (off-chain) information in their functioning (see John et al. 2023). Due to these constraints, DeFi lending relies on an exogenous interest rate function which sets the borrowing and lending rates strictly as a function of the observed ratio of borrowed-to-available loanable funds, referred to as *the utilization rate*. We study how this interest rate-setting mechanism affects the performance of DeFi lending when compared to a competitive lending market. Our contribution is threefold. We first prove that any PLF interest rate function that is increasing in the utilization rate always admits a unique equilibrium. Second, we characterize the informational frictions in the interest rate-setting mechanism and show that the existence of simultaneous shocks to borrowing demand and lending supply imply that it is infeasible to design a PLF function under which the DeFi lending market always clears without rationing. This rationing, due to excess supply or demand at the equilibrium PLF interest rates, leads to a welfare loss when compared to the competitive equilibrium outcome. Finally, we investigate the optimal design of the PLF interest rate function and show that the expected welfare loss due to inefficient equilibrium interest rates can be made arbitrarily small under the appropriate design. As a consequence, a well-designed PLF function can generate equilibrium interest rates that are arbitrarily close to the market clearing rates for any realization of the underlying market conditions, thereby rendering the technical friction inherent to DeFi lending inconsequential to welfare.

A Primer on PLF Interest Rates A DeFi lending market is characterized by a Protocol for Loanable Funds, which serves the role of a intermediary within a traditional lending market. The PLF is a set of smart contracts that encodes transparent rules for setting interest rates, which is the feature of DeFi lending that we focus on in this paper. The PLF rules are typically defined by a single pre-determined interest rate function which sets the borrower interest rate directly and the lender interest rate indirectly. More explicitly, the borrower interest rate corresponds to a specific point on the PLF interest rate function, whereas the lender interest rate is the implied rate from passing borrower payments onto lenders. The direct pass-through of borrower interest rate payments to lenders is considered desirable because it contrasts with traditional settings (e.g., bank deposits) in which the interest rate intermediation markups are typically high, reflecting the large market power of banks (see e.g., Drechsler et al. 2017).

A PLF sets the borrower interest rate by inputting on-chain information regarding lending supply and borrowing demand into the previously referenced interest rate function. More precisely, the PLF sets its borrowing interest rate as a function of the *utilization rate*, defined as follows:

$$\text{Utilization Rate} = \frac{\text{Observed Borrowing}}{\text{Observed Lending}} \tag{1.1}$$

In turn, the borrower interest rate is determined as:

$$\text{Borrower IR} = \rho(\text{Utilization Rate}) \tag{1.2}$$

where ρ denotes the exogenous interest rate function that defines the PLF.

The interest rate function, ρ, is specified as an increasing function of the utilization rate, so that borrower interest rates increase with observed borrowing demand and decrease with observed lending supply. Then, since the PLF passes borrower payments onto lenders, the PLF lender interest rate must satisfy the following equation:[1]

[1] In practice, a PLF generally passes a fixed proportion of payments onto the lender. That proportion is generally close to 100%, so we abstract from this detail and assume a 100% pass-through.

$$\underbrace{\text{Observed Lending} \cdot \text{Lender IR}}_{\text{Lender Payments}} = \underbrace{\text{Observed Borrowing} \cdot \text{Borrower IR}}_{\text{Borrower Payments}} \tag{1.3}$$

Combining these features yields the mechanical rule for determining lender interest rates in a PLF as a function of the observed utilization rate:

$$\text{Lender IR} = \text{Utilization Rate} \cdot \rho(\text{Utilization Rate}) \tag{1.4}$$

Finally, we note that given the pseudonymous nature of blockchain accounts, the current form of DeFi lending is collateralized lending and, therefore, analogous to margin lending or repo. As mentioned above, our focus in this paper is on the interest rate-setting mechanism, which is determined independently of the collateral mechanism.

Model Preview We derive a model of DeFi lending in order to study the interest rate-setting procedure of a PLF. More explicitly, we assume that in each period, a mass of potential borrowers and a mass of potential lenders randomly arrive. Potential borrowers arrive with no capital but have an investment opportunity that generates heterogeneous returns for each such borrower. Potential lenders arrive with a unit of capital with which they can lend to the platform or invest in an alternative outside option, also with heterogeneous returns for each such lender. Given the design of the PLF, the borrowers whose investment opportunities generate a higher return than the PLF borrowing rate will optimally borrow from the PLF. Similarly, lenders with outside options that generate lower returns than the PLF lending rate will optimally lend to the PLF.[2] Importantly, although Equations (1.2) and (1.4) fully specify a PLF by specifying rules for setting both borrower and lender interest rates, it is not immediately clear that the PLF procedure is consistent with equilibrium. In particular, the utilization rate cannot be specified arbitrarily in equilibrium because it itself depends upon supply and demand for loanable funds (see Equation 1.1), whereas supply and demand depend upon interest rates. Our first contribution is to formally establish the existence and uniqueness of a DeFi lending equilibrium by establishing that an equilibrium utilization rate

[2]These assumptions are made only to generate downward-sloping demand curves and upward-sloping supply curves for loanable funds.

always exists and that this utilization rate is always unique for a PLF function that is monotonically increasing in the utilization rate.

We then turn to analyze the performance of a DeFi lending scheme. First, we show that for any fixed PLF interest rate function, the DeFi lending equilibrium generates a lower level of welfare when compared to a perfectly competitive market-clearing equilibrium. To understand this result, first note that the first best outcome is achieved only when lending supply and borrowing demand are equalized, which is equivalent to requiring a utilization rate of unity. Nonetheless, since the PLF sets interest rates as a function of the utilization rate, a PLF utilization rate of unity would therefore imply that the PLF borrower and lender interest rates become static at $\rho(1)$. In turn, a DeFi lending market cannot simultaneously ensure that lending supply equates with borrowing demand *and* that interest rates respond to shocks to lending supply and borrowing demand. In more detail, when borrowing demand is high relative to lending supply, the PLF obtains a utilization rate of unity, but interest rates become static at $\rho(1)$ because it is not possible to borrow more funds than are available at the PLF. In turn, PLF interest rates become unresponsive to further positive borrowing demand shocks and negative lending supply shocks in this case. In contrast, when borrowing demand is low relative to lending supply, the utilization rate falls below unity and interest rates adjust but are sub-optimal precisely because supply is in excess of demand at the prevailing rate whenever the utilization rate is less than unity.

We then study the optimal design of the PLF interest rate function. We show that it is possible to generate DeFi equilibrium interest rates and welfare that are arbitrarily close to the interest rates and welfare from a competitive equilibrium. Although interest rates at a PLF only adjust when the utilization rate is below unity, it is nonetheless not necessary for equilibrium utilization rates to deviate far from unity in order to generate adjustments in the PLF interest rates that are necessary to match the changes in market conditions. In particular, we show that it is possible to design the PLF interest rate function to ensure that the equilibrium utilization rate for *any* market condition is arbitrarily close to unity and that, as a consequence, the PLF interest rate is also arbitrarily close to the competitive lending equilibrium rate. Such a design involves setting the interest rate function with a steep curvature to ensure that interest rates will move sufficiently fast as utilization rates move in a narrow range near unity. In that case, the PLF will only require small variations in utilization rates in order to produce large variations in the realized interest rates. Thus, by designing the PLF function to concentrate all utilization rates arbitrarily close (but not equal) to one, the PLF can generate DeFi equilibrium interest rates, and therefore welfare, that is arbitrarily close to the competitive equilibrium.

Our results generate important implications for DeFi lending markets. In particular, by properly designing the PLF interest rate function, it is possible to generate an outcome that approximates the competitive equilibrium outcome. Therefore, utilizing such designs can generate welfare gains in DeFi lending which can help to facilitate the use and growth of DeFi lending applications.

Literature Our paper contributes to the literature that studies the economics of blockchain technology. While much of the early work in that literature examines economic security (see, e.g., Biais et al. 2019, Easley et al. 2019, and Saleh 2021), we abstract from such concerns and focus instead upon economic implications that arise from a secure blockchain. We specifically examine a secure blockchain with smart contract functionality (e.g., Ethereum) and study welfare implications arising from a prominent application, DeFi lending facilitated by a PLF. The economics literature that studies blockchains with smart contract functionality is young and growing (see John et al. 2023) - topics of study within this literature include tokenomics (see, e.g., Cong et al. 2021 and Mayer 2022), stablecoins (see, e.g., d'Avernas et al. 2022 and Li and Mayer 2022), and decentralized exchanges (see, e.g., Capponi and Jia 2021, Park 2021, Lehar and Parlour 2022a, and Hasbrouck et al. 2023). We add to the smart contracts literature by providing a formal economic model of a PLF with a particular focus on the PLF's interest rate setting mechanism. Other notable papers that also examine PLFs include Aramonte et al. (2022), Chiu et al. (2022), Lehar and Parlour (2022b), and Mueller (2022). Aramonte et al. (2022) highlight that PLFs serve largely as a vehicle for leveraged cryptoasset trading. Chiu et al. (2022) theoretically model a PLF while focusing on asymmetric information between borrowers and lenders. Lehar and Parlour (2022b) and Mueller (2022) provide empirical insights regarding PLFs. Our work differs from prior PLF papers in that we focus upon the interest rate setting mechanism and the associated welfare implications. Crucially, Aramonte et al. (2022) point out that the mechanics of a PLF can be applied even to non-cryptoassets through tokenization of real assets and that such tokenization could have important welfare implications for SMEs; our analysis formalizes that assertion, clarifying the welfare implications of PLFs if integrated with traditional finance (see also John et al. 2023).

2 Model

We examine a discrete time infinite horizon model in which time is indexed by $t \in \mathbb{N}$. At the beginning of each period, a random measure of lenders and a random measure of borrowers arrive to a DeFi lending market, which consists of a single PLF. Lenders possesses a unit of capital and heterogeneous investment opportunities so that each lender may either lend to the PLF or invest in her alternative investment opportunity instead. Borrowers possess no capital but have heterogeneous investment opportunities which they can borrow funds from the PLF to invest in, if they find it optimal to do so.[3] We assume that the PLF sets borrower and lender interest rates, as in practice, and lenders and borrowers decide whether to participate in the PLF, knowing how those interest rates are determined. At the end of each period, each borrower who borrows from the PLF repays her loan with interest, and the PLF allocates the borrower payments to pay off the lenders who lent to the PLF. The next period then commences with a new random measure of borrowers and lenders arriving to the DeFi lending market.

