Dilutive Financing*

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Abstract

This paper builds a dynamic model of corporate financing where financial slack arises from bargaining. When financiers with bargaining power extract rent from cash-strapped firms, firms—despite the absence of fixed transaction costs or search frictions—finance in lumps to bargain infrequently, and typically before exhausting internal funds to strengthen outside option. Continuation value directly amplifies rent, rationalizing large cash-holdings of 'growth' firms. Firms with robust financing access preserve internal funding capacity that substantially exceeds the size of investment opportunities, whereas firms relying on concentrated financiers may externally finance investment despite sufficient funds yet forgo investment with even more funds.

JEL Classification: E22, E44, G32

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Why do firms hold cash reserves at all? This seemingly innocuous question points directly into the heart of financial market imperfections. Suppose hypothetically that any excess expenses could be frictionlessly financed on demand and incrementally from external sources. Firms, then, would always promptly expend any earnings, either through reinvestment or dividend payout, instead of retaining them. Furthermore, the pecking order of financing would dominate. That is, firms would fully exhaust more flexible sources of internal funding, most saliently cash and cash equivalents and also short-term debt and lines of credit, before tapping into costlier ones such as long-term debt and equity issuance. It would be unwise to deliberately underutilize cheaper funds and reserve them for future occasions.

Departure from the above scenarios, often called *financial slack* in the literature, is plainly prevalent. Many firms, despite having long-term borrowing with high interest rates or facing high cost of equity, hold a sizable amount of cash and tradable securities and often leave unused spare capacity for convenient funding with low required yields. As Graham (2022) documents through extensive (and pre-pandemic) surveys, corporate managers across different firm sizes consider financial flexibility as *the* primary factor in capital structure decisions, and its weakening, prompted by low current profits and cash-holdings, as the main driver of underinvestment. It is evident that understanding the origin and nature of financial slack has important economic implications concerning corporate investment and stock returns.

Theoretically, the present question is, at its core, a classic problem of return dominance, where agents hold an asset position that is strictly dominated in return by another. The canonical explanation, first proposed by Baumol (1952) and Tobin (1956), is that there is a *fixed transaction cost* associated with trading a higher-return illiquid asset for a lower-return liquid one[1]; consequently, agents trade for the liquid asset in intermittent "lumps"—rather than incrementally on demand—to avoid incurring the cost too frequently. While exogenous fixed cost has since been widely adopted across various fields of economics and finance as a standard modeling tool to generate lumpiness, it fails to explain the widespread violation of the pecking order, i.e. raising external financing before fully exhausting internal funds, except by specifying exogenous stochastic variation of the exogenous fixed cost to generate precautionary incentives.

In this paper, I propose an alternative model of financing that explains financial slack as arising solely from *bargaining* in financial markets. When a firm encounters cash rundowns and approaches business 'termination,' such as severe financial distress including bankruptcy, they have large surplus to be generated from a successful financing that averts the forced termination. If financiers have nontrivial bargaining power, then firms must give them a portion of the financing surplus in return, on top of the fair value of the funds being contributed. Due to this excess compensation from bargaining, financing dilutes equity value and thereby becomes *endogenously* frictional to the firm, even when the ideal allocation of frequent financing with zero internal funds is technologically feasible and efficient. Firms,

[1] Bank deposits and pocket cash, respectively, in their original setup; unused capacities for costly external financing and for flexible internal funding, respectively, in the current context.

then, respond by raising a large amount of funds at each financing transaction to reduce the frequency of bargaining, i.e. lumpy financing.

Financial slack, however, not only delays the next bargaining through lumpiness. Firms may also finance "early," i.e. raising funds externally even before running out of more convenient internal funding sources, to reduce the size of rent extraction. To understand why, note that it is proximity to unwanted business termination that enables financiers to extract rent from firms, because without their financing, the proximity is real. The funding cushion left untapped at financing exactly reduces this proximity; it allows firms to respond to unsuccessful bargaining with financiers, off equilibrium path, by pursuing backstop strategies—such as finding other financiers, cutting investment down to mitigate expenses, or even divesting capital to obtain funds—to avoid or delay termination. In other words, sparing the capacity to fund a viable backup plan reduces the portion of firm value that depends on the success of the *current* bargaining, thereby strengthening firms' outside option as they settle the terms of financing vis-à-vis financiers. Consequently, financiers can extract less rent in dilution when firms are financing early with funding cushion—i.e. 'financial flexibility.'

Optimal financial slack, then, counterbalances the two prongs of reducing financing dilution, one in frequency through lumpy financing and the other in size through early financing, against the opportunity cost of financial slack. This may include carry costs[2] of corporate cash reserves, term spreads on bonds with different maturities, equity premium relative to debt financing, etc. In short, financial slack is firms' *costly bargaining tool* against financiers.

This framework has two key predictions. First, financial slack increases in '*price-to-earnings*' ratio. In other words, firms with higher continuation value relative to the current cash inflow maintain relatively greater financial flexibility and raise financing less frequently. Such firms have greater surplus from financing that averts damage to the high value due to financial distress, and thus face greater rent extraction, *ceteris paribus*, as a fraction of the larger surplus. As such, these 'growth' firms employ greater financial slack to reduce dilution.

And the effect is more pronounced if firms are investing intensively. Investment is an act of reducing current cash flow to raise continuation value. It thus simultaneously makes dilution greater and more frequent. Financial slack, therefore, is even greater for firms with a similar price-to-earnings ratio but higher investment expenses.

Second, early financing, i.e. retention of internal funding capacity when financing externally, endogenously compresses the size of dilution and, as such, may arise even without any precautionary motive to 'avoid financing when its cost spikes up.' Even when there is no risk of such an impending liquidity crisis, firms may still finance early because doing so creates distance to termination enabling them to pursue backstop strategies upon bargaining failure, so that their outside option is strengthened against financiers in bargaining.

Two auxiliary predictions of the model further enrich the key predictions above. First, at external financing firms may maintain funding cushion that *strictly exceeds* contingent

[2]Firms' inefficiency in directly managing a portfolio of liquid financial securities—either low skill for asset management or presence of agency frictions—relative to the overall financial sector.

funding needs for lumpy investment opportunities. For instance, when a firm is waiting for a randomly arriving and fleeting opportunity to acquire a target firm at a given cost, it may always preserve strictly more internal funding capacity than the cost. To clarify, the model allows firms to raise financing right upon the arrival of opportunities, and so the funding cushion would *not* be to prevent missing the opportunity. Rather, the fleeting nature of such an opportunity—merger and acquisition deals are often highly time-sensitive as rivals are also targeting—means that if a firm needs external financing to fund the lumpy investment, the financing becomes highly dilutive because its surplus encompasses the total returns to investment. Moreover, even if the firm earmarked exactly the required amount in internal funding for stochastic investment opportunities, the arrival of an opportunity when its funding capacity is close to the preserved cushion would mean that once it has funded the investment internally, the firm would have nearly run out of funding. The subsequent financing *post investment* would thus become, again, highly dilutive. Therefore, firms may find it optimal to preserve enough financial flexibility so that they have large funding capacity remaining even after internally funding the investment and despite immediate post-investment financing.

On the flip side, firms without much ability to find alternative financiers may choose to forgo investment opportunities if the decision to invest would place firms close to funding depletion that forces external financing. As discussed, financing is highly dilutive for these firms due to their weak outside option. When they are *already* quite close to funding depletion due to accumulated losses, however, the arrival of investment opportunities offers firms a chance to 'turn around' by promptly financing just once to achieve the double goals of funding the investment *and* restoring target funding capacity.

In summary, the framework that I propose seamlessly rationalizes why firms with *robust* access to external financing (e.g. can find alternative financiers in, say, about a week) tend to internally fund sizable investment while maintaining funding capacity in substantial excess of such needs, whereas firms with limited financing access may still occasionally *finance* investment—even though they also forgo investment on other occasions.

Second, dilution may be endogenously amplified when both financial market depth and capital trade liquidity dry up. To understand why, recall that firms finance early when doing so allows firms to respond to bargaining failure by pursuing backstop strategies, thereby strengthening their outside option and improving the bargaining outcome. Such strategies involve finding alternative financiers and/or selling off capital to generate cash. When neither is feasible, there may be little benefit from financing early. Firms, then, may finance highly infrequently and only when internal funding is exhausted, amplifying the size of dilution.

But the very occurrence of such amplification can be avoided if firms maintain robust business fundamentals, in particular revenue streams and *internal* investment. The combination of both factors makes the investment irreversibility constraint *dynamically* less binding in the bargaining context. As such, a positive funding cushion, even when irreversibility statically binds at it, may retain its strategic value. Firms, then, continue financing early, such that the size of dilution remains small and financing still occurs relatively frequently.

Methodologically, the framework I propose is highly tractable, and I establish a tight equilibrium characterization for a rich set of firm technologies, including state dependence. I focus on the key aspect of outside option within the context of financing bargaining in that finding *alternative* financiers takes *nonzero* time duration. The modeling construction of stochastic duration with Poisson rate enables a parsimonious representation of this dynamic bargaining friction such that solution requires keeping track of just a single reservation value function in conjunction with the main value function on equilibrium path. Empowered by the model's tractability, I obtain general comparative statics of financial slack in the key parameters of bargaining power, and establish additional comparative statics in business characteristics for stylized examples with closed-form equilibrium solutions.

In terms of application, it is noteworthy that even minimal bargaining power by financiers results in a substantial increase in financial slack and underinvestment compared to perfectly competitive financial markets. As such, this paper's framework is applicable to a broad class of firms. It can be applied to small firms that face heterogeneous degrees of search friction and finance mostly from a few specialized investors, such as venture capital funds, that they can access with ease. It can also be applied to large firms that, in fulfilling their sizable financing needs, rely on concentrated investment banks that bring together a broad pool of investors through superior firm valuation technology. Bargaining power might vary substantially across firm size and financing environment, but this model's mechanism is likely at work throughout the financial markets as long as they exhibit nontrivial market power.

Main contribution. As discussed, the conventional theoretical view on cash-holdings in economics goes back to the classic theory of money demand by Baumol (1952) and Tobin (1956). In their framework, financial slack arises from fixed transaction costs of withdrawing cash from high-yield sources—bank deposits in their original setup. The trade-off between increasing yield from bank deposits and decreasing the frequency of fixed transaction costs induces lumpy cash withdrawal and the concomitant cash-holdings. The idea of fixed transaction cost has since become the most readily adopted modeling device to generate lumpiness.

In specific application of this standard framework to equity financing, Décamps, Mariotti, Rochet and Villeneuve (2011) study corporate cash-holdings and dividend policy given exogenous cash flow and fixed transaction costs of financing. Bolton, Chen and Wang (2011) add investment into the framework à la Hayashi (1982) to show that marginal value of liquidity, arising due to transaction costs, suppresses optimal investment. These models generate lumpy financing as expected, but do not rationalize early financing. Bolton, Chen and Wang (2013) introduce stochastic transaction cost, replicating early financing during normal times as a precaution to avoid the excessive cost of financing during crisis.

This paper contributes to the literature on equity financing—and also, by implication, to the vast literature that studies lumpiness in general—with a novel understanding of financial slack as indicating the presence of *strategic* frictions in the market. In this new framework that I propose, the 'fixed transaction cost' derives organically from the underlying market structure, specifically the bargaining power of trading counterparties. Put differently, this paper 'microfounds' the standard modeling tool through the lens of bargaining. And it does so in a way that preserves tractability.

Moreover, this paper delivers a novel understanding of early financing. Under the standard fixed-cost framework, early financing is only rationalized as a precautionary choice when the cost is stochastic. Here, firms may finance early even without any exogenous variation in parameters, because doing so may improve their outside option, thereby reducing the financing cost itself. In other words, early financing is what may *reduce* the financing cost in the first place, a novel direction of causality that is absent in the standard framework.