2.1 Lenders and Borrowers

At the beginning of each period, a measure λ_t of lenders and a measure μ_t of borrowers arrive where both $\{\lambda_t\}_{t=1}^{\infty}$ and $\{\mu_t\}_{t=1}^{\infty}$ constitute strictly positive sequences with i.i.d elements and finite first moments. As we shall see, λ_t and μ_t affect equilibrium solutions only through their ratio (i.e., $\frac{\mu_t}{\lambda_t}$) so that it is convenient to define an additional random sequence, $\{\theta_t\}_{t=1}^{\infty}$, as:

[3]Note that we abstract from collateral given that the focus of this paper is on the PLF interest rate mechanism which is entirely orthogonal to the collateral mechanism. All of our results would hold if we required borrowers to post collateral to secure their loans.

$$\theta_t := \frac{\mu_t}{\lambda_t} \tag{2.1}$$

Note that θ_t captures the *credit market tightness* from the borrower's perspective with high values of θ_t corresponding to high market tightness — relatively more borrowers as compared to lenders (i.e., high μ_t or low λ_t) — and low values of θ_t corresponding to low market tightness — relatively few borrowers as compared to lenders (i.e., low μ_t or high λ_t). Consequently, we hereafter refer to θ_t as the market tightness in period t. Moreover, as a simplifying regularity condition, we require that θ_t possesses a strictly increasing and continuously differentiable distribution, G, which is supported on $(0, \infty)$.

2.1.1 Lenders

We uniquely identify each lender by the ordered pair $(j, t) \in [0, \lambda_t] \times \mathbb{N}$ where t denotes the period in which the lender arrives and j denotes the index of each lender among the lenders within period t. We assume that Lender (j, t) possesses a unit of capital and has access to an investment opportunity with net expected return $r_{l,(j,t)} \sim F_l[0, \infty)$ where $F_l \in \mathcal{C}^1[0, \infty)$ denotes a continuously differentiable and strictly increasing cumulative distribution function with a finite first moment. Lender (j, t) may invest her capital in the PLF or invest it in her investment opportunity. As a consequence, denoting by $l_t \geq 0$ the PLF lending rate in period t, Lender (j, t)'s utility, $\mathcal{U}_{l,(j,t)}$, is given as follows:

$$\mathcal{U}_{l,(j,t)}(b_t, l_t) = \max\{l_t, r_{l,(j,t)}\} \tag{2.2}$$

Moreover, the period t lender supply curve for the PLF, $\mathcal{S}_t(l_t)$, is given explicitly as follows:

$$\mathcal{S}_t(l_t) = \lambda_t \cdot \mathbb{P}(r_{l,(j,t)} \leq l_t) = \lambda_t \cdot F_l(l_t) \tag{2.3}$$

2.1.2 Borrowers

We uniquely identify each borrower by the ordered pair $(i, t) \in [0, \mu_t] \times \mathbb{N}$ where t denotes the period in which the borrower arrives and i denotes the index of the borrower among the borrowers within period t. We assume that Borrower (i, t) possesses no capital and has access to an investment opportunity with net expected return $r_{b,(i,t)} \sim F_b[0, \infty)$ where $F_b \in \mathcal{C}^1[0, \infty)$ denotes a continuously differentiable and strictly increasing cumulative distribution function with a finite first moment. We assume that Borrower (i, t) may invest up to one unit of capital in her investment opportunity, implying that it is optimal for her to borrow a unit of capital from the PLF so long as $r_{b,(i,t)} \geq b_t$ with $b_t \geq 0$ denoting the PLF borrowing rate in period t. As a consequence, the demand curve for the PLF, $\mathcal{D}_t(b_t)$, is given as follows:

$$\mathcal{D}_t(b_t) = \mu_t \cdot \mathbb{P}(r_{b,(i,t)} \geq b_t) = \mu_t \cdot (1 - F_b(b_t)) \tag{2.4}$$

As a practical matter, the PLF cannot lend out funds beyond the funds that it possesses; in particular, whenever borrowing demand exceeds lending supply (i.e., $\mathcal{D}_t(b_t) > \mathcal{S}_t(l_t)$), then the PLF cannot fulfill all borrowing demand. In turn, we assume that the PLF lends out its funds pro-rata whenever borrowing demand exceeds lending supply, implying that each borrower seeking a unit of capital receives $\frac{\mathcal{S}_t(l_t)}{\mathcal{D}_t(b_t)}$ in that case. Then, the utility of Borrower (i, t), $\mathcal{U}_{b,(i,t)}$, is given as follows:

$$\mathcal{U}_{b,(i,t)}(b_t, l_t) = \begin{cases} \max\{r_{b,(i,t)} - b_t, 0\} & \text{if } \mathcal{D}_t(b_t) \leq \mathcal{S}_t(l_t) \\ \max\{\frac{\mathcal{S}_t(l_t)}{\mathcal{D}_t(b_t)} \cdot (r_{b,(i,t)} - b_t), 0\} & \text{if } \mathcal{D}_t(b_t) > \mathcal{S}_t(l_t) \end{cases} \tag{2.5}$$

2.2 Protocol for Loanable Funds (PLF)

A PLF is defined by an exogenous interest rate function, $\rho : [0, 1] \mapsto [0, \infty)$, which is continuously differentiable, strictly increasing and satisfies $\rho(0) \geq 0$. Fixing the exogenous interest rate function, ρ, the PLF sets the borrower's interest rate in period t, b_t, as follows:

$$b_t = \rho(U_t) \tag{2.6}$$

where U_t is the utilization rate in period t. The utilization rate is defined as the proportion of available lending that has been borrowed; more explicitly, U_t is given as follows:

$$U_t = \frac{\text{Total Borrowing}}{\text{Total Lending}} = \frac{\min\{\mathcal{D}_t(b_t), \mathcal{S}_t(l_t)\}}{\mathcal{S}_t(l_t)} = \min\{\ \frac{\mathcal{D}_t(b_t)}{\mathcal{S}_t(l_t)}\ ,\ 1\ \} \tag{2.7}$$

where the min in the numerator of Equation (2.7) arises because, as previously discussed, the PLF cannot lend out funds beyond those received from lenders - that is, total borrowing equals the minimum of $\mathcal{D}_t(b_t)$ and $\mathcal{S}_t(l_t)$ because it is not feasible for the PLF to lend in excess of $\mathcal{S}_t(l_t)$.

Turning to the procedure for setting the PLF lender rate, the PLF passes borrower payments directly onto lenders, thereby implying:

$$\underbrace{\mathcal{S}_t(l_t) \cdot l_t}_{\text{Payments To Lenders}} = \underbrace{\min\{\mathcal{S}_t(l_t), \mathcal{D}_t(b_t)\} \cdot b_t}_{\text{Payments From Borrowers}} \tag{2.8}$$

In turn, dividing both sides of Equation (2.8) by total lending, $\mathcal{S}_t(l_t)$, and applying the definition of the utilization rate in Equation (2.7) yields the PLF's mechanical rule for setting the lender interest rate:

$$l_t = U_t \cdot \rho(U_t) \tag{2.9}$$

2.3 Equilibrium

Given an interest rate function, ρ, a DeFi lending equilibrium is a sequence of borrower interest rates, $\{b_t^\star\}_{t=1}^{\infty}$, lender interest rates, $\{l_t^\star\}_{t=1}^{\infty}$, and utilization rates, $\{U_t^\star\}_{t=1}^{\infty}$, such that Equations (2.5) - (2.9) hold for all $t \in \mathbb{N}$. Of note, solving for such an equilibrium reduces to solving for the sequence of utilization rates given that, as per Equations (2.6) and (2.9), equilibrium interest rates in a given period are uniquely determined by the utilization rate in that period.

To determine equilibrium utilization rates, we apply the borrowing rate rule in Equation (2.6) and the lending rate rule in Equation (2.9) to the definition of the utilization rate in Equation (2.7), yielding:

$$U_t = \min\{ \frac{\mathcal{D}_t(\rho(U_t))}{\mathcal{S}_t(U_t \cdot \rho(U_t))} , 1 \} \tag{2.10}$$

Additionally, applying the demand curve in Equation (2.4) and the supply curve in Equation (2.3) to Equation (2.10) yields a fixed-point problem in U_t:

$$U_t = \min\{ \theta_t \cdot \frac{1 - F_b(\rho(U_t))}{F_l(U_t \cdot \rho(U_t))} , 1 \} \tag{2.11}$$

which implies that an equilibrium exists if and only if there exists a sequence of utilization rates $\{U_t^\star\}_{t=1}^{\infty}$ that satisfy Equation (2.11) for all $t \in \mathbb{N}$. Moreover, such an equilibrium is unique if and only if the sequence of utilization rates is unique. In Section 3, we proceed to demonstrate existence and uniqueness of a DeFi lending equilibrium precisely by demonstrating that Equation (2.11) possesses a unique solution, as a function of θ_t, for each t.

3 Existence and Uniqueness of a DeFi Lending Equilibrium

Our first result states that a DeFi lending equilibrium always exists and that it is always unique:

Proposition 3.1. *DeFi Lending Equilibrium*

There always exists a unique DeFi lending equilibrium. Within this equilibrium, in each period t, the borrower interest rate b_t, the lender interest rate l_t, and the utilization rate U_t depend only on the market tightness $\theta_t = \frac{\mu_t}{\lambda_t}$ with the equilibrium solutions being given explicitly as follows:

$$U_t^\star = U^\star(\theta_t), \qquad b_t^\star = b^\star(\theta_t) \equiv \rho(U^\star(\theta_t)), \qquad l_t^\star = l^\star(\theta_t) \equiv U^\star(\theta_t) \cdot \rho(U^\star(\theta_t)) \tag{3.1}$$

where $U^\star : [0, \infty) \mapsto [0, 1]$ is the point-wise unique function satisfying:

$$U^\star(\theta) = \min\{ \theta \cdot \frac{1 - F_b(\rho(U^\star(\theta)))}{F_l(U^\star(\theta) \cdot \rho(U^\star(\theta)))} , 1 \} \tag{3.2}$$

Proposition 3.1 clarifies that PLFs generate a stable and unambiguous economic outcome (i.e., a unique equilibrium). This result arises because a PLF purposefully adjusts interest rates according to market tightness (i.e., $b_t^\star$ and $l_t^\star$ vary with μ_t and λ_t through θ_t). In particular, since supply and demand curves are not directly observable in practice, a PLF employs the utilization rate as a proxy for market tightness and sets interest rates to be increasing in the utilization rate (see Equation 2.6 and 2.9). In turn, positive borrowing demand shocks (i.e., increases in μ_t) increase PLF interest rates because more demand implies a higher utilization rate which then mechanically increases PLF interest rates. Similarly, positive lending supply shocks (i.e., increases in λ_t) imply decreases in PLF interest rates because increases in supply decrease the utilization rate which then mechanically decreases PLF interest rates. We formalize the aforementioned point with the following supplementary result:

Proposition 3.2. *Lending, Borrowing, and Utilization Rates Increase in Market Tightness*
If $\theta_t < \theta_{t'}$, then the following results hold:

$$U_t^\star \leq U_{t'}^\star, \qquad b_t^\star \leq b_{t'}^\star, \qquad l_t^\star \leq l_{t'}^\star \tag{3.3}$$

Additionally, if $U_t^\star < 1$, then the inequalities are all strict.

4 Sub Optimality of DeFi Lending Equilibrium

Having established the existence and uniqueness of a DeFi lending equilibrium, we now examine how the welfare from this unique equilibrium differs from the welfare of a competitive lending equilibrium. We define welfare at time t as the sum of expected utility of the borrowers and lenders in period t. As a consequence, the realized period t welfare in a DeFi lending equilibrium, $\mathcal{W}_t^{DeFi}$, is given as follows:

$$\mathcal{W}_t^{DeFi} = \mathcal{W}(b_t^\star, l_t^\star; \mu_t, \lambda_t) \equiv \underbrace{\int_0^{\mu_t} \mathbb{E}[\mathcal{U}_{b,(i,t)}(b_t^\star, l_t^\star)]\ di}_{\text{Borrower Welfare}} + \underbrace{\int_0^{\lambda_t} \mathbb{E}[\mathcal{U}_{l,(j,t)}(b_t^\star, l_t^\star)]\ dj}_{\text{Lender Welfare}} \tag{4.1}$$

In contrast, the realized period t welfare in a competitive lending equilibrium, $\mathcal{W}_t^{CE}$, is given by the sum of utility of the borrowers and lenders in the case where the borrowing and lending rates are both equal to the competitive equilibrium interest rate, r_t^C. In particular,

$$\mathcal{W}_t^{CE} = \mathcal{W}(r_t^C, r_t^C; \mu_t, \lambda_t) \equiv \underbrace{\int_0^{\mu_t} \mathbb{E}[\mathcal{U}_{b,(i,t)}(r_t^C, r_t^C)]\ di}_{\text{Borrower Welfare}} + \underbrace{\int_0^{\lambda_t} \mathbb{E}[\mathcal{U}_{l,(j,t)}(r_t^C, r_t^C)]\ dj}_{\text{Lender Welfare}} \tag{4.2}$$

where the competitive equilibrium interest rate is defined as the rate that equates lending supply and borrower demand:

$$\mathcal{D}_t(r_t^C) = \mathcal{S}_t(r_t^C) \tag{4.3}$$

As a preliminary result, we establish that the competitive equilibrium interest rate, r_t^C, always exists, is unique, and increases with the credit market tightness θ_t:

Proposition 4.1. *The Competitive Equilibrium Interest Rate is Unique*

There always exists a unique competitive equilibrium lending equilibrium. Within this equilibrium, in each period t, the borrower interest rate b_t^C, the lender interest rate l_t^C, and the utilization rate U_t^C only depend on the market tightness $\theta_t = \frac{\mu_t}{\lambda_t}$ with the equilibrium solutions being given explicitly as follows:

$$U_t^C = U^C(\theta_t) = 1 \qquad b_t^C = l_t^C = r_t^C = r^C(\theta_t)$$

with $r^C : [0, \infty) \mapsto [0, \infty)$ the is unique function satisfying:

$$F_l(r^C(\theta)) = \theta \cdot (1 - F_b(r^C(\theta))) \tag{4.4}$$

guaranteed to exist. Furthermore, r^C is one-to-one, onto, and strictly increasing in the market tightness θ_t.

4.1 The PLF identification Problem

We have shown in Proposition 4.1 that for any level of market tightness θ_t there exists a unique market clearing rate $r^C(\theta_t)$. Our next result demonstrates the PLF identification problem, which precludes PLF's from setting the competitive equilibrium interest rate for each possible realization of market tightness θ_t. We will present this result for a more general class of PLF interest rate functions which map the realized supply, $\mathcal{S}_t$, and demand, $\mathcal{D}_t$, in each period to an interest rate $\rho(\mathcal{S}_t, \mathcal{D}_t)$.

Proposition 4.2. *Consider a general PLF interest rate function $\rho : \mathbb{R}_+^2 \to \mathbb{R}_+$ that sets an interest rate, $\rho(\mathcal{S}_t, \mathcal{D}_t)$, as a function of the realized supply $\mathcal{S}_t$ of loanable funds and borrowing demand $\mathcal{D}_t$ in each period. Further assume that μ_t and λ_t are continuously distributed. Then, for any such function ρ,*

$$Pr(\rho(\mathcal{S}_t, \mathcal{D}_t) = r_t^C) = 0$$

This result demonstrates the difficulty that PLFs face when trying to match the first best lending rate. In particular, given that μ_t and λ_t are both random, we show that it is impossible for the realized supply and demand to identify the market condition θ_t, which is necessary to know in order to set the market clearing rate. We illustrate the identification problem in Figure 1 where we

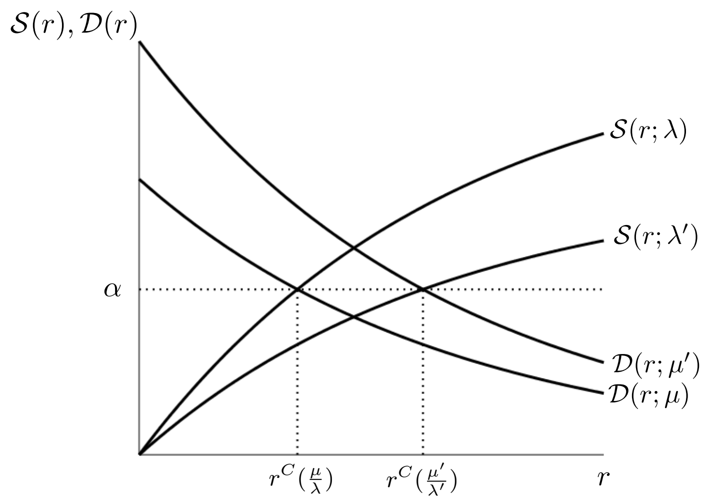


Figure 1: An illustration of the PLF identification problem when $F_l(x) = F_b(x) = 1 - e^{-15x}$ for two different pairs of market conditions $(\mu, \lambda) = (.4, .6)$ and $(\mu', \lambda') = (.6, .4)$.

plot the supply and demand as a function of the interest rate r under two different pairs of market conditions $(\mu, \lambda) = (.4, .6)$ and $(\mu', \lambda') = (.6, .4)$. Importantly, it can be seen that at the market clearing rates, $r^C(\frac{\mu}{\lambda})$ and $r^C(\frac{\mu'}{\lambda'})$, the market clearing supply (equal to demand) is equal to α under both market conditions (μ, λ) and (μ', λ'). Ideally, in this case the PLF would like to specify that when realized supply is equal to realized demand and both are equal to α then the PLF interest rate is set equal to the market clearing rate associated with that level of realized supply and demand (for all possible α). Yet, as can be seen from Figure 1 it is not possible to determine what is the correct market clearing rate when observing realized supply equal to realized demand equal to α without knowing whether the market condition is (μ, λ) or (μ', λ'). Further, this is demonstrated for only two pairs of (μ, λ) and (μ', λ') but we can replicate this argument for infinitely many pairs, each with their own unique market clearing rate. Hence, when arbitrarily specifying an interest rate function, the probability of setting the correct market clearing rate is zero.

We note here that while Proposition 4.2 is derived using the most general PLF interest rate function, the intention here was to demonstrate that it is not the fact that PLFs in practice only adjust to the utilization rate that creates the identification problem. Therefore, a more general PLF design that uses all available information (i.e., realized supply and demand) will also suffer from this problem. In what follows we will revert back to studying the class of PLF interest rate functions presented in Section 2.2, as our next results will show that even this subclass of PLF interest rate functions (which have the added benefit of practical simplicity) can still perform well when compared to the competitive equilibrium provided that the interest rate function, ρ, is selected appropriately.

4.2 Welfare Results

As demonstrated above, due to the PLF identification problem the DeFi lending equilibrium will always generates lower welfare than the competitive equilibrium. We formalize this in our next result.

Proposition 4.3. Sub-Optimal Welfare For DeFi Lending Equilibrium

For any level of the market tightness θ_t, the DeFi lending equilibrium generates weakly lower welfare than the competitive lending equilibrium so that $\mathcal{W}_t^{DeFi} \leq \mathcal{W}_t^{CE}$ for all t.

The DeFi lending equilibrium is sub-optimal because welfare is determined entirely by interest rates (see Equations 4.1 and 4.2), and the PLF is not able to perfectly match the competitive market equilibrium interest rate for all realizations of the market tightness. Importantly, we prove in our next result that the PLF interest rate function can match the competitive equilibrium lending rates and welfare for only a single realization of market tightness θ_t.

Proposition 4.4. *DeFi Equilibrium Matches CE at a Unique* θ_t

There exists a unique realization of the market tightness θ_t such that the DeFi equilibrium matches the competitive lending equilibrium in period t. More formally, there exists a unique $\tilde{\theta}$ such that:

$$\mathcal{W}_t^{DeFi} = \mathcal{W}_t^{CE} \text{ if and only if } \theta_t = \tilde{\theta} \tag{4.5}$$

For this value of $\tilde{\theta}$, PLF interest rates and the competitive equilibrium interest rate all equal $\rho(1)$:

$$r_t^C = b_t^\star = l_t^\star = \rho(1) \iff \theta_t = \tilde{\theta} \tag{4.6}$$

The aforementioned unique value of $\tilde{\theta}$ is given explicitly as follows:

$$\tilde{\theta} = (r^C)^{-1}\Big(\rho(1)\Big) \tag{4.7}$$

where $(r^C)^{-1}$ refers to the inverse function of r^C.