Other related literature. This paper provides a coherent theoretical framework that can be used to explain many empirically documented patterns in corporate cash-holdings—the most salient component of firms' internal funds—in the U.S. For example, Opler, Pinkowitz, Stulz and Williamson (1999) find that firms with higher growth prospects hold more cash. Bates, Kahle and Stulz (2009) show that corporate cash-holdings significantly increased from 1980 to 2006 due not only to increased risk but also to firms becoming more R&D intensive. They document that firms with low cash flow and high Tobin's q have exhibited the greatest increase in cash-holdings, and additionally report that, regarding the hypothesis that an increase in agency problems—à la Jensen (1986) for instance—explains the trend, no consistent evidence in favor can be found. Graham and Leary (2018) document the increased cross-sectional divergence in cash-holdings since 1980s. In particular, smaller firms and those in tech/health industries exhibit greater cash-holding ratios. As discussed, Graham (2022) reports through (pre-pandemic) surveys of chief financial officers that firms consider financial flexibility as the primary factor in capital structure decisions and its weakening as the main driver of underinvestment. The findings of this paper suggest that the theoretically novel insight of bargaining concerns in corporate financing can be a key driver of these observations.

Although constructed primarily in the lingo of equity financing, the strategic framework proposed in this paper offers new insights into firms' *debt* financing as well, especially in terms of maturity management and early refinancing. Froot, Scharfstein and Stein (1993) build a framework of dynamic risk management where firms with stronger investment opportunities refinance early to have funds available for investment. Rampini and Viswanathan (2010) study the collateral channel whereby constrained firms with less collateralizable capital exhaust their debt capacity to maximize current investment and do not engage in maturity management. Mian and Santos (2018) empirically find that more creditworthy firms refinance early with greater procyclicality and early refinancing predicts a particularly strong investment growth. In this paper's setup, the ability of creditworthy firms to easily find alternative creditors implies an early refinancing strategy that is highly sensitive to future firm value, and inability to switch creditors may preclude maturity management *unless* the firm can divest capital efficiently—not for collateral enforcement but as part of backstop cash generation.

On the bargaining side of corporate debt financing is the literature on debt renegotiation.

In Hart and Moore (1998), inefficiency of asset liquidation deteriorates the bargaining outcome for its beneficiary—creditors in case of asset seizure—such that the borrowing firm can strategically default to induce favorable renegotiation. Bolton and Scharfstein (1996) study a setup where the structure of multiple creditors raises the liquidation proceeds in the presence of asset complementarity, but also lowers the chance of finding a firm willing to buy despite information asymmetry. This paper suggests a complementary view wherein the efficiency of liquidation may also influence whether firms wait for financing until default is imminent.

This paper complements the literature on capital structure under dynamic contract. DeMarzo and Fishman (2007a, 2007b) characterize optimal contract on firm financing and investment under agency problems. The predictions of their model include, among others, delayed dividend payout and positive relation, which is strong for small firms in particular, between current investment and payoff-irrelevant past cash flows. DeMarzo, Fishman, He and Wang (2012) expand the framework with the q theory of investment à la Hayashi (1982) to endogenize financing friction. They find that financial slack, not current cash flow, is the valid proxy for the friction. This paper provides a novel and highly tractable mechanism that reinforces the above patterns.

Hugonnier, Malamud and Morellec (2014) study search frictions in financial markets. In their model, financial flexibility has liquidity value of sustaining business until successful match. Search friction directly generates lumpy financing and also early financing for precautionary motives.[3] Their model also implies greater financial slack for firms with higher equity value, exactly because these firms have more to lose from failure to finance. In comparison, this paper is more relevant for firms with direct but limited access to certain financiers, either the so-called 'bulge-bracket' investment banks or a venture capital fund with specialized expertise in a particular industry and startup stage.

Theoretically, this paper contributes to the literature on dynamic bargaining with a disciplined framework to endogenize dynamic outside options. Recently, McClellan (2024) studies the classic "split-the-pie" contracting with full commitment when the agent's outside option evolves exogenously and finds, somewhat similar to the present paper, that when the outside option improves, the agent receives a promise of greater reward on equilibrium path and also simultaneously secures more time before having to decide acceptance or termination. In the present paper with limited commitment, the financing environment endogenously places a plausible structure to the evolution of the outside option.

This paper also relates to the literature on strategic tensions between different classes of firm stakeholders. Myers (1977) shows that corporate debt overhang dilutes equity value and suppresses profitable investment. Rajan (1992) posits a firm choosing between a more financially sophisticated creditor possessing bargaining power and a price-taking creditor that lacks additional financial expertise to boost firm value and shows that shareholders' optimal

[3] The optimal financing strategy under search friction alone is to always finance upon any match. Naturally, such a model does not feature either endogenous inactivity—financing occurs intermittently exactly because of the search friction—or a single well-defined financing threshold as firms' nontrivial strategy—funding cushion is whatever amount of internal funds that remain upon any match.

choice is interior. More recently, Admati, DeMarzo, Hellwig and Pfleiderer (2018) show that in absence of commitment to leverage policy, firms ratchet leverage up because deleveraging benefits senior debt at the cost to equity. DeMarzo and He (2021) consider leverage ratchet with endogenous response in credit spreads to generate slow mean reversion in leverage. Donaldson, Gromb and Piacentino (2020) find that a firm's ability to dilute existing creditors with new debt induces a collateral overhang that causes underinvestment. Dangl and Zechner (2021) analyze the trade-off between the benefit of leveraging-reducing commitment device that short-term debt offers and its frequent transaction costs to rationalize negative relations between risk and average maturity.

In addition, this paper's framework provides a complementary perspective to the literature on investment irreversibility, financing friction and productivity. Caggese (2007) explores how misallocation of production factors with varying degrees of irreversibility exacerbates and is exacerbated by the collateral constraint for financing during recessions. Kurlat (2013) focuses on adverse selection in the context of capital reallocation and financial markets, and shows that a negative productivity shock worsens the adverse selection through general equilibrium effects. In the framework of Lanteri (2018), recessions exacerbate irreversibility for downsizing firms because, due to user specificity of capital, the wedge that expanding firms face between the returns to new investment and the value of used capital increases. More recently in Cui (2022), unproductive firms delay divestment and exhaust their debt capacity to optimally time for stochastic liquidation costs; adverse financial shocks increase delays in capital liquidation, deteriorating allocative efficiency. This paper adds to the discussion by illuminating how small variation in productivity may either amplify or suppress—directly even in partial equilibrium—the interaction between irreversibility and financing friction that leads to drastic consequences such as a long-lasting breakdown of financial markets.

On a methodological side, this paper's modeling setup suggests an intriguing innovation to the canonical search friction models of decentralized asset markets. The literature—initiated in the current main framework with the seminal work by Duffie, Gârleanu and Pedersen (2005) and having vastly expanded in many important ways including heterogeneous Nash bargaining weight in Farboodi, Jarosch, Menzio and Wiriadinata (2019)—mostly assumes a stochastic match that instantaneously dissolves. Hendershott, Li, Livdan and Schürhoff (2020) study the effects on trading liquidity of a non-dissolving 'relationship' between a client and a dealer through which multiple transactions take place via Nash bargaining over time, where a bargaining failure, similar to this paper, dissolves the relationship. The underlying lumpiness of trading, however, is exogenously modeled through the standard search friction in the interdealer market and stylized valuation shocks that directly induce clients to seek to trade a unit quantity. It would be interesting to explore how the durability of a match endogenously amplifies lumpiness, including *strategic* non-trading despite gains from trading, under a richer preference heterogeneity of traders such as Üslü (2019) in asset markets.

Overview. The rest of the paper is organized as follows. In Section 1, I describe the core mechanism with a simplified deterministic two-period model as a primer. I then formally set up the model with exogenous cash flow in Section 2. In Section 3, I proceed to characterize the equilibrium, including comparative statics in strategic bargaining parameters, to analyze the mechanism in depth. The analysis is then illustrated graphically in Section 4, ending with comparative statics results that directly segue into introducing investment.

In Sections 5 and 6, I introduce two extensions of endogenous cash flow with investment choice. Section 5 features a stochastic arrival of lumpy investment opportunities, and Section 6 smooth investment with convex adjustment cost. The first extension illustrates why firms with robust financing access maintain financial flexibility in substantial excess of contingent investment needs, while the second explores what factors may drive or prevent endogenous amplification of dilution. Section 7 concludes the paper.

1 Core Mechanism

In this section, I present a stylized and analogical model that, despite its simplicity, precisely captures the core of the mechanism being proposed. Notably, the analytic conditions that induce slack in this basic model will be replicated with the main model in Sections 2 and 3.

Setup. There are two periods with three dates $t \in \{0, 1, 2\}$, where $t = 2$ is the terminal date. Time is not discounted. There is a *crop* that lives for the two periods. In each period, the crop requires a unit amount of *fertilizer* to survive. With any less fertilizer input, the crop dies. A 'long-lived' agent called *farmer* owns the crop. She seeks to sustain the crop until harvest at the terminal date $t = 2$, when it can be sold for a price $\overline{v} > 0$. The crop has no value other than through harvest.

The farmer cannot manufacture fertilizer by herself. There are two *chemists*, each coming to visit the farmer at $t \in \{0, 1\}$ to make fertilizer for her. Each of them incurs a marginal cost, normalized to unity, to manufacture fertilizer. Each chemist possesses bargaining power over fertilizer production when he is visiting the farmer and, when requested to manufacture fertilizer, demands a $1 - \theta$ fraction, $\theta \in (0, 1)$, of surplus from that period's fertilization on top of the manufacturing cost. The farmer has an imperfect technology to store fertilizer: a unit of stored fertilizer at t decays down to $\beta \in (0, 1)$ at $t + 1$. Lastly, I assume $\overline{v} \geq 1 + \frac{1}{\theta} > 2$ so that the farmer chooses to farm.[4]

Denote h_t as the farmer's inventory of stored fertilizer at date t. By backward induction, the farmer purchases from the second chemist $\max\{0, 1 - h_1\}$ units of additional fertilizer

[4] The lower bound is set at $1+1/\theta > 2$ simply to streamline exposition. With $\overline{v} \in (2, 1+1/\theta)$, farming is censored—i.e. the farmer quits farming though it is efficient not to—when $h_0 = 0$ and $\beta < \frac{1}{\overline{v}-1}$. The censoring arises *jointly* from the finite horizon *and* the fact that the second chemist demands rent from the bargaining surplus without deducting the sunk cost of the first-period fertilizer. No censoring arises if either (less plausibly) the second chemist demands $1 - \theta$ of *net* surplus after deducting the first-period fertilizer, or (as in the main model) the setup is stationary so that the total continuation value at the second bargaining is also compressed by the same non-deduction going forward. Even under the less plausible scenario of sunk-cost deduction, slack arises when $\overline{v}$ and β are high.

at date $t = 1$. If $h_1 \geq 1$, then autarky is feasible until harvest. If $h_1 < 1$, then the farmer compensates the second chemist for the manufacturing cost plus his rent $(1-h_1)+(1-\theta)\big(\overline{v}-(1-h_1)\big) = (1-\theta)\overline{v} + \theta(1-h_1)$. The farmer's continuation payoff, therefore, is

$$\theta\big(\overline{v} - (1 - h_1)\big);$$

that is, they retain a θ fraction of the fertilizing surplus $\overline{v} - (1 - h_1)$. Due to the storage cost, $h_1 \in \{0, 1\}$ in equilibrium as long as $h_0 \leq 1$. The farmer's continuation payoff at the second period is, therefore, $\overline{v}$ if $h_1 = 1$ and $\theta(\overline{v} - 1)$ if $h_1 = 0$. Denote $v_0^2 := \theta(\overline{v} - 1)$.