It is important to note that Proposition 4.4 holds for *any* interest rate function ρ. Therefore, independently of how the interest rate function ρ is chosen, there always exists a unique level of market tightness, $\tilde{\theta}$, whereby the the interest rate function ρ generates the highest possible level of welfare. To see how Proposition 4.4 arises, note that in order for both the PLF borrower and lender interest rates, $b_t^\star$ and $l_t^\star$, to be equal to the competitive equilibrium interest rate, r_t^C, then it must be the case that PLF borrower and lender interest rates are identical. However, per Equations (2.6) and (2.9), the PLF borrower and lender interest rates are identical only when the PLF achieves full utilization (i.e., when $U_t^\star = 1$). Then, since the PLF sets interest rates as a mechanical function of the utilization rate, full utilization sets both the PLF borrower and lender interest rates at $\rho(U_t^\star) = \rho(1)$, and the DeFi lending equilibrium therefore achieves the competitive lending equilibrium welfare only if the competitive equilibrium interest rate, r_t^C, equals $\rho(1)$. Finally, since the competitive equilibrium interest rate, r_t^C, is an increasing function of the market tightness θ_t (see Proposition 4.1), the equivalence of the competitive equilibrium interest rate, r_t^C, and the PLF interest rates only occurs for one realization of the market tightness $\theta_t = \tilde{\theta}$, and this is the unique realization of market tightness satisfying $r^C(\tilde{\theta}) = \rho(1)$ hence Equation (4.7).

To better understand the inefficiency within DeFi lending markets, we offer the following supplementary result which clarifies how PLF interest rates vary from the competitive equilibrium interest rate when the market tightness is such that they are not identical (i.e., when $\theta_t \neq \tilde{\theta}$):

Proposition 4.5. *DeFi Interest Rates Deviate From CE*

When the market tightness θ_t entails either excess borrowers or a shortage of lenders relative to $\tilde{\theta}$ (i.e., when $\theta_t > \tilde{\theta}$), then the competitive equilibrium interest rate, r_t^C, strictly exceeds both the PLF borrower interest rate, $b_t^\star$, and the PLF lender interest rate, $l_t^\star$:

$$\theta_t > \tilde{\theta} \implies r_t^C > b_t^\star \text{ and } r_t^C > l_t^\star \tag{4.8}$$

When the market tightness θ_t entails either a shortage of borrowers or excess lenders relative to $\tilde{\theta}$ (i.e., when $\theta_t < \tilde{\theta}$), then the PLF borrower interest rate, $b_t^\star$, strictly exceeds the competitive equilibrium interest rate, r_t^C:

$$\theta_t < \tilde{\theta} \implies b_t^\star > r_t^C \tag{4.9}$$

whereas the relationship between the competitive equilibrium interest rate, r_t^C, and the PLF lender interest rate, $l_t^\star$, is ambiguous.

The first part of Proposition 4.5 establishes that both the PLF borrower interest rate, $b_t^\star$, and PLF lender interest rate, $l_t^\star$, are strictly lower than the competitive equilibrium interest rate, r_t^C, when borrowing demand is high or lending supply is low relative to $\tilde{\theta}$ (i.e., when $\theta_t > \tilde{\theta}$). In contrast, the second part of Proposition 4.5 demonstrates that the PLF borrower interest rate, $b_t^\star$, strictly exceeds the competitive equilibrium interest rate, r_t^C, when borrowing demand is low or lending supply is high relative to $\tilde{\theta}$ (i.e., when $\theta_t < \tilde{\theta}$). In the latter case, the relationship between the competitive equilibrium interest rate, r_t^C, and the PLF lender interest rate, $l_t^\star$, is ambiguous.

5 Optimal PLF Design

In this section we study the optimal design of the PLF interest rate function, ρ. What we will show is that by properly designing the interest rate function, it is possible for the DeFi equilibrium to achieve welfare that is arbitrarily close to the competitive equilibrium welfare. In that sense, the optimally designed PLF will be approximately efficient.

We begin by first showing that it is possible to design the interest rate function ρ such that that the equilibrium utilization rates, $U^\star(\theta_t)$, are arbitrarily close to one for all realizations of market tightness θ_t.

Proposition 5.1. *Suppose that $\theta_t \sim G[\underline{\theta}, \infty)$ where $\underline{\theta} > 0$. Then, for any sufficiently small $\epsilon > 0$ there exists an interest rate function ρ such that $U^\star(\theta) \in (1-\epsilon, 1)$ for all $\theta \in [\underline{\theta}, \infty)$.*

What this result states is that by properly designing the interest rate function, the PLF can guarantee that *all* equilibrium utilization rates are arbitrarily close to 1. The way that this interest rate function is designed is by first *targeting* a minimal equilibrium utilization rate $1-\epsilon$. Next, for the lowest realization of market tightness, $\underline{\theta}$, we solve for the value x that satisfies the following equation and demonstrate that a unique such x always exists:

$$1 - \epsilon = \underline{\theta} \frac{1 - F_b(x)}{F_l((1-\epsilon) \cdot x)} \tag{5.1}$$

Next, we set $\rho(1-\epsilon) = x$ which implies by construction that $U^\star(\underline{\theta}) = 1 - \epsilon$. Finally, we set ρ as any increasing function such that $\rho(U) \to +\infty$ as $U \to 1$.[4] We then prove that this design will ensure that $U^\star(\theta) \in (1-\epsilon, 1)$ for all $\theta \in [\underline{\theta}, +\infty)$ and use the fact that ϵ was arbitrarily chosen to guarantee that we can design ρ to ensure that all equilibrium utilization rates are arbitrarily close to 1.

Finally, we leverage Proposition 5.1 to prove our next result, that when using the properly designed interest rate function, DeFi equilibrium welfare can be made arbitrarily close to the competitive equilibrium welfare.

[4] Note that this is not necessary if the distribution for θ is bounded above, in which case we can choose ρ such that $\rho(1) = B$ for some sufficiently large constant B.

Proposition 5.2. *Suppose that $\theta_t \sim G[\underline{\theta}, \infty)$ where $\underline{\theta} > 0$. For any $\delta > 0$ there exists an interest rate function ρ such that $\mathbb{E}[\mathcal{W}_t^{DeFi}] \geq \mathbb{E}[\mathcal{W}_t^{CE}] - \delta$.*

This last result arises because the PLF borrower and lender rates, $b_t^\star$ and $l_t^\star$, can both be made arbitrarily close to the competitive lending equilibrium rate, $r^C(\theta)$, and thus welfare being a continuous function of interest rates enables us to establish that the DeFi lending equilibrium welfare can be made arbitrarily close to competitive lending equilibrium welfare. In more detail, we generate a sequence of PLF interest rate functions such that the utilization rate converges to unity via Proposition 5.1, and we demonstrate that when utilization rates converge to unity, then the PLF borrower and lender interest rates must both converge to the competitive equilibrium lending rate (i.e., $r_t^\star = \rho(U^\star(\theta_t)) \to r^C(\theta_t)$ and $l_t^\star = U^\star(\theta_t) \cdot b_t^\star \to r^C(\theta_t)$ when $U^\star(\theta_t) \to 1$). In turn, invoking continuity of welfare in interest rates (see Equations 4.1 and 4.2), we establish that the DeFi lending equilibrium welfare from the constructed sequence converges to the competitive lending equilibrium welfare, implying that the there exists PLF interest rate functions that can generate welfare within any arbitrary amount of the competitive lending equilibrium welfare.

5.1 Approximately Optimal Linear/Non-Linear PLF Interest Rate Functions

Finally, we will demonstrate the results of Proposition 5.1 and Proposition 5.2 with two examples. First, we will illustrate the results of Proposition 5.1 by showing that we can generate equilibrium utilization rates that are arbitrarily close to one using only linear interest rate functions.

Example 1. Optimal Linear Interest Rate Functions

Consider the class of linear PLF interest rate functions of the form $\rho(U) = a + b \cdot U$ with $a, b \in \mathbb{R}$ and suppose that $\theta \sim G[\underline{\theta}, \bar{\theta}]$.[5] Then, for any sufficiently small $\epsilon > 0$ we can choose constants $a_\epsilon, b_\epsilon \in \mathbb{R}$ in order to guarantee that for all $\theta \in [\underline{\theta}, \bar{\theta}]$ the equilibrium utilization rates $U^\star(\theta) \in (1-\epsilon, 1)$. In this case, the optimal linear interest rate function chooses $a, b \in \mathbb{R}$ to satisfy

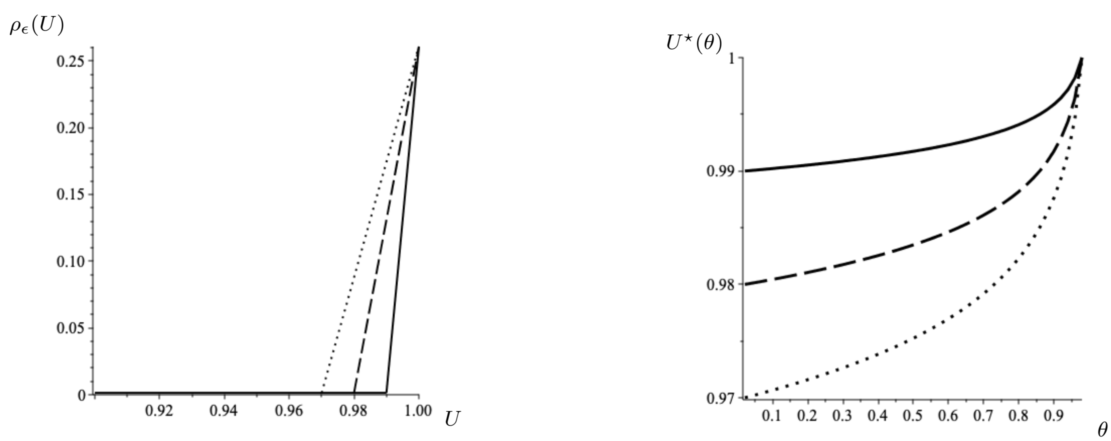
$$\rho_\epsilon(1) = a_\epsilon + b_\epsilon = r^C(\bar{\theta}) \quad \text{and} \quad \rho(1-\epsilon) = a_\epsilon + b_\epsilon \cdot (1-\epsilon) = x_\epsilon$$

[5]The results are easier to demonstrate with linear functions when the support of θ is bounded above. Choosing an arbitrarily large bound $\bar{\theta}$ will also guarantee that the inefficiency incurred if $\theta > \bar{\theta}$ is minimized whenever F_l and F_b are continuous distributions with support $[\underline{\theta}, +\infty)$.

where x_ϵ is chosen to solve (5.1). Equivalently, this implies that

$$a_\epsilon = \frac{1}{\epsilon}(x_\epsilon - (1-\epsilon)r^C(\overline{\theta})) \quad \text{and} \quad b_\epsilon = \frac{1}{\epsilon}(r^C(\overline{\theta}) - x_\epsilon)$$

We plot the equilibrium utilization rates as a function of θ for $\epsilon \in \{.01, .02, .03\}$ assuming $[\underline{\theta}, \overline{\theta}] = [\frac{1}{50}, \frac{49}{50}]$ and $F_l(x) = F_b(x) = 1 - e^{-15x}$. In this case, note that $r^C(\overline{\theta}) \approx .2608$ while $(x_{.01}, x_{.02}, x_{.03}) \approx (.001374, .001402, .00143)$. This implies that $(a_{.01}, a_{.02}, a_{.03}) \approx (-25.68, -12.71, -8.39)$ while $(b_{.01}, b_{.02}, b_{.03}) \approx (25.94, 12.97, 8.65)$. In order to avoid negative lending rates, we could also set $\rho_\epsilon(U) = 0$ for all $U < 1 - \epsilon$ (although this is not necessary). Figure 2 (a) demonstrates the optimal design of the interest rate functions $\rho_\epsilon(U)$ for $\epsilon \in \{.01, .02, .03\}$. As can be see from Figure 2 (b) when utilizing the optimal linear interest rate functions, all equilibrium utilization rates $U^\star(\theta) \geq 1 - \epsilon$ for each $\epsilon \in \{.01, .02, .03\}$. The distinguishing feature of these interest rate functions is that as ϵ becomes smaller (i.e., you want to target higher utilization rates) the slope of the interest rate function increases to be able to exploit more variation in realized interest rates as a function of utilization rates.

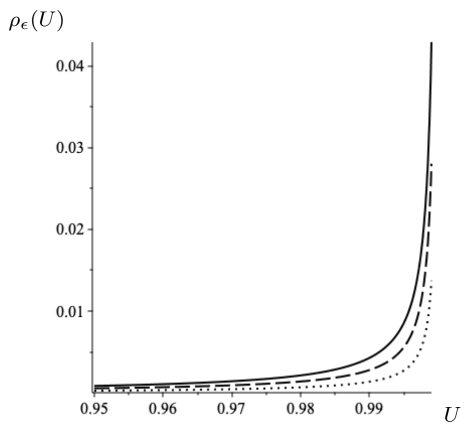


(a) The optimal linear interest rate functions $\rho_\epsilon(U)$ for $\epsilon = .01$ (solid), $\epsilon = .02$ (dashed), and when $\epsilon = .03$ (dotted).

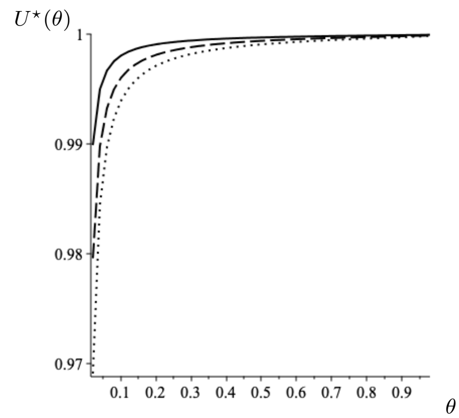
(b) Equilibrium utilization rates $U^\star(\theta)$ under the optimal linear $\rho_\epsilon(U)$ for $\epsilon = .01$ (solid), $\epsilon = .02$ (dashed), and when $\epsilon = .03$ (dotted).

Figure 2: A demonstration of Proposition 5.1 for the optimal linear interest rate functions.

Next, we will demonstrate the same insights as in Example 1 but for the case of non-linear functions. In this setting it is not possible to determine the optimal design of non-linear functions, but we can show how a very simple class of non-linear functions can still achieve arbitrarily high utilization rates. Further, we exploit the non-linearity to demonstrate that this class of functions can perform strictly better than the class of linear interest rate functions.



(a) Approximately optimal non-linear interest rate functions $\rho_\epsilon(U)$ for $\epsilon = .01$ (solid), $\epsilon = .02$ (dashed), and when $\epsilon = .03$ (dotted).



(b) Equilibrium utilization rates $U^\star(\theta)$ under the approximately optimal non-linear $\rho_\epsilon(U)$ for $\epsilon = .01$ (solid), $\epsilon = .02$ (dashed), and when $\epsilon = .03$ (dotted).

Figure 3: A demonstration of Proposition 5.1 for the approximately optimal non-linear interest rate functions of Example 2.

Example 2. Optimal Non-Linear Interest Rate Functions

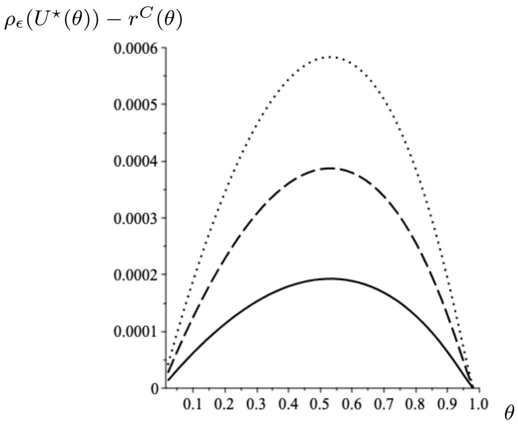
Consider the class of non-linear PLF interest rate functions of the form $\rho(U) = \frac{a}{1-U}$ with $a \in \mathbb{R}$ and suppose that $\theta \sim G[\underline{\theta}, \bar{\theta}]$. Then, again for any sufficiently small $\epsilon > 0$ we can choose the constant $a_\epsilon \in \mathbb{R}$ in order to guarantee that for all $\theta \in [\underline{\theta}, \overline{\theta}]$ the equilibrium utilization rate $U^\star(\theta) \in (1-\epsilon, 1)$. In this case, the interest rate function is uniquely determined by the value of a_ϵ which is chosen to satisfy

$$\rho(1-\epsilon) = \frac{a_\epsilon}{\epsilon} = x_\epsilon \quad \text{or equivalently} \quad a_\epsilon = \epsilon \cdot x_\epsilon$$

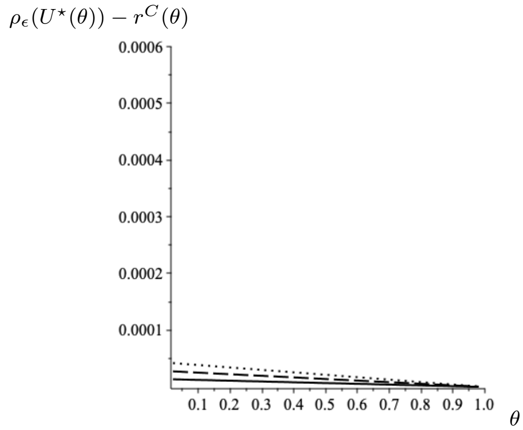
where x_ϵ is chosen (the same way as in Example 1) in order to solve (5.1).

Figure (3) (a) plots this class of non-linear interest rate functions $\rho_\epsilon(U)$ for $\epsilon \in \{.01, .02, .03\}$ while Figure (3) (b) plots the equilibrium utilization rates as a function of θ when using each respective interest rate functions and the same distributional assumptions of Exercise 1. Similar to Exercise 1 it can be seen that the utilization rate for all realizations of θ are above the targeted thresholds $1 - \epsilon$ for each $\epsilon \in \{.01, .02, .03\}$. More importantly, it can be seen that the utilization rates go from the target of $1 - \epsilon$ to 1 at a much faster rate than when utilizing the class of linear interest rate functions.

Finally, we will demonstrate the results of Proposition 5.2 by plotting in Figure 4 the difference between the DeFi equilibrium interest rates and the market clearing interest rate for each market condition θ. As can be seen, given our choice of $\epsilon \in \{.01, .02, .03\}$ the difference between the DeFi equilibrium interest rates and the market clearing rate is very small in magnitude for all of the six interest rate functions we characterize but much smaller for the class of non-linear interest rate functions than linear interest rate functions. Therefore, (as we prove) the welfare generated by the DeFi equilibrium will be approximately equal to the welfare generated by the competitive market.



(a) Interest rate differential between the optimal linear $\rho_\epsilon(U)$ and the competitive rate for $\epsilon = .01$ (solid), $\epsilon = .02$ (dashed), and when $\epsilon = .03$ (dotted).



(b) Interest rate differential between the optimal non-linear $\rho_\epsilon(U)$ and the competitive rate for $\epsilon = .01$ (solid), $\epsilon = .02$ (dashed), and when $\epsilon = .03$ (dotted).

Figure 4: A demonstration of Proposition 5.2 for both linear (a) and non-linear (b) interest rate functions.

6 Conclusion

In this paper we study an economic model of a DeFi lending market facilitated by a Protocol for Loanable Funds (PLF). We focus on the design of the interest rate setting mechanism of the PLF in order to describe the inefficiency of PLFs when compared to competitive lending markets. We first show that the design of PLF interest rate setting function, and the fact that it only relies on the observed ratio of borrowed to lent funds, supports a unique DeFi lending equilibrium. Subsequently, we compare DeFi equilibrium welfare to the competitive lending equilibrium welfare. The PLF faces a natural disadvantage when compared to a competitive lending market in that the PLF cannot incorporate off-chain information when setting interest rates. Despite this informational disadvantage, we show that it is possible to design the PLF interest rate function to ensure that the resulting DeFi equilibrium achieves a level of welfare that is arbitrarily close to the welfare generated by the competitive lending market equilibrium. This design achieves this objective by ensuring that the PLF interest rate function guarantees equilibrium interest rates that are arbitrarily close to the competitive rates for all possible realizations of borrower demand and lender supply. We then demonstrate a relatively simple procedure for designing approximately optimal interest rate functions. These results therefore contribute to the practical design of PLF interest rate functions which will be useful to maximize the use and growth of DeFi lending markets.

References

Aramonte, S., S. Doerr, W. Huang, A. Schrimpf, et al. 2022. DeFi lending: intermediation without information? Tech. rep., Bank for International Settlements.

Biais, B., C. Bisière, M. Bouvard, and C. Casamatta. 2019. The Blockchain Folk Theorem. *Review of Financial Studies* 32(5):1662–1715.