Lumpy purchase. First suppose that the farmer has zero initial endowment of fertilizer $h_0 = 0$. At date 0, the farmer decides to purchase from the first chemist either (i) just enough fertilizer to sustain that period, or (ii) enough to sustain both periods. If she chooses (i), then she buys exactly unit fertilizer from the first chemist so that $h_1 = 0$, securing v_0^2 in value. If she chooses (ii), then she buys $1 + 1/\beta$ units of fertilizer, securing $\overline{v}$ at $t = 1$ because bargaining is no longer necessary. Either way, the farmer retains θ of the fertilizing surplus—i.e. the difference between continuation value at $t = 1$ and the cost of fertilizer production at $t = 0$—and so her payoff in either scenario (i) or (ii) is:

$$\text{(i)}: \ \theta\Big(v_0^2 - 1\Big), \quad \text{(ii)}: \ \theta\left(\overline{v} - \left(1 + \frac{1}{\beta}\right)\right).$$

Therefore, the farmer finds it optimal to buy fertilizer (ii) once at $t = 0$ instead of (i) twice if

$$\overline{v} - v_0^2 \geq \frac{1}{\beta} \iff (1-\theta)(\overline{v} - 1) \geq \frac{1}{\beta} - 1, \tag{1}$$

i.e. when the rent that the second chemist will demand $(1 - \theta)(\overline{v} - 1)$ upon (i) is higher than the storage cost $1/\beta - 1$ upon (ii).

Despite wasting $1/\beta - 1$ units of fertilizer, the farmer retains greater surplus by bargaining once instead of twice. The total payoffs for the three agents have decreased by $1/\beta - 1$ relative to the efficient choice of (i). By making fertilizer provision infrequent and lumpy, however, she effectively appropriates the second chemist's surplus for herself (and the first chemist).

Early purchase. Next, suppose instead that $h_0 = 1$, so that the farmer needs to purchase fertilizer only to sustain the second period. If she chooses to (iii) buy at $t = 1$, then her total payoff is $v_0^2 = \theta(\overline{v} - 1)$; note that in the second-period bargaining, she is at the brink of losing her crop, i.e. her reservation value is zero. If she chooses to (iv) buy early at $t = 0$, the farmer is no longer placed at the brink. Instead, the consequence of a failure of the present bargaining is simply a deterioration in her bargaining position in the next date; i.e. her reservation value is now $v_0^2 > 0$. At the same time, early purchase involves a greater cost of fertilizer production $\frac{1}{\beta} > 1$ due to inefficient storage.

Under either scenario (iii) or (iv), the farmer's continuation value upon the one-time bargaining is her reservation value plus her θ share of the fertilizing surplus, i.e.

$$\begin{aligned}
&\text{(iii)} : 0 + \theta\Big(\overline{v} - 0 - 1\Big) &&= \theta\Big(\overline{v} - 1\Big) + (1-\theta)0 &&=: v_0^2, \\
&\text{(iv)} : v_0^2 + \theta\left(\overline{v} - v_0^2 - \frac{1}{\beta}\right) &&= \theta\left(\overline{v} - \frac{1}{\beta}\right) + (1-\theta)v_0^2.
\end{aligned}$$

Therefore, the farmer buys (iv) $1/\beta$ units early at $t = 0$ instead of (iii) one unit at $t = 1$ if

$$\begin{aligned}
v_0^2 - 0 &\geq \theta\left[\Big(v_0^2 - 0\Big) + \Big(\frac{1}{\beta} - 1\Big)\right] \\
\iff (1-\theta)\Big(v_0^2 - 0\Big) &\geq \theta\left(\frac{1}{\beta} - 1\right), \qquad (2)
\end{aligned}$$

This time, the same waste of fertilizer $1/\beta - 1$ occurs as in the case of $h_0 = 0$, when the above inequality holds. Inequality (2) is algebraically equivalent to Inequality (1) but has the important difference in interpretation as previously explained, highlighted through a somewhat redundant expression. If she chooses an early purchase, the farmer raises her reservation value by $v_0^2 - 0$, and lowers the fertilizing surplus by the same amount plus the storage cost $\big(v_0^2 - 0\big) + \big(\frac{1}{\beta} - 1\big)$; she only bears a $\theta \in (0,1)$ fraction of the surplus reduction.

Conclusion. Both forms of 'fertilizing slack'—i.e. choice of (ii), (iv) over (i), (iii) respectively, each of which involves wasted resources—may arise despite the absence of fixed transaction costs or search frictions. The farmer buys enough at a time to sustain multiple periods to reduce the frequency of dilution, and purchases early to boost her bargaining position and reduce the size of dilution by preserving the option to bargain at a later date. In addition, the slack increases in future payoff $\overline{v}$ for both scenarios because dilution is the sharing of it. With a higher terminal payoff, dilution becomes costlier to the farmer even as the cost of wasted fertilizer $1/\beta - 1$ is fixed.

This model captures the essence of the framework that I propose in this paper, but the derivation quickly loses tractability as the number of periods increases or stochastic elements are introduced. Therefore, I transition to the formal setup in continuous time to enable more precise and effective analysis. Despite the generalized setup, I recover both conditions for slack—(1) asymptotically in Section 3.1 and (2) in the exact same form in Section 3.2.

2 Model

In this section, I formally set up a theoretical model of corporate financing with dilution given exogenous business cash flow profile—a Lucas tree with cash endowments. After the setup, I will then analytically characterize the equilibrium in Section 3 and graphically illustrate it in Section 4, ending with core insights that motivate introducing endogenous cash flow with investment choice in Sections 5 and 6.

Environment. Time is continuous and infinite $t \in [0, \infty)$. Every agent is risk neutral and has a common time discount rate $\rho > 0$. There is a *business* owned by a group of agents called *shareholders*. The business has an underlying cash flow profile, to be discussed shortly in Section 2.1, and holds *internal funds* (or 'funds') $h_t \geq 0$ to which cash flow accrues. Internal funds earn a yield at rate $r \in [0, \rho)$.

The spread $\rho - r > 0$ is the carry cost of internal funds. With h_t interpreted as holdings of cash and cash equivalents, the cost may be either literal inefficiency of financial asset management or a reduced-form representation of the presence of agency problems due to corporate managers' ability to stash cash, or both. With h_t viewed as unused capacity to raise funds relatively cheaply and flexibly (i.e. free of bargaining), the cost represents the difference in the required yields, e.g. term structure across debt maturities or equity premium.

Shareholders may frictionlessly receive *non-negative* dividend. If the internal funds are depleted without immediate financing, the business terminates with zero liquidation value.[5]

Financing bargaining. To avoid termination, shareholders must regularly finance the business. Shareholders are assumed penniless, and hence must raise additional business funds from deep-pocketed outside agents called *financiers*. The financing market is modeled as a bilateral Nash bargaining without direct search frictions. That is, shareholders can exactly choose when to bargain with financiers for funds. Shareholders have *Nash bargaining weight* $\theta \in (0, 1)$ and financiers $1 - \theta$. To abstract from debt structure and financing history, I assume that financiers receive a proportional ownership stake in the business in compensation for the funds that they contribute; accordingly, financiers upon financing join the group of shareholders on a pro-rata basis and become penniless.[6]

Outside option. If shareholders can walk out from bargaining and immediately find alternative financiers to raise funds from, bargaining is trivialized as they possess a credible 'take-it-or-leave-it' technology. Shareholders are therefore assumed to wait a strictly positive time lag until the next financing if they walk out from the current bargaining. I use a technical term *exclusion* to denote this time lag for alternative financing. Exclusion can be either perfect or imperfect. If perfect, excluded shareholders will continue the business until internal funds are depleted, at which point the business has to terminate. Generally, excluded shareholders become *re-included* into the financing market at a Poisson arrival rate $\gamma \geq 0$ that parametrizes the *accessibility of alternative financing*. That is, shareholders face a stochastic time lag until finding another financing counterparty. Re-inclusion means regaining the ability to finance, and re-included shareholders may choose not to finance immediately.

The above modeling construction calls for a couple of comments. First, while the exclusion assumption may appear somewhat arbitrary and its necessity contingent on the continuous-

[5] More general liquidation will be addressed in Section 6 in the context of divestment.

[6] The assumption of penniless shareholders and deep-pocketed financiers who, upon financing, become penniless allows the model to focus on *external* financing. It is without loss as long as financiers' commitment to fund the business has an upper bound and involves inefficient interim fund management.

time setup, it has a concrete conceptual counterpart in a discrete-time version of the model, such as the one in Section 1. In discrete time, shareholders still possess a take-it-or-leave-it offer if they can re-bargain with other financiers in the same time period. A natural modeling choice would be to exclude shareholders for that period if they walk out and re-include them in the next period. In continuous time, I could similarly let exclusion last a deterministic time lag to avoid trivializing the bargaining. The stochastic duration of exclusion, given as Poisson arrival, achieves the same goal and additionally generates tremendous tractability by requiring the model to keep track of just one more value function—i.e. reservation value—in addition to the main one on equilibrium path.

The second comment concerns the introduction of what is essentially search friction only for off equilibrium path. It can be thought of as a reduced-form way of allowing corporate managers to predict whether they will need financing in a near future and thus start engaging with financial intermediaries, such as investment banks, in advance. At the same time, firms are prohibited from engaging with two or more intermediaries for the purpose of pitting one against another to induce Bertrand competition. The combination of the visibility of short-term cash forecast and the prohibition of double engagement rationalizes suppressed search friction on equilibrium path but its latency still influencing the on-path bargaining outcome.

More concretely, this latent search friction can be interpreted as either (i) direct access to only a handful of specialized financiers, such as venture capital funds with specialty in a particular sector and particular startup stage, or (ii) time lag in financing that arises because of necessary due diligence process. These two interpretations can also be jointly employed in capturing how large firms finance from the concentrated investment banks that syndicate dispersed investors through their superior firm valuation technology.

It is worth noting that the present modeling construction of exclusion embeds double-layered conservatism with respect to the above interpretations. First, all financiers are assumed to have the same cost of funding ρ. In practice, alternative financiers are likely to have higher funding costs, consistent with their *not* being primary financiers in the first place. Second, supposing shareholders have walked out from bargaining—off equilibrium path—and found alternative financiers, the outside option at this off-path bargaining still involves the same stochastic time lag γ until finding the *next* alternative financiers. In practice, it is reasonable that the time lag of finding the next alternative financiers should increase in the number of past bargaining failures. In short, the present setup with $\gamma > 0$ gives an upper bound on shareholders' outside option relative to firms in practice that can expect to find '*second*-best' financiers in $1/\gamma < \infty$ time periods. Later in Sections 5 and 6 where I let $\gamma = 26$ or 52 so that alternative financiers can be found in 'a week or two,' the model's predictions of financial slack and, in Section 6, underinvestment will therefore be a *lower* bound.

Conceptually, one may think of shareholders' Nash bargaining weight θ and accessibility of alternative financing γ as their *static* and *dynamic bargaining power*, respectively. θ is static in essence since it takes as given how willing shareholders are to walk out from unfavorable bargaining terms. γ is dynamic as this willingness to walk out is strengthened by the ease of finding an alternative bargaining counterparty in the future. Similarly, $1-\theta$ and $1/\gamma$ can be considered financiers' static and dynamic bargaining power. The distinction between static and dynamic nature of θ and γ will be revisited in Section 3 with discussion of Proposition 3.

2.1 Cash flow profile

The business has an exogenously given underlying cash flow profile. The time-t cash flow is

$$\mu \, dt + \sigma \, dB_t,$$

where $\mu \in \mathbb{R}$, $\sigma \geq 0$, and B_t is a standard Brownian motion representing cash flow volatility; one of the two main examples in Sections 2 and 3, introduced right below, will have $\sigma = 0$, i.e. no Brownian motion. In addition, the business may 'succeed' at a random time following Poisson rate $\lambda \geq 0$. Upon success, shareholders receive a one-time liquidating dividend

$$\Pi + h_t,$$

where $\Pi \in \mathbb{R}$ is the terminal payoff.[7] For concreteness, consider two stylized examples to be explored graphically in Section 4. I intentionally modify notations for μ in each example to better align with their respective interpretations.

1. The business is a *startup* that incurs a constant flow *expense* $\kappa \, dt$ with $\kappa > 0$, until success arrives at a Poisson rate $\lambda > 0$. Upon success at t, the business earns a terminal payoff $\Pi > 0$ and ends with a liquidating dividend $\Pi + h_t$.
   - $\mu =: -\kappa < 0, \ \sigma = 0, \ \Pi > 0.$
2. The business is an *operating firm* with flow profit $\pi \, dt + \sigma \, dB_t$, where $\pi, \ \sigma > 0$.
   - $\mu =: \pi > 0, \ \lambda = \Pi = 0.$

Π in startups captures the future value of business that is orthogonal to the current cash flow, whereas π in operating firms determines both future value and current cash flow. It will generate some interesting contrast in comparative statics in Section 4, motivating the introduction of investment choice in Sections 5 and 6.

The following assumption is made on the cash flow parameters $(\mu, \sigma, \lambda, \Pi)$.

Assumption 1. *$\mu + \lambda\Pi > 0$. If $\sigma = 0$, then $\mu < 0$ and $r < -\frac{\mu}{\Pi}\left(1 + \frac{\rho}{\lambda}\right)$.*

The first part, which can be phrased Profitability Assumption, gives a positive net present value $\frac{1}{\rho+\lambda}\left(\mu + \lambda\Pi\right) > 0$, ensuring that the business is inherently worth undertaking. The second part ensures that in the absence of cash flow volatility $\sigma^2 = 0$, (i) flow loss still occurs with nonzero probability($= 1$) $\mu < 0$, and (ii) it is never optimal to prevent fund depletion

[7] If $\Pi < \frac{\mu}{\rho}$, then 'success' would be a misnomer.

entirely by maintaining sufficiently high internal funds so that $rh_t \geq -\mu$. Note that $\mu < 0$ implies $\lambda, \Pi > 0$ by Part 1; otherwise it is optimal to immediately terminate the business.

I will refer to $(\mu, \sigma, \lambda, \Pi)$ as *business parameters* and (θ, γ) as *strategic parameters*.

2.2 Dividend payout and HJB equation

By risk-neutrality, optimal dividend policy is a payout threshold, i.e. $\overline{h} \geq 0$ such that dividend payout equals $dD_t = \max\{0, h_t - \overline{h}\}$, where D_t is cumulative dividend. Letting V denote shareholders' value function, $\overline{h}$ satisfies $V'(\overline{h}) = 1$ and $V''(\overline{h}) = 0$ by smooth pasting and super contact conditions,[8] respectively. Concretely, $V'(\overline{h}) = 1$ equalizes the marginal values of internal funds and dividend payout, and $V''(\overline{h}) = 0$ means that risk-neutral shareholders choose to receive dividend only when they are indeed *locally* risk-neutral with respect to volatility in the firm's funding capacity.

Next, shareholders' inactive value function, i.e. when they are neither financing nor receiving dividend, satisfies the following HJB:

$$\rho V(h) - rhV'(h) = \mathcal{H}(V)(h), \text{ where}$$
$$\mathcal{H}(V)(h) := \lambda\Big(\Pi + h - V(h)\Big) + \mu V'(h) + \frac{1}{2}\sigma^2 V''(h). \tag{3}$$

Here, the term $rhV'(h)$ denotes the change in value due to the yield on internal funds, and $\mathcal{H}(V)$ represents the cash flow profile; the first term is expected change in value from terminal liquidating dividend upon λ, and the last two terms the expected change in value due to non-terminal running cash inflow.

Note that $\mathcal{H}$ is a linear operator on value function, because cash flow is exogenous. Linearity makes analysis substantially transparent by canceling out $\mathcal{H}$ entirely in the context of Nash bargaining, as Section 3 will demonstrate in the workhorse proof of Lemma 2.

2.3 Nash bargaining for financing

Suppose that at time t shareholders have engaged in bargaining with financiers for funds. Denote by V_o shareholders' reservation value function ('outside option'), and by $x \in [0, 1]$ their retained ownership fraction post bargaining. Nash bargaining then solves

$$\max_{x \in [0,1],\ \overline{h} \geq 0} \Big(xV(\overline{h}) - V_o(h_t)\Big)^{\theta} \Big((1-x)V(\overline{h}) - (\overline{h} - h_t)\Big)^{1-\theta}.$$

[8] Super contact condition does not apply when $\sigma = 0$.

The first-order conditions yield straightforward solutions:

$$\begin{aligned}
\overline{h} &\in \arg\max_{h} \; V(h) - h \quad (\implies V'(\overline{h}) = 1), \\
x(h)V(\overline{h}) &= V_o(h) + \theta\Big(V(\overline{h}) - V_o(h) - (\overline{h} - h)\Big) \\
&= \theta\big(V(\overline{h}) - (\overline{h} - h)\big) + (1-\theta)V_o(h). \qquad (4)
\end{aligned}$$

Simply put, $\overline{h}$ maximizes the total net value $V-h$ and x splits the business ownership so that shareholders' retained value $x(h)V(\overline{h})$ equals their reservation value $V_o(h)$ plus a θ fraction of the financing surplus. By risk neutrality and the absence of any cost—in particular, variable—associated with fund injection and dividend payout, $\overline{h}$ here coincides with the dividend payout threshold briefly discussed in Section 2.2. None of this paper's results depend on this identity in any substantive manner, but it enhances exposition by reducing the number of endogenous objects to keep track of. Note the implicit restriction to Markov perfect equilibrium in that the newly financed business value $V(\overline{h})$ derives from the ahistorical value function $V(\cdot)$.

Crucially, recall that shareholders can choose when to finance. Therefore, if it is optimal to finance at h, then $V(h) = x(h)V(\overline{h})$ i.e. their continuation value is given by immediate financing. This is not true for canonical search friction models, because the value function is formulated without a present match and so does not encompass the surplus from the ability to bargain immediately. In this paper, if it is optimal to finance at h, Nash bargaining (4) implies that $V(h)$ is a convex combination of $h \mapsto V(\overline{h}) - (\overline{h} - h)$ and $h \mapsto V_o(h)$. The convex combination that Nash bargaining *without* explicit search friction represents means that all the linear HJB terms in Equation (3) $\mathcal{H}(V)(h)$—that is, the entire right-hand side of (3)—can be conveniently cancelled out, enabling substantial transparency of analysis in Section 3.

But what does this convex combination represent in terms of economics? In equilibrium, shareholders' value when they *have chosen* optimally to finance at h is determined *exclusively* by (i) the value of the newly financed firm $V(\overline{h})$ net of fund cost $\overline{h} - h$ and (ii) their reservation value at the current funding capacity $V_o(h)$—i.e. what happens upon bargaining success versus failure. And these two different scenarios involve the same exogenous running cash inflow to the business as described in Section 2.1. Conceptually, as Nash (1950) formalized in the axiom of invariance to positive affine transformation, anything that does not depend on the success or failure of a bargaining must not affect its outcome. As will be concretely analyzed shortly in Section 3.2, there is a sense in which, due to shareholders' ability to optimally time the financing, the exogenous running cash inflow serves *locally* as a parallel vertical shift of the two values. Hence, the bargaining outcome is, under the appropriate context, invariant to it. Convex combination in Equation (4) will mathematically replicate this axiomatic invariance.

3 Analysis

In this section, I analytically characterize the equilibrium, including comparative statics in key strategic parameters. Equilibrium analysis begins with existence and preferably uniqueness, and the following lemma establishes both.

Lemma 1 (Unique existence)**.** *There uniquely exist V and V_o, the equilibrium value function of shareholders and their reservation value function.*

Proof. Immediate from contraction mapping theorem. See Appendix A.1 for details. □

With the assurance of existence and non-multiplicity, I proceed by demonstrating lumpiness of financing in Section 3.1, characterizing optimal financing strategy in Section 3.2, analyzing costs and benefits of financial slack in Section 3.3, and establishing comparative statics of financial slack with respect to strategic parameters in Section 3.4.

The analysis in the present section will be graphically illustrated in Section 4.

3.1 Lumpy financing

Let $\overline{h} = \inf\{\arg\max_h \ V(h) - h\}$. Whenever $h_t > \overline{h}$, the business will immediately pay out dividend $h_t - \overline{h}$ to shareholders.[9] So $h \mapsto V(h) - h$ is constant above $\overline{h}$, and therefore choosing $\overline{h}$ as the infimum is without loss of generality. Next, define $B \subset [0, \overline{h}]$ as the set of internal funds h at which shareholders find it optimal to finance in equilibrium. Obviously, $0 \in B$ since $\theta > 0$ whereas liquidation value upon termination is zero. The following result establishes that financing is always 'lumpy,' indicating the emergence of an endogenous friction.

Proposition 1 (Lumpy financing)**.** *Financing is lumpy and intermittent, i.e.* $\sup B < \overline{h}$.

Proof. Suppose not, i.e. at $h_t = \overline{h}$, shareholders find it optimal to finance. By Equation (4),

$$V(\overline{h}) = x(\overline{h})V(\overline{h}) = \theta V(\overline{h}) + (1 - \theta)V_o(\overline{h}).$$

Since $\theta < 1$, the above is equivalent to $V(\overline{h}) = V_o(\overline{h})$. But this cannot be true since $\theta > 0$ and $\gamma < \infty$. This is because (i) there is always some finite time interval over which the probability of internal funds being depleted at some point within it, barring additional fund injection, is nonzero by Assumption 1, and (ii) for any finite time interval, the probability that no re-inclusion occurs within it is also nonzero since $\gamma < \infty$. Therefore, there is always a positive probability that today's exclusion brings about depletion-prompted business termination before a finite time interval, while without exclusion, a strictly positive value would have been retained even at depletion since $\theta > 0$. Hence, $V(\overline{h}) > V_o(\overline{h})$, a contradiction. □

[9] This includes the new financier-turned shareholders, pro rata to the newly acquired ownership stake.

Financial friction and the resulting lumpiness in this paper's framework arise solely due to bargaining. This is clear from writing financiers' rent using Equation (4) as

$$(1-\theta)\Big[\underbrace{\Big(V(\overline{h})-V(h)-(\overline{h}-h)\Big)}_{Y}+\underbrace{\Big(V(h)-V_o(h)\Big)}_{Z}\Big].$$

In the above, Y is the social surplus from financing and Z the cost of exclusion; preclusion of 'take-it-or-leave-it' directly implies that $Z > 0$. Financiers receive as their surplus the $1-\theta$ fraction of not just Y, but also Z. Since $(1-\theta)Z > 0$ is bounded away from zero,[10] the financing rent above does not vanish[11] with the financing amount $\overline{h}-h \to 0$, even though Y does. At the same time, $\overline{h}-h \to 0$ implies infinite financing frequency, i.e. paying a $1-\theta$ fraction of $Z > 0$ infinitely often. Incremental financing, therefore, is never optimal in the continuous-time limit.[12]

Note that the strict positivity of financing rent derives exactly from nontrivial bargaining. In other words, if *either* $\theta = 1$ *or* $\gamma \to \infty$ so that bargaining is trivialized, then there is no rent. It is obvious from the above expression that $\theta = 1$ implies zero rent. Suppose $\theta \in (0,1)$ but $\gamma \to \infty$ so that $Z(h) \to 0$ pointwise for any $h > 0$. If it is optimal to finance at $h > 0$,

$$V(h) = x(h)V(\overline{h}) = V_o(h) + \theta\big(Y(h) + Z(h)\big) \to V(h) + \theta Y(h),$$

where the last limiting expression is because $Z(h) \to 0$ if and only if $V_o(h) \to V(h)$.[13] Consequently, $Y(h) \to 0$ as well, and the rent $(1-\theta)\big(Y(h) + Z(h)\big)$ vanishes for all $h > 0$.