Capponi, A., and R. Jia. 2021. The Adoption of Blockchain-based Decentralized Exchanges. *Columbia University Working Paper* .

Chiu, J., E. Ozdenoren, K. Yuan, and S. Zhang. 2022. On the Fragility of DeFi Lending. *Available at SSRN 4328481* .

Cong, L. W., Y. Li, and N. Wang. 2021. Tokenomics: Dynamic Adoption and Valuation. *Review of Financial Studies* 34(3):1105–1155.

Drechsler, I., A. Savov, and P. Schnabl. 2017. The Deposits Channel of Monetary Policy. *The Quarterly Journal of Economics* 132:1819–1876.

d'Avernas, A., V. Maurin, and Q. Vandeweyer. 2022. Can Stablecoins be Stable? *Working Paper* .

Easley, D., M. O'Hara, and S. Basu. 2019. From Mining to Markets: The Evolution of Bitcoin Transaction Fees. *Journal of Financial Economics* 134(1):91–109.

Hasbrouck, J., T. Rivera, and F. Saleh. 2023. The Need for Fees at a DEX: How Increases in Fees Can Increase DEX Trading Volume. *Working Paper* .

John, K., L. Kogan, and F. Saleh. 2023. Smart Contracts and Decentralized Finance. *Annual Review of Financial Economics* Forthcoming.

Lehar, A., and C. Parlour. 2022a. Decentralized Exchanges. *Working Paper* .

Lehar, A., and C. A. Parlour. 2022b. Systemic fragility in decentralized markets. *Available at SSRN* .

Li, Y., and S. Mayer. 2022. Money Creation in Decentralized Finance: A Dynamic Model of Stablecoin and Crypto Shadow Banking. *Working Paper* .

Mayer, S. 2022. Token-Based Platforms and Speculators. *Working Paper* .

Mueller, P. 2022. DeFi Leveraged Trading: Inequitable Costs of Decentralization. *Available at SSRN 4241356* .

Park, A. 2021. The Conceptual Flaws of Constant Product Automated Market Making. *University of Toronto Working Paper* .

Saleh, F. 2021. Blockchain Without Waste: Proof-of-Stake. *Review of Financial Studies* 34(3):1156–1190.

Appendices

A Proofs

A.1 Proof of Proposition 3.1

Proof. To prove existence of $U^\star$, first note that

$$\frac{\partial}{\partial U}[\min\{x \cdot \frac{1-F_b(\rho(U))}{F_l(U \cdot \rho(U))}, 1\}] \leq 0 \tag{A.1}$$

which comes directly from the fact that

$$\frac{\partial}{\partial U}[x \cdot \frac{1-F_b(\rho(U))}{F_l(U\rho(U))}] = -x \cdot \frac{F_b'(U)\rho'(U)F_l(U\rho(U)) + F_l'(U\rho(U)) \cdot (\rho(U) + U\rho'(U)) \cdot (1-F_b(\rho(U)))}{(F_l(U\rho(U)))^2} < 0$$

Further, note that

$$0 < \lim_{U \to 0} \min\{x \cdot \frac{1-F_b(\rho(U))}{F_l(U \cdot \rho(U))}, 1\} = 1 \qquad \text{and} \qquad 1 \geq \min\{x \cdot \frac{1-F_b(\rho(1))}{F_l(\rho(1))}, 1\}$$

and therefore, for all $x \in [0, \infty)$, there must exist U such that

$$U = \min\{x \cdot \frac{1 - F_b(\rho(U))}{F_l(U \cdot \rho(U))}, 1\}$$

To prove uniqueness, suppose by contradiction that for some $x \in [0, \infty)$ there exists $U \neq U'$ such that

$$U = \min\{x \cdot \frac{1 - F_b(\rho(U))}{F_l(U \cdot \rho(U))}, 1\} \qquad \text{and} \qquad U' = \min\{x \cdot \frac{1 - F_b(\rho(U'))}{F_l(U' \cdot \rho(U'))}, 1\}$$

Further, assume without loss of generality that $U > U'$. Then, $U - U' > 0$ which implies that

$$\min\{x \cdot \frac{1 - F_b(\rho(U))}{F_l(U \cdot \rho(U))}, 1\} - \min\{x \cdot \frac{1 - F_b(\rho(U'))}{F_l(U' \cdot \rho(U'))}, 1\} > 0$$

Yet, (A.1) implies that

$$\min\{x \cdot \frac{1 - F_b(\rho(U))}{F_l(U \cdot \rho(U))}, 1\} - \min\{x \cdot \frac{1 - F_b(\rho(U'))}{F_l(U' \cdot \rho(U'))}, 1\} \leq 0$$

whenever $U > U'$, a contradiction. □

A.2 Proof of Proposition 3.2

Proof. Let z be defined as

$$z(U) = \theta \cdot \frac{1 - F_b(\rho(U))}{F_l(U \cdot \rho(U))}$$

Then, consider $\theta < \theta'$ and denote by

$$U = \min\{\theta \cdot z(U), 1\} \qquad \text{and} \qquad U' = \min\{\theta' \cdot z(U'), 1\}$$

with the existence and uniqueness of U and U' guaranteed by Proposition 3.1. We further know from the proof of 3.1 that $z'(U) < 0$ for all $U \in [0, 1]$. Now, by contradiction suppose that $U > U'$. Then,

$$U' = \min\{\theta' \cdot z(U'), 1\} \geq \min\{\theta \cdot z(U'), 1\} \geq \min\{\theta \cdot z(U), 1\} = U$$

where the last inequality comes from the fact that $U > U'$ implies $z(U') > z(U)$. This presents a contradiction to the fact that we have assumed $U > U'$. Therefore, we have shown that $\theta_t < \theta_{t'}$ implies that $U_t^\star \leq U_{t'}^\star$.

Finally, suppose that $U < 1$ but $U = U'$. Then,

$$U = \theta \cdot z(U) < \min\{\theta' \cdot z(U), 1\} = \min\{\theta' \cdot z(U'), 1\} = U'$$

where the first inequality comes from the fact that $\theta \cdot z(U) < 1$ and $\theta < \theta'$. Again, this presents a contradiction given that we have assumed that $U = U'$.

In order to prove that $b_t^\star \leq b_{t'}^\star$ and $l_t^\star \leq l_{t'}^\star$ whenever $\theta_t < \theta_{t'}$ we note that $b_t^\star = \rho(U_t^\star)$ and $l_t^\star = U_t^\star \cdot \rho(U_t^\star)$. Therefore, noting that $\rho'(U) > 0$ and $\frac{d}{dU}[U \cdot \rho(U)] = U \cdot \rho'(U) + \rho(U) > 0$ implies that whenever $U_t^\star \leq U_{t'}^\star$ then $b_t^\star \leq b_{t'}^\star$ and $l_t^\star \leq l_{t'}^\star$ with all inequalities strict whenever $U_t^\star < U_{t'}^\star$ which we have shown is the case whenever $U_t^\star < 1$. □

A.3 Proof of Proposition 4.1

Proof. The existence of $r_{CE}^\star$ comes immediately from the fact that $F_l'(r) > 0$ and $F_b'(r) > 0$ for all $r \in [0, +\infty)$ and

$$F_l(0) = 0 \leq x(1 - F_b(0)) = x \qquad \text{and} \qquad \lim_{r \to +\infty} F_l(r) = 1 > \lim_{r \to +\infty} x(1 - F_b(r)) = 0$$

Uniqueness comes from the fact that $F_l(r)$ is strictly increasing while $x(1 - F_b(r))$ is strictly decreasing. Existence implies that $r_{CE}^\star$ is onto while uniqueness implies that $r_{CE}^\star$ is one-to-one. □

A.4 Proof of Proposition 4.2

Proof. If λ and μ are continuously distributed then for any fixed $\alpha > 0$:

$$|\{(\lambda, \mu) : \alpha = S(r^C(\frac{\mu}{\lambda})) = D(r^C(\frac{\mu}{\lambda}))\}| = +\infty$$

In particular, suppose that $D(r^C(\frac{\mu}{\lambda})) = \mu(1 - F_b(r^C(\frac{\mu}{\lambda}))) = \alpha$. In order to prove our claim, we will show that for all $\mu' \neq \mu$ there exists $\lambda' \neq \lambda$ such that $S(r^C(\frac{\mu'}{\lambda'})) = D(r^C(\frac{\mu'}{\lambda'})) = \alpha$. In order to do so, first take any $\mu' \neq \mu$. Then, using the fact that $r^C(\frac{\mu}{\lambda})$ is continuous, strictly increasing in μ, strictly decreasing in λ, and unbounded we know that if

$$\mu'(1 - F_b(r^C(\frac{\mu'}{\lambda}))) > \alpha$$

then there must exist $\lambda' < \lambda$ such that

$$\mu'(1 - F_b(r^C(\frac{\mu'}{\lambda'}))) = \alpha$$

Similarly, if

$$\mu'(1 - F_b(r^C(\frac{\mu'}{\lambda}))) < \alpha$$

then there must exist $\lambda'' > \lambda$ such that

$$\mu'(1 - F_b(r^C(\frac{\mu'}{\lambda''}))) = \alpha$$

and therefore we have proven our claim.

Using this result, we know that for a fixed $\alpha = D(r^C(\frac{\mu}{\lambda}))$ there exists infinitely many unique pairs (λ', μ') such that $\alpha = D(r^C(\frac{\mu}{\lambda})) = S(r^C(\frac{\mu}{\lambda})) = D(r^C(\frac{\mu'}{\lambda'})) = S(r^C(\frac{\mu'}{\lambda'}))$. Further, if $D(r^C(\frac{\mu'}{\lambda'})) = D(r^C(\frac{\mu}{\lambda}))$ then it must be the case that

$$\mu(1 - F_b(r^C(\frac{\mu}{\lambda}))) = \mu'(1 - F_b(r^C(\frac{\mu'}{\lambda'})))$$

which implies that if $\mu' > \mu$ then $\frac{\mu'}{\lambda'} > \frac{\mu}{\lambda}$ and therefore $r^C(\frac{\mu'}{\lambda'}) > r^C(\frac{\mu}{\lambda})$. Similarly, if $\mu > \mu'$ then it must be the case that $\frac{\mu'}{\lambda'} < \frac{\mu}{\lambda}$ and therefore $r^C(\frac{\mu'}{\lambda'}) < r^C(\frac{\mu}{\lambda})$. This implies that there are an infinite number of market conditions (μ, λ) that generate realized supply and demand equal to α at the market clearing rate, with each pair (μ, λ) having a unique market clearing rate $r^C(\frac{\mu}{\lambda})$. Hence, when observing $\mathcal{S}_t = \mathcal{D}_t = \alpha$ the interest rate function cannot possibly set $\rho(\alpha, \alpha) = r^C(\frac{\mu}{\lambda})$ without precisely knowing what the true market condition (μ, λ) is. Further, if $\rho(\alpha, \alpha)$ is arbitrarily specified then this implies that there is a unique pair (μ, λ) such that $\rho(\alpha, \alpha) = r^C(\frac{\mu}{\lambda})$. Namely, while multiple pairs (μ, λ) can generate the same market clearing interest rate, we have shown above that no two such pairs can generate the same realized demand at that market clearing rate. Hence, for any interest rate function $\rho(\mathcal{S}_t, \mathcal{D}_t)$, it must be the case that $Pr(\rho(\alpha, \alpha) = r^C(\frac{\mu}{\lambda})) = 0$ for all $\alpha > 0$.