3.2 Early financing and backstop strategy

Next, let us characterize the optimal financing strategy. Note that since there is no search friction for financing and the cash flow profile does not exhibit discrete jumps conditional on no exogenous termination, any $h < \sup B$ does not occur on equilibrium path after the initial financing. Nevertheless, the shape of B affects shareholders' reservation value at the boundary $V_o(\sup B)$ through subgame perfection, and hence the bargaining *outcome* on equilibrium path. Therefore, it is crucial for the purpose of analysis to characterize the structure of B.

The following lemma establishes that B is an interval, such that optimal financing strategy

[10]The proof of Proposition 1 employs the argument of nonzero probability of depletion before alternative financing to establish the nonzero bound. Nonzero bound is still valid when exclusion has a deterministic duration instead, as in a discrete-time case; the argument just needs to be somewhat more involved, though still with a similar flavor.

[11]As will be seen in Section 4, this limit is always positive but typically quite small.

[12]In relation to Section 1, shareholders are now unconstrained in terms of the financing frequency. In discrete time, there is the natural upper bound on frequency—up to one per period—and so comparison between frequency of rent extraction and carry cost of infrequency is not trivial, as Inequality (1) shows. Continuous time allows infinite frequency, and the marginal carry cost of reducing frequency vanishes asymptotically. The comparison as the frequency asymptotes, therefore, gives an interior solution.

[13]There is some abuse of notation here: in principle, all expressions to the right of the limiting arrow '$\to$' must be expressed as functions of γ and their limits as $\gamma \to \infty$.

is monotone in internal funds h. While this is a technical result, I include an abridged proof of the lemma in the article because its derivation points at the core of the model's transparency.

Lemma 2 (Monotone financing strategy)**.** *If $h \in B$, then $[0, h] \subset B$.*

Proof. Since the optimal strategy on B is immediate financing, it must be that for $h \in B$,

$$V(h) = x(h)V(\overline{h}) = \theta\big(V(\overline{h}) - \overline{h} + h\big) + (1 - \theta)V_o(h). \tag{5}$$

Suppose $B \supsetneq \{0\}$. Since immediate financing at $h_t \in B \setminus \{0\}$ is optimal, it must be preferred to an instantaneously delayed financing. That is, loss from time discounting must be greater than instantaneous yield from drifting value along the cash flow. Therefore, for $h \in B \setminus \{0\}$,

$$\rho V(h) - rhV'(h) \geq \mathcal{H}(V)(h), \tag{6}$$

where $\mathcal{H}(V)(h) := \lambda\big(\Pi + h - V(h)\big) + \mu V'(h) + \frac{1}{2}\sigma^2 V''(h)$ is the HJB operator that represents the exogenous cash flow profile. Lastly,

$$\rho V(\overline{h}) - r\overline{h} = \mathcal{H}(V)(\overline{h}), \tag{7}$$

$$\rho V_o(h) - rhV_o'(h) = \mathcal{H}(V_o)(h) + \gamma\big(V(h) - V_o(h)\big), \tag{8}$$

where $V'(\overline{h}) = 1$ and V_o has the additional HJB term representing re-inclusion upon γ.

Now, let us substitute (5), (7) and (8) into (6). Because $\mathcal{H}(V)(h) = \theta\mathcal{H}(V)(\overline{h}) + (1 - \theta)\mathcal{H}(V_o)(h)$, the substitution cancels out the entire $\mathcal{H}(V)(h)$ from (6). The result is a compact expression of the inequality that holds on B:

$$(1 - \theta)\gamma\big(V(h) - V_o(h)\big) \geq \theta(\rho - r)\big(\overline{h} - h\big). \tag{9}$$

It then takes a few straightforward steps, albeit on top of a somewhat technical proof for strict concavity of V_o, to show that this inequality holds in a single-crossing fashion, such that $h \in B$ implies $[0, h] \subset B$. For details including the single-crossing and strict concavity of V_o (and also of V as an implication), see Appendix A.1. □

Why can the model deliver the cancellation of all exogenous cash flow terms? Suppose that shareholders find it optimal to finance at $h_t > 0$. This implies, at least, that when they compare bargaining immediately and delaying bargaining by an instant dt as a one-shot deviation, they prefer the former. Let us analyze this comparison of immediate financing against instantaneous delay. First, the risk of running out of funds and facing termination in financing due to the dt delay is negligible because $h_t > 0$, and hence irrelevant to the comparison. Furthermore, the instantaneous running cash inflow to the business during $(t, t + dt]$ is not altered; in principle, whether to bargain before or after receiving a given cash flow is, in the canonical terminology of Nash bargaining, simply a question of whether to shift the entire feasible set as well as the disagreement point up or down along that player's dimension, for which the axiom of invariance to positive affine transformation applies.

Immediate financing does create the following three nontrivial changes relative to instantaneous delay: (1) the instantaneous cash inflow during the dt interval induces an instantaneous variation in carry cost because it affects both financing and dividend payout differently in either scenario; (2) there is an additional carry cost because funds are injected at t instead of $t + dt$; (3) shareholders' reservation value at the immediate bargaining improves because of the chance of instantaneous access to alternative financing during $(t, t + dt]$. Note that the first change is of order $(dt+dB_t)\cdot dt$, which vanishes as $dt \to 0$. Shareholders, therefore, assess the effects from the second and third changes, which do not vanish, in deciding whether to delay financing by a dt instant.

And that is exactly what Inequality (9) is. On one hand, the benefit of immediate financing is the chance of instantaneous access to alternative financing $\gamma(V(h) - V_o(h))\ dt$, improving their bargaining outcome by a factor of $1 - \theta$; it raises their reservation value and lowers the total financing surplus by the same amount, but shareholders bear only a θ fraction of the surplus reduction. On the other hand, immediate financing involves an instantaneous carry cost $(\rho - r)(\overline{h} - h)\ dt$, of which shareholders again bear a θ fraction. If the benefit is greater, shareholders choose to finance immediately.[14]

On a more pragmatic note, Lemma 2 allows the equilibrium to be fully characterized by the 's-S' bounds $(\underline{h}, \overline{h})$ where $\underline{h} \equiv \sup B$. At the financing threshold $\underline{h}$, the business restores the target funding capacity $\overline{h}$. Going forward, I will refer to $\underline{h}$ both as 'financing threshold'—mostly when highlighting the dynamics of financing—and 'funding cushion'—when addressing optimal financial slack. 'Early financing' will denote its strict positivity $\underline{h} > 0$. I will also refer to $\overline{h}$ and $\Delta h := \overline{h} - \underline{h}$ as 'funding target' and 'buffer stock' of funds, respectively.

Corollary 1. *Given other parameters, there exists* $\underline{\gamma} \in (0, \infty)$ *such that* $\underline{h} = 0$ *if and only if* $\gamma \leq \underline{\gamma}$. *In particular,* $\gamma = 0$ *always implies* $\underline{h} = 0$.

Proof. It follows mainly from observing that if $\gamma = 0$, then Inequality (9) is strictly reversed for any $h < \underline{h}$, in particular at $h = 0$. See Appendix A.1 for more detailed reasoning. □

Corollary 1 formalizes the logical implication from the proof and discussion of Lemma 2 in that the only reason that shareholders would choose to finance early $\underline{h} > 0$ is that this cushioned funds can be used, off equilibrium path, to secure time to pursue a *sufficiently viable* backstop strategy upon exclusion. Since cash flow is exogenous in the main framework of Section 2, the only available backstop strategy is to access alternative financing. When the backstop strategy is not effective enough $\gamma \leq \underline{\gamma}$, shareholders always wait until depletion for financing. As a preview, underinvestment can also be a backstop strategy. Hence, with investment choice, $\gamma = 0$ does not imply $\underline{h} = 0$, as Section 6 will show.

[14]Despite the general cash flow profile in the current setup, Inequality (9) exactly coincides with Inequality (2) in the two-period model of Section 1; the (net) drop in reservation value from delay is $v_0^2 - 0$ in Section 1 when alternative financing was guaranteed in the next period, and $\gamma\big(V(h) - V_o(h)\big)\ dt$ here when it arrives with a $\gamma\ dt$ probability over $(t, t + dt]$.

3.3 Costs and benefits of financial slack

This part illustrates how shareholders optimize financial slack against dilution. Given a general cash flow profile from Section 2.1, posit the equilibrium $(\underline{h}, \overline{h})$ along with the implied ownership retention $\underline{x}$ at each financing, occurring at the fixed threshold $\underline{h}$. Initiate the dynamics with endowed funds $h_0 = \overline{h}$. Define a counting process $\{n_t\}_{t\geq 0}$ by $n_0 = 0$ and

$$dn_t = \mathbb{1}(h_{t^-} = \underline{h}),$$

where $h_{t^-} := \lim_{s\to t^-} h_s$. The process n_t tracks how many times financing (hence dilution) has occurred over the time interval $(0, t]$. Also define $\{\tau_m\}_{m\in\mathbb{N}}$ by

$$\tau_m := \inf\{t \geq 0 \mid n_t \geq m\}$$

as the associated increasing sequence of stopping times for m^{th} financing; that is, the first financing occurs at $t = \tau_1 > 0$, and so on. Let τ be the stopping time for the terminal success arriving at a Poisson rate λ with terminal payoff Π.[15]

Then, shareholders' net value $V - h$ can be decomposed as: given $h_0 = \overline{h}$,

$$\begin{aligned} V(\overline{h}) - \overline{h} &= \mathbb{E}_0\left[\int_0^\tau e^{-\rho t}\underline{x}^{n_t}\Big(\big(\mu - (\rho - r)h_t\big)\ dt + \sigma\ dB_t\Big) + e^{-\rho\tau}\underline{x}^{n_\tau}\Pi\right] \\ &= \mathbb{E}_0\left[\int_0^\tau e^{-\rho t}\mu\ dt + e^{-\rho\tau}\Pi - e^{-\rho t}(\rho - r)h_t\ dt\right] - \mathbb{E}_0\left[\sum_{m=1}^{n_\tau} e^{-\rho\tau_m}(1 - \underline{x})\Big(V(\overline{h}) - \overline{h}\Big)\right]. \end{aligned}$$

As a reminder, $\underline{x} \in [0, 1]$ is shareholders' retained share fraction at each financing in equilibrium, from Section 2.3. Note that inside the last expectation term on the last line, both $1 - \underline{x}$ and $V(\overline{h}) - \overline{h}$ are constant in equilibrium for all $\{\tau_m\}_{m\in\mathbb{N}}$ due to the time-invariant financing threshold and funding target $\underline{h}$, $\overline{h}$, respectively. Rearranging the equation gives

$$V(\overline{h}) - \overline{h} = \frac{\text{NPV} - \mathcal{C}}{1 + \mathcal{D}}, \tag{10}$$

where

$$\begin{aligned} \text{NPV} :=& \mathbb{E}_0\left[\int_0^\tau e^{-\rho t}\mu\ dt + e^{-\rho\tau}\Pi\right] = \frac{\mu + \lambda\Pi}{\rho + \lambda}, \\ \mathcal{C} :=& \mathbb{E}_0\left[\int_0^\tau e^{-\rho t}(\rho - r)h_t\ dt\right] \equiv \underline{\mathcal{C}} + \mathcal{C}_\Delta \end{aligned}$$

with $\underline{\mathcal{C}} := (\rho - r)\frac{\underline{h}}{\rho+\lambda}$ and $\mathcal{C}_\Delta := (\rho - r)\mathbb{E}_0\left[\int_0^\tau e^{-\rho t}(h_t - \underline{h})\ dt\right]$, and

$$\mathcal{D} := (1 - \underline{x})\mathbb{E}_0\left[\sum_{m=1}^{n_\tau} e^{-\rho\tau_m}\right].$$

[15] $\tau_m = \infty$ if and only if $m > n_\tau$, where n_τ counts the total financing instances before exogenous business termination. Also, $\lambda = 0$ if and only if $\tau \overset{a.s.}{=} \infty \overset{a.s.}{=} n_\tau$.

As Equation (10) shows, net equity value is lower than net present value due to carry cost of financial slack $\mathcal{C}$ and dilution $\mathcal{D}$. Shareholders choose financial slack $(\underline{h}, \Delta h)$ to maximize net value $V(h) - h$, balancing the mitigation of dilution $\mathcal{D}$ against the carry cost $\mathcal{C}$. On one hand, both funding cushion $\underline{h}$ and buffer stock $\Delta h = \overline{h} - \underline{h}$ lower dilution $\mathcal{D}$, by reducing its size $(1-\underline{x})$ and frequency $\mathbb{E}_0\left[\sum_{m=1}^{n_\tau} e^{-\rho\tau_m}\right]$, respectively.[16] On the other hand, financial slack involves carry cost $\mathcal{C} = \underline{\mathcal{C}} + \mathcal{C}_\Delta$, with $\underline{\mathcal{C}}$ for funding cushion $\underline{h}$ and $\mathcal{C}_\Delta$ for buffer stock Δh. As Proposition 1 has shown, $\Delta h \to 0$ fails to drive the size of dilution $1 - x(h)$ down to zero and yet blows its frequency $\mathbb{E}_0\left[\sum_{m=1}^{n_\tau} e^{-\rho\tau_m}\right]$ up to infinity. Shareholders' dynamic optimization between dilution and carry cost, therefore, yields an interior solution $\mathcal{D} > 0$, $\mathcal{C} \geq \mathcal{C}_\Delta > 0$.

Note that funding cushion $\underline{h}$, viewed as a one-dimensional Markov strategy, incurs a greater marginal carry cost $\frac{\rho-r}{\rho+\lambda}$ than buffer stock Δh. This is because $(\rho - r)\underline{h}$ is a fixed flow cost whereas $(h_t - \underline{h}) < \Delta h$ almost always so that $\mathcal{C}_\Delta < (\rho - r)\frac{\Delta h}{\rho+\lambda}$. Consequently, when a marginal increase in $\underline{h}$ fails to sufficiently compress $1 - \underline{x}$ by boosting $V_o(\underline{h})$, shareholders let $\underline{h} = 0$ and resort to $\Delta h > 0$ exclusively in reducing dilution $\mathcal{D}$, as Corollary 1 shows.

Then, when exactly is funding cushion employed, and what does its use imply about financing rent? The following proposition provides an explicit answer.

Proposition 2 (Funding cushion and financing rent)**.** $\underline{h} > 0$ *if and only if*

$$(1-\theta)\gamma > \frac{(\rho - r)\,\overline{h}}{V(\overline{h}) - \overline{h}}, \tag{11}$$

in which case

$$\gamma\big(V(\overline{h}) - V(\underline{h}) - \Delta h\big) = (\rho - r)\Delta h. \tag{12}$$

Proof. For (11), evaluate Inequality (9) at $h = 0$ given $V(0) = \theta\big(V(\overline{h}) - \overline{h}\big)$. For (12), enforce equality on (9) at $h = \underline{h} > 0$ given $(1-\theta)\big(V(\underline{h}) - V_o(\underline{h})\big) = \theta\big(V(\overline{h}) - V(\underline{h}) - \Delta h\big)$. □

Proposition 2 first shows that funding cushion is employed despite its higher marginal cost when the bargaining-adjusted effectiveness of backstop strategy $(1-\theta)\gamma$ is higher than the total relative carry cost burden $\frac{(\rho-r)\overline{h}}{V(\overline{h})-\overline{h}}$. In Inequality (11), the right-hand side does not have the factor of θ as in (9) because it is evaluated at zero funds in the limit. With zero reservation value $V_o(0) = 0$, the flow yield from instantaneous alternative access is $\gamma\big(V(0) - 0\big) = \gamma\theta\big(V(\overline{h}) - \overline{h}\big)$, which is being compared against shareholders' θ share of instantaneous carry cost $(\rho - r)\overline{h}$. Consequently, shareholders with $h_t \to 0$ can simply compare *total* instantaneous carry cost against *total* net value in deciding whether to delay financing until exact depletion.

Second, Proposition 2 demonstrates that with a positive funding cushion, shareholders in equilibrium pay financiers an optimized rent of $\frac{\rho-r}{\gamma}\Delta h$ at each financing; in Equation (12), $V(\overline{h}) - V(\underline{h}) - \Delta h = (1-\underline{x})V(\overline{h}) - \Delta h$ is exactly the excess compensation to financiers above

[16] The dichotomy is heuristic; $1 - \underline{x}$ represents financiers' total compensation, i.e. their rent plus Δh.

the fair value of funds being contributed. This result is due to the fact that

$$(1-\theta)\big(V(\underline{h}) - V_o(\underline{h})\big) = \theta\big(V(\overline{h}) - V(\underline{h}) - \Delta h\big)$$

from Nash bargaining equation (4), with the interpretation that financing rent is determined by the cost of exclusion $V - V_o$ and static bargaining weight θ. Multiplying both hand sides by $\gamma\, dt$ shows that shareholders' gain from immediate financing at $\underline{h}$ is $\theta\gamma\big(V(\overline{h}) - V(\underline{h}) - \Delta h\big)\, dt$, i.e. reducing financing rent by a factor of $\theta\gamma\, dt$. Immediate financing costs shareholders $\theta(\rho - r)\Delta h\, dt$ relative to an instantaneous delay. Optimal *interior* financing threshold equalizes marginal rent reduction and marginal carry cost burden, which gives (12).

To illustrate the trade-off to early financing in more concrete terms, reformulate[17] Inequality (11) as follows: $\underline{h} > 0$ if and only if

$$\frac{(1-\theta)\gamma}{\rho + \lambda + (1-\theta)\gamma} > \frac{(\rho - r)\overline{h}}{\mu + \lambda\Pi}. \tag{13}$$

The right-hand side is the flow carry cost from financing instantaneously earlier than $h = 0$, relative to the average flow of frictionless business value $\mu + \lambda\Pi > 0$. The left-hand side is a fraction that represents the relative effectiveness of dynamic bargaining.

Specifically, there is the time preference $\rho > 0$ that inclines shareholders to delay the cost of dilution. There is also the chance, $\lambda \geq 0$, that the business will exogenously terminate before financing, which has a real option value of averting dilution entirely. These two forces incentivize shareholders to postpone financing as much as possible, i.e. have zero funding cushion. The last force, $(1-\theta)\gamma$, reflects the effectiveness of dynamic bargaining and inclines shareholders towards early financing. This force is strong when financiers have a large *on-the-table* bargaining power, i.e. low θ, but shareholders possess a viable backstop strategy *off-the-table* of finding an alternative bargaining counterparty, i.e. high γ. A sufficiently high $(1-\theta)\gamma$ implies $\underline{h} > 0$ because a backstop strategy is feasible only if shareholders come to the table with positive internal funds; otherwise, immediate termination precludes its implementation. Note that if financiers have a small on-the-table bargaining power i.e. high θ, the cost of dilution is already small, in which case the first two forces may dominate despite high γ.

3.4 Comparative statics of financial slack

Next, let us explore how financial slack varies with strategic parameters (θ, γ). The following proposition establishes tight comparative statics that will be illustrated in Section 4 with the two stylized examples introduced in Section 2.1.

Proposition 3 (Comparative statics in θ and γ)**.**

[17] Solve the HJB (3) for $V(\overline{h})$, given $V'(\overline{h}) = 1,\ V''(\overline{h}) = 0$. Then substitute it into (11).

1. *$\overline{h}$ decreases*[18] *in θ. $\overline{h}$ is constant in γ when $\gamma < \underline{\gamma}$ and decreasing otherwise.*

2. *$\underline{h}$ decreases in θ when $\underline{h} > 0$. $\underline{h} = 0$ is constant in θ above some $\underline{\theta} < 1$.*

3. *When $\underline{h} > 0$, Δh is constant in θ if $r = 0$ and increasing if $r \in (0, \rho)$. When $\underline{h} = 0$, $\Delta h = \overline{h}$ decreases in θ. Δh is constant in γ when $\gamma < \underline{\gamma}$. When $\gamma \geq \underline{\gamma}$, Δh is decreasing in γ if $r = 0$.*

4. *$\overline{h} \to 0$ as either $\theta \to 1$ or $\gamma \to +\infty$.*

First off, note from Part 4 of Proposition 3 that an extremely viable backstop financing strategy $\gamma \to +\infty$ is a near-perfect substitute for a perfect Nash bargaining weight $\theta \to 1$, i.e. first-best. With $\gamma \to +\infty$, shareholders need only minimal funding cushion to access alternative financing, *off* equilibrium path, with minimal time delay and nearly absent risk of fund depletion, thereby extracting almost all surplus on equilibrium path.

In intermediate parameter ranges, these two dimensions of shareholders' bargaining power operate similarly in terms of reducing the 'total' financial slack $\overline{h} = \underline{h} + \Delta h$, as Part 1 of Proposition 3 shows. Since they both directly reduce dilution, either a higher γ (above $\underline{\gamma}$) or θ mitigates the incentive to pile up internal funds, hence incurring the carry cost, before receiving dividend.

But there are interesting contrasts in terms of how each of θ and γ differentially affect (1) funding cushion $\underline{h}$ and (2) buffer stock Δh.

(1) Funding cushion $\underline{h}$. Funding cushion always decreases in static bargaining power θ by a similar reasoning as $\overline{h}$. That is, if surplus extraction $1 - \theta$ is less, shareholders find less incentive to incur carry cost of the funding cushion $(\rho - r) \cdot \underline{h}$ to boost their bargaining position $V_o(\underline{h})$.

In terms of the accessibility of alternative financing γ, however, funding cushion exhibits non-monotonicity. In a neighborhood above $\underline{\gamma}$, funding cushion $\underline{h}$ rises with greater accessibility γ, although it eventually vanishes. With a rising γ above but near $\underline{\gamma}$, the equilibrium is now pushed away from the corner solution of $\underline{h} = 0$. Backstop strategy is still not that viable so that a substantial funding cushion is needed to sufficiently boost the reservation value. At some point, as alternative financing becomes more and more accessible, a smaller and smaller funding cushion is needed to boost the reservation value similarly or more.

(2) Buffer stock Δh. This element clearly illuminates a key but subtle contrast between bargaining weight θ and accessibility of alternative financing γ in that the former is inherently *static* while the latter *dynamic*. Let us restrict the following discussion to $\gamma \geq \underline{\gamma}$ since otherwise Δh would simply equal $\overline{h}$. For clarity of exposition, consider the case of $r = 0$.[19]

[18] Here, 'decreasing'/'increasing' is reserved for strict monotonicity.