□

A.5 Proof of Proposition 4.3

Proof. We first note that

$$\mathcal{W}_t^{DeFi} = \mu_t \cdot (1 - F_b(b_t)) \cdot \min\{\frac{\mathcal{S}(l_t)}{\mathcal{D}(b_t)}, 1\} \cdot (\mathbb{E}[r_b | r_b \geq b_t] - b_t) + \lambda_t \cdot (F_l(l_t) \cdot l_t + (1 - F_l(l_t)) \cdot \mathbb{E}[r_l | r_l \geq l_t])$$

and

$$\mathcal{W}_t^{CE} = \mu_t \cdot (1 - F_b(r_t^C)) \cdot (\mathbb{E}[r_b | r_b \geq r_t^C] - r_t^C) + \lambda_t \cdot (F_l(r_t^C) \cdot r_t^C + (1 - F_l(r_t^C)) \cdot \mathbb{E}[r_l | r_l \geq r_t^C])$$

Further, using the fact that r_t^C guarantees that supply equals demand for all θ_t implies $\mu_t \cdot (1 - F_b(r_t^C)) = \lambda_t \cdot F_l(r_t^C)$ and therefore

$$\mathcal{W}_t^{CE} = \mu_t \int_{r_t^C}^{+\infty} r dF_b(r) + \lambda_t \int_{r_t^C}^{+\infty} r dF_l(r)$$

for all θ_t.

Now, let $\tilde{\theta} = (r^C)^{-1}(\rho(1))$ guaranteed to exist given that r^C is a bijection. Whenever $\theta_t = \tilde{\theta}$ then $l_t^\star = b_t^\star = r_t^C$ so that the DeFi market supply is equal to demand and therefore $\mathcal{W}_t^{DeFi} = \mathcal{W}_t^{CE}$. What will show is that whenever $\theta_t \neq \tilde{\theta}$ then $\mathcal{W}_t^{DeFi} < \mathcal{W}_t^{CE}$. We proceed with two cases.

Case 1: $\theta_t > \tilde{\theta}$: By construction, note that $\theta_t > \tilde{\theta}$ implies that $r_t^C > \rho(1)$. Further, given Proposition 3.2 we know that $U_t^\star$ is weakly increasing in θ_t but cannot exceed one. Therefore, it must be the case that $U_t^\star = 1$ and $b_t^\star = l_t^\star = \rho(1)$ whenever $\theta_t > \tilde{\theta}$. Once this is the case, we know that $\theta_t > \tilde{\theta}$ implies that $\mathcal{D}(r_t^C) = \mathcal{S}(r_t^C)$ while $\mathcal{D}(b_t^\star) > \mathcal{S}(l_t^\star)$. Therefore, whenever $\theta_t > \tilde{\theta}$ then using the fact that

$$\mu_t(1 - F_b(b_t)) \cdot \frac{\mathcal{S}(l_t)}{\mathcal{D}(b_t)} = \lambda_t \cdot F_l(l_t)$$

we can see that in this case,

$$\mathcal{W}_t^{DeFi} = \frac{\lambda_t \cdot F_l(\rho(1))}{1 - F_b(\rho(1))} \cdot \int_{\rho(1)}^{+\infty} r dF_b(r) + \lambda_t \int_{\rho(1)}^{+\infty} r dF_l(r)$$

Furthermore, denote by

$$\Phi(\rho) := \frac{\lambda_t \cdot F_l(\rho)}{1 - F_b(\rho)} \cdot \int_{\rho}^{+\infty} r dF_b(r) + \lambda_t \int_{\rho}^{+\infty} r dF_l(r)$$

then

$$\frac{d}{d\rho}\Phi(\rho) = \frac{\lambda_t(F_l'(\rho)(1 - F_b(\rho)) + F_b'(\rho)F_l(\rho))}{(1 - F_b(\rho))^2} \int_{\rho}^{+\infty} r dF_b(r) - \lambda_t(\frac{F_l(\rho)}{1 - F_b(\rho)} F_b'(\rho) \cdot \rho + F_l'(\rho) \cdot \rho) >$$

$$\frac{\lambda_t(F_l'(\rho)(1 - F_b(\rho)) + F_b'(\rho)F_l(\rho))}{(1 - F_b(\rho))^2} \cdot (1 - F_b(\rho)) \cdot \rho - \lambda_t(\frac{F_l(\rho)}{1 - F_b(\rho)} F_b'(\rho) \cdot \rho + F_l'(\rho) \cdot \rho) = 0$$

Therefore, given that $r_t^C > \rho(1)$ and $\Phi(\rho)$ is increasing in ρ, then recalling that r_t^C clears the market so that $\lambda_t F_l(r_t^C) = \mu_t(1 - F_b(r_t^C))$ implies

$$\mathcal{W}_t^{DeFi} < \frac{\lambda_t \cdot F_l(r_t^C)}{1 - F_b(r_t^C)} \cdot \int_{r_t^C}^{+\infty} r dF_b(r) + \lambda_t \int_{r_t^C}^{+\infty} r dF_l(r) = \mu_t \int_{r_t^C}^{+\infty} r dF_b(r) + \lambda_t \int_{r_t^C}^{+\infty} r dF_l(r) = \mathcal{W}_t^{CE}$$

Case 2: $\theta_t \leq \tilde{\theta}$: In this case, we will prove that setting $l_t = b_t = r^C$ maximizes welfare subject to the constraint $\mathcal{S}(l_t) \geq \mathcal{D}(b_t)$ which must hold whenever $\theta_t \leq \tilde{\theta}$. In particular, when $\theta_t \leq \tilde{\theta}$ then for any tuple (l_t, b_t), welfare is given by:

$$\mathcal{W}(l_t, b_t) = \mu_t \int_{b_t}^{\infty} r dF_b(r) - \mu_t \int_{b_t}^{\infty} b_t dF_b(r) + \lambda_t \int_{0}^{l_t} l_t dF_l(r) + \lambda_t \int_{l_t}^{\infty} r dF_l(r)$$

Further noting that,

$$\mathcal{W}(l_t, b_t) - \lambda_t \int_{0}^{\infty} r \; dF_l(r) = \mu_t \cdot \int_{b_t}^{\infty} (r - b_t) dF_b(r) + \lambda_t \cdot \int_{0}^{l_t} (l_t - r) dF_l(r)$$

Then it can be seen that maximizing welfare with respect to (b_t, l_t) is equivalent to solving the following optimization problem:

$$\max_{l_t, b_t} \mu_t \cdot \int_{b_t}^{\infty} (r - b_t) dF_b(r) + \lambda_t \cdot \int_{0}^{l_t} (l_t - r) dF_l(r)$$

s.t. $\lambda_t \cdot l_t \cdot F_l(l_t) = \mu_t \cdot b_t \cdot (1 - F_b(b_t))$, $\mathcal{S}(l_t) \geq \mathcal{D}(b_t)$ where the first constraint arises due to the fact that all interest paid by the borrowers is passed through to the lenders. Note that combining the first constraint with the second implies that $l_t \leq b_t$. Further, using the first constraint, this optimization problem is equivalent to

$$\max_{l_t, b_t} \mu_t \cdot \int_{b_t}^{\infty} r \cdot dF_b(r) - \lambda_t \cdot \int_{0}^{l_t} r \cdot dF_l(r) \tag{A.2}$$

s.t. $\lambda_t \cdot l_t \cdot F_l(l_t) = \mu_t \cdot b_t \cdot (1 - F_b(b_t))$, $\mathcal{S}(l_t) \geq \mathcal{D}(b_t)$. Next note that

$$\frac{d}{db_t} [\mu_t \cdot \int_{b_t}^{\infty} r \cdot dF_b(r) - \lambda_t \cdot \int_{0}^{l_t} r \cdot dF_l(r)] = -\mu_t b_t F_b'(b_t) < 0$$

and therefore letting l_t' and b_t' denote the solution to (A.2), then it must be the case that $l_t' = b_t' = r$. Further, whenever $l_t' = b_t' = r$ then the pass through condition implies $\lambda_t \cdot r \cdot F_l(r) = \mu_t \cdot r \cdot (1 - F_b(r))$ which can only be the case if $r = 0$ or $r = r^C$. Finally, we note that if $r = 0$ then $\mathcal{D}(0) = \mu_t > 0 = \mathcal{S}(0)$ which cannot be the case when $\theta_t \leq \tilde{\theta}$ and therefore welfare is optimized at $l_t' = b_t' = r^C$.

Finally, noting that $\theta_t < \tilde{\theta}$ implies that $\mathcal{S}(l_t^\star) > \mathcal{D}(b_t^\star)$ so that $l_t^\star = \frac{\mathcal{D}(b_t^\star)}{\mathcal{S}(l_t^\star)} \cdot b_t^\star < b_t^\star$, and therefore it must be the case that $\mathcal{W}_t^{DeFi} < \mathcal{W}_t^{CE}$.