[19] A positive internal yield $r \in (0, \rho)$ introduces another channel through which both θ and γ can indirectly affect Δh. This new channel is $\overline{h}$. When $\overline{h}$ goes up, the increased cash inflow from internal yield implies that the same Δh can delay next financing more than before. Therefore, shareholders will re-optimize by lowering Δh somewhat. Since $\overline{h}$ is decreasing in both $\gamma \geq \underline{\gamma}$ and θ, their increase exerts

Shareholders' continuation value $V(h)$ with $h \in (\underline{h}, \overline{h}]$, *in equilibrium*, is determined by two different trade-offs against carry cost of internal funds: how much to delay next financing, and how much to reduce ownership dilution at each financing. Because cash flow is exogenously given in the current setup, financing frequency is completely and mechanically determined by buffer stock Δh. Optimal frequency balances the carry cost of buffer stock $\mathcal{C}_\Delta$ against the given financing rent $\frac{\rho-r}{\gamma}\Delta h$ from Proposition 2. Nash bargaining weight θ does not influence this dynamic trade-off that determines optimal financing frequency. θ is already 'optimized away,' so to speak, by the optimal reservation value $V_o(\underline{h})$ at the interior funding cushion $\underline{h} > 0$, whose marginal cost is constant $\frac{\partial \mathcal{C}}{\partial \underline{h}} = \frac{\rho-r}{\rho+\lambda}$.

Such a dichotomy fails to hold, however, with a change in the viability of backstop strategy $\gamma \geq \underline{\gamma}$. This is because it additionally affects the marginal effective carry cost of improving reservation value $V_o(\underline{h})$ and thereby reducing dilution $1 - \underline{x}$. A rise in γ causes a drop in this marginal cost, bringing about income and substitution effects. First, through the income effect, funding target $\overline{h} = \underline{h} + \Delta h$ falls, as Part 1 of Proposition 3 demonstrates. But this channel, which is also present when θ changes, does not affect Δh by itself. Second, through the substitution effect, the part of the funding target reserved for reducing financing frequency, i.e. Δh, is now reallocated towards reducing the size of dilution, i.e. towards $\underline{h}$. The marginal rise in the frequency of dilution is now better compensated by the marginal drop in the size of dilution, and so Δh optimally falls through this dynamic substitution.

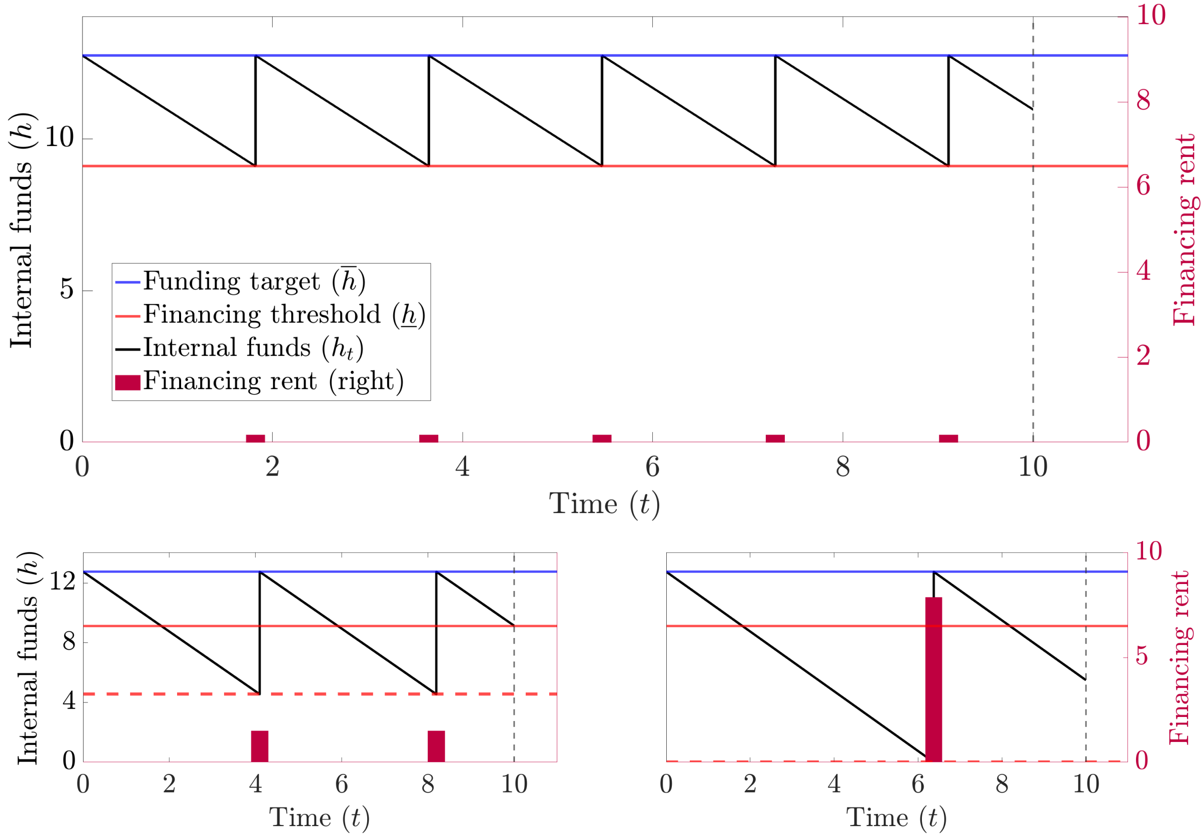
4 Graphical Illustration

Let us now illustrate the above analysis more concretely with the two examples introduced in Section 2.1. In Section 4.1, I discuss the cost and benefit of financial slack in their effect to the frequency and size of dilution. In Section 4.2, I discuss comparative statics in bargaining parameters (θ, γ) as well as business parameters $(\mu, \sigma, \lambda, \Pi)$. In particular, tractability of the setup allows the respective equilibria to be solved analytically through the steps in Appendix B.2, enabling proof of additional comparative statics with respect to the business parameters. It leads to an interesting contrast between current profitability and future payoff, segueing into the investment extensions in Sections 5 and 6.

4.1 Financial slack and size/frequency of dilution

Let us start with the startup example. As a reminder, a startup incurs a fixed expense $\kappa\ dt$ until success arrives at a Poisson rate λ with a terminal payoff Π. Baseline parameters are $\rho = 0.05,\ \theta = 0.5,\ \gamma = 1,\ \lambda = 0.1,\ \Pi = 50,\ \kappa = 2$. Note that alternative financing is allowed $\gamma = 1$. Lastly, $r = 0$ to enable explicit equilibrium solutions.

an upward pressure on Δh through this channel. Since the baseline channel for θ is neutral, Δh now increases in θ. As for γ, this new channel counteracts the baseline channel of dynamic substitution and so gives rise to potential non-monotonicity.



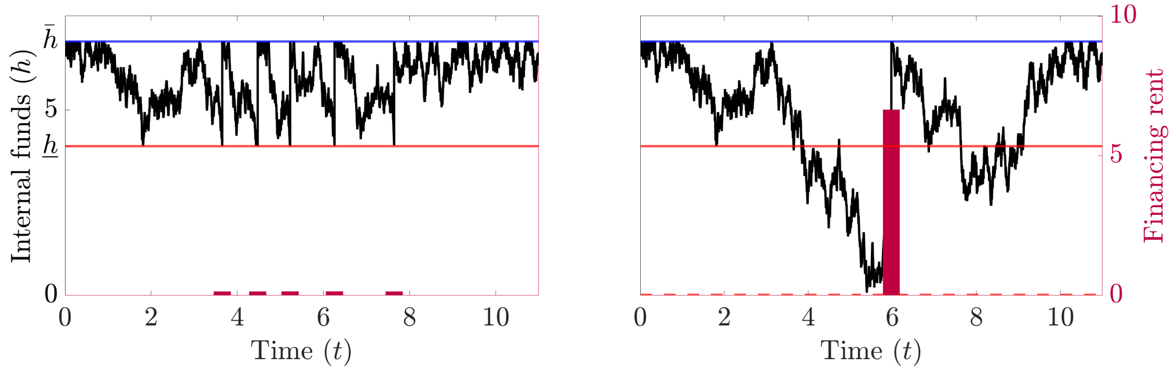
(a) Halved funding cushion $\widetilde{h} = 0.5\underline{h}$ (b) Finance only at depletion $\widetilde{h} = 0$

Figure 1: Financial slack and dilution

The left axis tracks internal funds. The funding target in blue and financing threshold in red bound the evolution of internal funds, which is straightforward due to the cash flow structure of the startup. Whenever financing occurs, i.e. h_t jumps up to the blue line, shareholders incur dilution whose size is the height of the bars in purple measured in the right axis. The vertical dashed line in gray is when the business 'succeeds,' receiving the terminal payoff Π. Bottom two subplots show the financing frequency and the size of dilution with lower financing thresholds that are suboptimal.

Frequency-size trade-off. Figure 1 illustrates the relationship between financial slack and dilution. In the main plot that describes the optimal financing strategy, shareholders finance once every 1.8 periods with a large funding cushion $\underline{h} \approx 9.1$. Upon each financing, shareholders pay financiers the rent of 0.18 in value. If this cost were 'fixed,' the strategy that the main plot illustrates would be strictly dominated by lower financing thresholds $\widetilde{h} < \underline{h}$ as it reduces the frequency of the cost. The bottom two subplots, Figures 1a and 1b, show that these deviations are indeed not optimal. Although frequency indeed decreases, the size of financing cost endogenously magnifies, up to 1.5 with $\widetilde{h} = 0.5\underline{h}$ and even to 7.88 with $\widetilde{h} = 0$. And the pattern is not unique to the specific cash flow profile of startups: the 'operating firm' example with expected revenue $\pi = 1$, volatility $\sigma = 2$ and no 'success' $\lambda = \Pi = 0$ yields a similar pattern, as Figure 2 shows.

Why is financing rent amplified when shareholders raise financing without funding cushion? The answer lies in understanding shareholders' *outside option*. In Figure 3, I decompose valuation for the startup example, assuming a one-shot strategy of immediate financing at



(a) Optimal financial slack

(b) Finance only at depletion

Figure 2: Dilution for operating firms

each h across $[0, \overline{h}]$ on the horizontal axis. The right edge in blue is the funding target $\overline{h}$, and the red vertical line the financing threshold $\underline{h}$. As can be inferred, the immediate financing being considered is, on $(\underline{h}, \overline{h}]$, a deviation.