□

A.6 Proof of Proposition 4.4

Proof. We have shown that r^C is a bijection and therefore its inverse $(r^C)^{-1}$ exists. Now, let $\tilde{\theta} = (r^C)^{-1}(\rho(1))$. By construction, when $\theta_t = \tilde{\theta}$ then $U_t^\star = 1$ and $b_t^\star = l_t^\star = \rho(1) = r_t^C$ therefore, $\mathcal{W}_t^{CE} = \mathcal{W}_t^{DeFi}$. Furthermore, given that r^C is a bijection then it must be the case that $\tilde{\theta}$ is the unique value such that $\mathcal{D}_t(\rho(1)) = \mathcal{S}_t(\rho(1))$ and therefore by Proposition 4.1 the unique value of $\tilde{\theta}$ such that $\mathcal{W}_t^{CE} = \mathcal{W}_t^{DeFi}$. □

A.7 Proof of Proposition 4.5

Proof. Note that $\theta_t > \tilde{\theta}$ implies that $b_t^\star = l_t^\star = \rho(1)$ due to the fact that $b_t^\star \leq \rho(1)$, $l_t^\star \leq \rho(1)$, $b_t^\star = l_t^\star = \rho(1)$ when $\theta_t = \tilde{\theta}$ and $b_t^\star$ and $l_t^\star$ are weakly increasing in θ_t by Proposition 3.2. Further, we know that r_t^C is strictly increasing in θ_t and therefore $r_t^C > \rho(1) = b_t^\star = l_t^\star$ whenever $\theta_t > \tilde{\theta}$.

We know that whenever $\theta_t < \tilde{\theta}$, then $D(b_t^\star) < S(l_t^\star)$ so that $U_t^\star < 1$ and therefore $l_t^\star < b_t^\star$. In that case, we know

$$\frac{F_l(r_t^C)}{1 - F_b(r_t^C)} = \theta_t < \frac{F_l(l_t^\star)}{1 - F_b(b_t^\star)} < \frac{F_l(b_t^\star)}{1 - F_b(b_t^\star)}$$

and therefore $b_t^\star > r_t^C$. The ambiguity of the relationship between $l_t^\star$ and r_t^C comes from the fact that $b_t^\star = \rho(U_t^\star) > r_t^C$ does not necessarily imply any relationship between $U_t^\star \rho(U_t^\star)$ and r_t^C. □

A.8 Proof of Proposition 5.1

Proof. We proceed by construction. More explicitly, for any $\epsilon > 0$, we construct an interest rate function ρ_ϵ such that the equilibrium utilization rate arising from using this function as the PLF interest rate function (i.e., $\rho = \rho_\epsilon$) satisfies $U^\star(\theta) \in (1 - \epsilon, 1)$ for all $\theta \in [\underline{\theta}, \infty)$. We construct ρ_ϵ explicitly as follows:

$$\rho_\epsilon(U) = \frac{r^C(\underline{\theta}) + \delta_\epsilon}{1 - U} \cdot \epsilon \tag{A.3}$$

where δ_ϵ is defined as the unique solution to the following equation:

$$1-\epsilon = \underline{\theta}\frac{1-F_b(r^C(\underline{\theta})+\delta_\epsilon)}{F_l((1-\epsilon)\cdot(r^C(\underline{\theta})+\delta_\epsilon))} \tag{A.4}$$

Note that δ_ϵ is always uniquely well-defined because the right hand side of Equation (A.4) is larger than $1-\epsilon$ at $\delta_\epsilon = 0$ and strictly decreases to 0 as $\delta_\epsilon \to \infty$.[6]

To establish the stated result, we must prove that the equilibrium PLF Utilization rate, $U^\star(\theta)$, implied by $\rho = \rho_\epsilon$ satisfies $U^\star(\theta) \in (1-\epsilon, 1)$. To demonstrate this result, we define:

$$\Delta_\epsilon(x,\theta) = x - \min\{\ \theta \cdot \frac{1-F_b(\rho_\epsilon(x))}{F_l(x\cdot\rho_\epsilon(x))}\ ,\ 1\ \} \tag{A.5}$$

Note that $\Delta_\epsilon(1-\epsilon,\theta) < \Delta_\epsilon(1-\epsilon,\underline{\theta}) = 0$, that $\lim\limits_{x\to 1^-}\Delta_\epsilon(x,\theta) = 1$ and that $\Delta_\epsilon(x,\theta)$ is strictly increasing in x. In turn, continuity of Δ_ϵ in x implies that, for an arbitrary θ, there exists a unique solution $x^\star_\epsilon(\theta)$ to $\Delta_\epsilon(x^\star_\epsilon(\theta),\theta) = 0$ such that $x^\star_\epsilon(\theta) \in (1-\epsilon, 1)$. Then, from Proposition 3.1, recall that $U^\star(\theta)$ is the unique solution to $U^\star(\theta) = \min\{\ \theta \cdot \frac{1-F_b(\rho_\epsilon(U^\star(\theta)))}{F_l(U^\star(\theta)\cdot\rho_\epsilon(U^\star(\theta))}\ ,\ 1\ \}$ so that $U^\star(\theta) = \min\{\ \theta \cdot \frac{1-F_b(\rho_\epsilon(U^\star(\theta)))}{F_l(U^\star(\theta)\cdot\rho_\epsilon(U^\star(\theta))}\ ,\ 1\ \} \Leftrightarrow \Delta_\epsilon(U^\star(\theta),\theta) = 0$ implies $U^\star(\theta) = x^\star_\epsilon(\theta) \in (1-\epsilon,1)$ as desired.

□

A.9 Proof of Proposition 5.2

Proof. We apply Proposition 5.1 to generate a sequence of PLF interest rate functions, $\{\rho_{\epsilon_n}(U)\}_{n=1}^\infty$, where $\epsilon_n = \frac{1}{n}$ so that the associated utilization rate, $U^\star_n(\theta)$, satisfies $U^\star_n(\theta) \in (1-\frac{1}{n}, 1)$. In turn, we demonstrate that $\lim\limits_{n\to\infty}\mathbb{E}[\mathcal{W}^{DeFi}_{t,n}] = \mathbb{E}[\mathcal{W}^{CE}_t]$ where $\mathcal{W}^{DeFi}_{t,n}$ refers to the realized welfare from a PLF when the interest rate function is ρ_{ϵ_n}.

Note that Proposition 4.3 implies $\sup\limits_{n\in\mathbb{N}}|\mathcal{W}^{DeFi}_{t,n}| \le |\mathcal{W}^{CE}_t|$. In turn, since $\mathbb{E}[|\mathcal{W}^{CE}_t|] \le \mathbb{E}[|\mu_t|]\int\limits_0^{+\infty} r\, dF_b(r) +$

[6] The fact that the right hand side is larger than $1-\epsilon$ at $\delta_\epsilon = 0$ arises due to the definition of $r^C(\underline{\theta})$. More explicitly, as per Equation (4.4), $F_l(r^C(\underline{\theta})) = \underline{\theta}\cdot(1-F_b(r^C(\underline{\theta})))$ which implies $(1-\epsilon)\cdot F_l((1-\epsilon)\cdot r^C(\underline{\theta})) < \underline{\theta}\cdot(1-F_b(r^C(\underline{\theta})))$ and thus $1-\epsilon < \underline{\theta}\frac{1-F_b(r^C(\underline{\theta}))}{F_l((1-\epsilon)\cdot(r^C(\underline{\theta})))}$.

$\mathbb{E}[|\lambda_t|] \int_0^{+\infty} r \, dF_l(r) < \infty$, we apply Dominated Convergence Theorem to yield:

$$\lim_{n\to\infty} \mathbb{E}[\mathcal{W}_{t,n}^{DeFi}] = \mathbb{E}[\lim_{n\to\infty} \mathcal{W}_{t,n}^{DeFi}] \tag{A.6}$$

Then, since welfare is a continuous function of the borrowing and lending rates (see Equations 4.1 and 4.2), it is sufficient for this proof to demonstrate that the lending and borrowing rates at the PLF converge to the competitive equilibrium interest rate because then $\lim_{n\to\infty} \mathcal{W}_{t,n}^{DeFi} = \mathcal{W}_t^{CE}$ which, via Equation (A.6), further mplies $\lim_{n\to\infty} \mathbb{E}[\mathcal{W}_{t,n}^{DeFi}] = \mathbb{E}[\mathcal{W}_{t,n}^{CE}]$ as desired.

To demonstrate that the lending and borrowing rates at the PLF converge to the competitive equilibrium rate, note that, when the PLF interest rate function is ρ_{ϵ_n}, then the equilibrium PLF borrowing rate in period t, $b_n^\star(\theta_t)$, must satisfy:

$$U_n^\star(\theta_t) = \theta_t \cdot \frac{1 - F_b(b_n^\star(\theta_t))}{F_l(U_n^\star(\theta_t) \cdot b_n^\star(\theta_t))} \tag{A.7}$$

Recall that $U_n^\star(\theta_t) \in (1 - \frac{1}{n}, 1)$ implying that $\lim_{n\to\infty} U_n^\star(\theta_t) = 1$. In turn, taking limits as $n \to \infty$ on both sides of Equation (A.7) and using continuity of F_b and F_l implies:

$$1 = \theta_t \cdot \frac{1 - F_b(\lim_{n\to\infty} b_n^\star(\theta_t))}{F_l(\lim_{n\to\infty} b_n^\star(\theta_t))} \Leftrightarrow F_l(\lim_{n\to\infty} b_n^\star(\theta_t)) = \theta_t(1 - F_b(\lim_{n\to\infty} b_n^\star)(\theta_t)) \tag{A.8}$$

Equation (A.8) then implies $\lim_{n\to\infty} b_n^\star(\theta_t) = r^C(\theta_t)$ because this equation uniquely defines the competitive equilibrium interest rate as per Proposition 4.1. Moreover, recall that the PLF lending rate, $l_n^\star(\theta_t)$, is given explicitly by $l_n^\star(\theta_t) = U_n^\star(\theta_t) \cdot b_n^\star(\theta_t)$ so that $\lim_{n\to\infty} l_n^\star(\theta_t) = \lim_{n\to\infty} U_n^\star(\theta_t) \cdot \lim_{n\to\infty} b_n^\star(\theta_t) = 1 \cdot r^C(\theta_t) = r^C(\theta_t)$. Finally, $\lim_{n\to\infty} (b_n^\star(\theta_t), l_n^\star(\theta_t)) = (r^C(\theta_t), r^C(\theta_t))$ implies $\lim_{n\to\infty} \mathcal{W}_{t,n}^{DeFi} = \mathcal{W}_t^{CE}$ which implies $\lim_{n\to\infty} \mathbb{E}[\mathcal{W}_{t,n}^{DeFi}] = \mathbb{E}[\mathcal{W}_t^{CE}]$ as per Equation (A.6), thereby completing the proof.

□