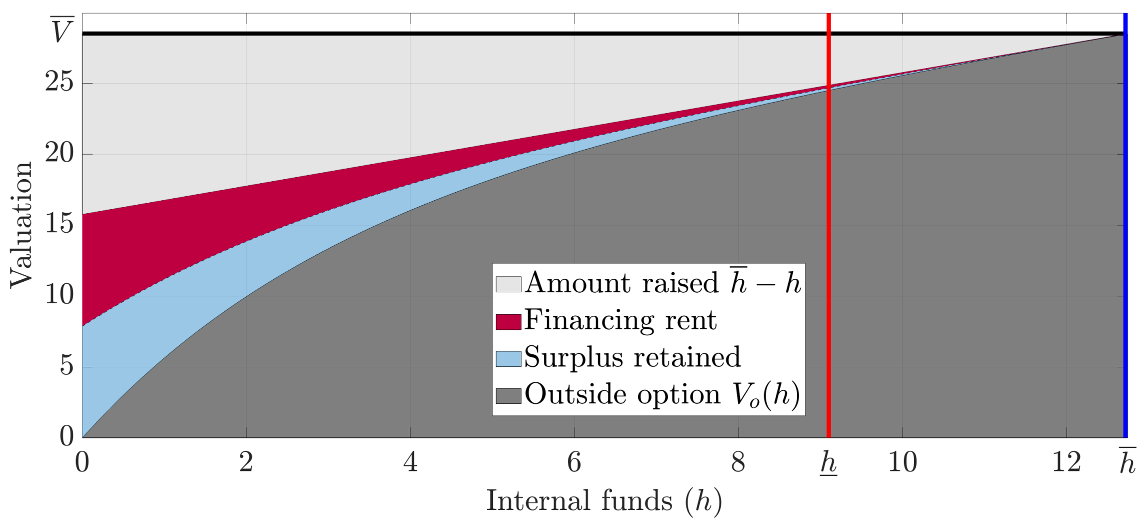


Figure 3: Outside option and dilution

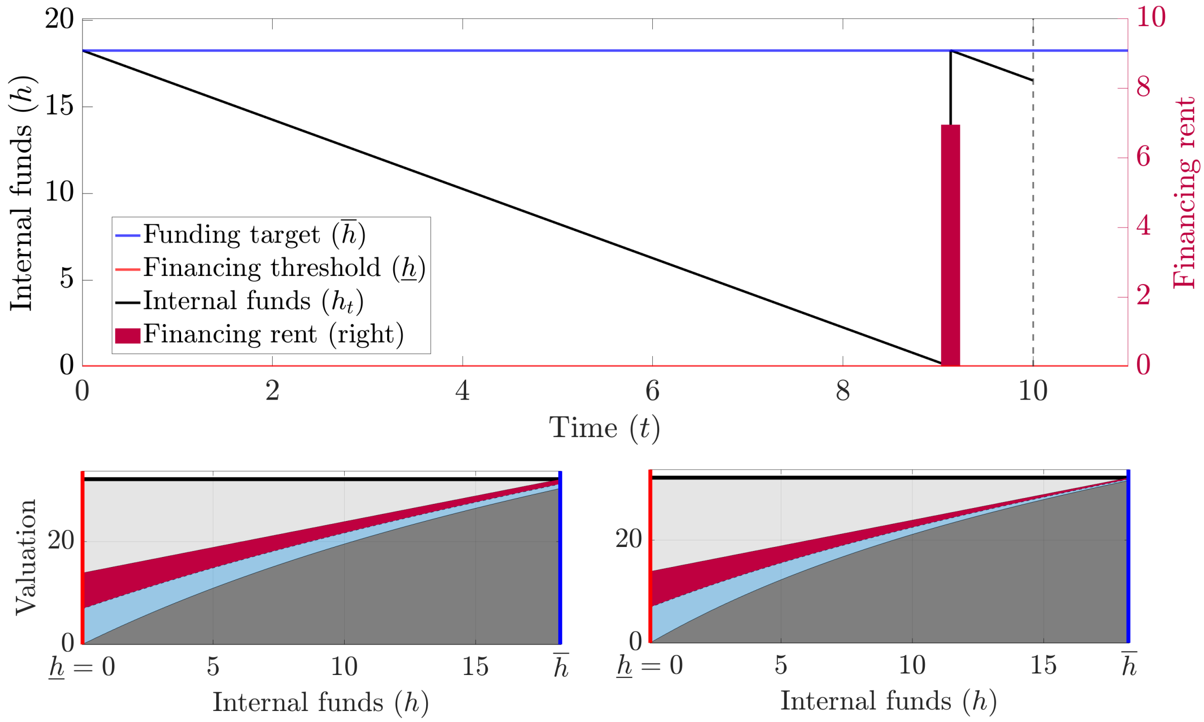
As h rises towards $\underline{h}$, the increase in reservation value $V_o(h)$ greatly compresses financing rent (purple). Shareholders' retained surplus (light blue) is on top of their reservation value (dark shade), which is not being shared with financiers.

The solid black line at the top is value of business right upon financing $\overline{V} \equiv V(\overline{h})$. To attain this post-money value, financiers must provide financing $\overline{h} - h$, represented as height of the light gray area right below it; this amount decreases in h at a unit slope. Financing surplus, however, is not simply the difference between post-money value $\overline{V}$ and 'money' $\overline{h} - h$. Shareholders at bargaining possess outside option $V_o(h)$, represented as height of the dark gray area at the bottom. With a backstop strategy of finding alternative financiers $\gamma = 1$, funding cushion preserved at financing raises the outside option steeply, especially near $h = 0$.

Financing surplus, which is therefore the two colored areas in the middle, is then divided according to the $(\theta, 1 - \theta)$ ratio into shareholders' portion in light blue and financiers' portion in purple. With $\theta = 0.5$, shareholders and financiers always split financing surplus in half.

But with a high h, shareholders' outside option compresses financing surplus. In other words, funding cushion allows shareholders to take a large portion of their value off the bargaining table, as it were, thereby reducing the rent that financiers can extract.

If funding cushion monotonically lowers the size of dilution (which is indeed the case), why is optimal financing threshold $\underline{h}$ not even higher? The answer is frequency. In the startup case, the financing frequency, conditional on no success, is $\frac{1}{\Delta h/\kappa} \approx 0.55$ per unit time period. A higher $\underline{h}$ decreases Δh and thus increases the frequency of dilution, while its size no longer decreases as steeply. As an aside, there is a strictly positive gap even at the top $V(\overline{h}) - V_o(\overline{h}) > 0$, which is merely 0.022—and so not quite visible on the plot—but certainly nonzero. As Proposition 1 shows, this non-vanishing cost of exclusion is the source of lumpy financing, as incremental financing blows the frequency up to infinity.



(a) No alternative financing $\gamma = 0$

(b) Breakeven accessibility $\gamma = \underline{\gamma} \approx 0.131$

Figure 4: Ineffective backstop strategy

The legends for the two subplots are identical to that for Figure 3. As an aside, post-money value $V(\overline{h})$ is higher with $\gamma \le \underline{\gamma} \approx 0.131$ than with $\gamma = 1$ in Figure 3 only because $\overline{h}$ is higher. Net value $V(\overline{h}) - \overline{h}$ is indeed higher with $\gamma = 1$.

Backstop strategy. When, then, is early financing *not* optimal? As Corollary 1 establishes, it is not optimal when backstop strategy is not effective. That is, if it takes very long to find alternative financiers $\gamma \in [0, \underline{\gamma}]$ for some $\underline{\gamma} > 0$, then funding cushion does not boost the outside option enough to justify increases in carry cost and financing frequency. Without early financing, γ is irrelevant since shareholders' outside option at financing threshold is zero $V_o(0) = 0$, and so equilibrium dynamics is invariant to $\gamma \in [0, \underline{\gamma}]$.

Figure 4 illustrates this situation. The main plot at the top describes when shareholders

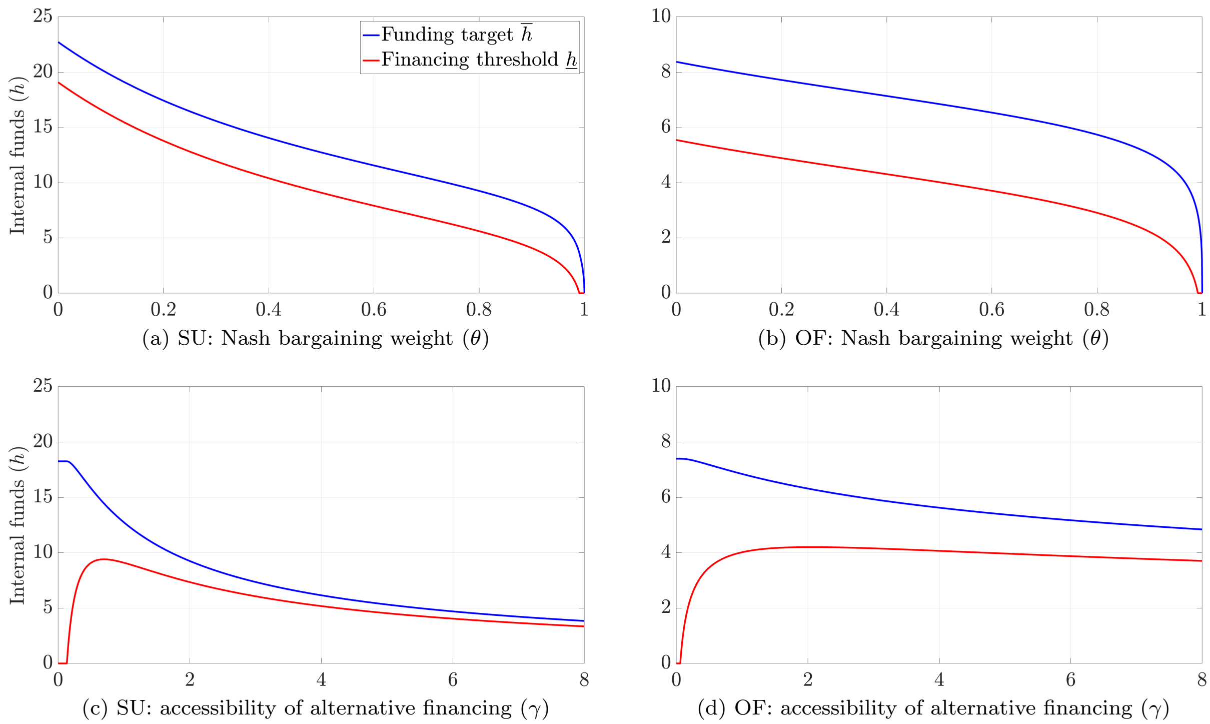


Figure 5: Comparative statics in strategic parameters

Horizontal axis: parameter being varied. Vertical axis: equilibrium $(\underline{h}, \overline{h})$. SU: startup, OF: operating firm. Baseline parameters: $\rho = 0.05,\ r = 0,\ \theta = 0.5,\ \gamma = 1$; startups — $\lambda = 0.1,\ \Pi = 50,\ \kappa = 2$; operating firms — $\pi = 1,\ \sigma = 2$.

find it *optimal* to wait until running out of funding to finance. Compared to Figure 1 where early financing is optimal, shareholders face much larger dilution. They optimize against it by increasing funding target even more $\overline{h} \approx 18.27 > 12.74$ such that this costly financing is necessitated much less frequently.

The bottom two subplots, 4a with $\gamma = 0$ and 4b with $\gamma = \underline{\gamma} \approx 0.131$, show why a low γ induces zero funding cushion $\underline{h} = 0$. Financing rent in either subplot does not diminish in h as sharply as in Figure 3. A lower γ implies a less steep rise in the reservation value V_o. Consequently, the marginal drop in rent at $h = 0$ is not enough to justify the marginal increase in frequency of financing and the greater marginal carry cost of funding cushion as discussed in Section 3.3. Note as an aside that compared to Figure 4a with $\gamma = 0$, Figure 4b with $\gamma = \underline{\gamma} \approx 0.131$ has a greater marginal drop in the size of financing rent at $h = 0$ that renders shareholders marginally indifferent between $\underline{h} = 0$ and a very small $\underline{h} = \varepsilon > 0$.

4.2 Comparative statics

Let us now proceed to comparative statics. I vary strategic parameters (θ, γ) first, and then consider the effect of varaition in business parameters $(\mu, \sigma, \lambda, \Pi)$.

Comparative statics in strategic parameters. Figure 5 illustrates Proposition 3 across both examples. Funding target $\overline{h}$ decreases monotonically when shareholders have better bargaining power, either statically θ or dynamically $\gamma \geq \underline{\gamma}$, eventually converging to zero slack $\overline{h} \to 0$ with either $\theta \to 1$ or $\gamma \to \infty$. Financing threshold $\underline{h}$ also behaves similarly with respect to θ. It shows, however, an interesting non-monotonicity with respect to γ. When the backstop strategy of alternative financing is not sufficiently feasible $\gamma \leq \underline{\gamma}$, funding cushion is not effective at reducing the rent, as Figures 4a and 4b have demonstrated. Financing threshold rises steeply once γ exceeds the breakeven level $\underline{\gamma} > 0$[20] so that shareholders choose to secure sufficient time to pursue the somewhat-feasible backstop strategy as part of the outside option. When the strategy becomes highly feasible, even a small funding cushion is enough to substantially raise the outside option.

Also, note from Figures 5a and 5b that financial slack is substantial even for a minimal but nonzero bargaining weight of financiers $1 - \theta$. In fact, as demonstrated in Appendix C (Propositions C.1.1 and C.2.1[21]), $\partial\overline{h}/\partial\theta \to -\infty$ as $\theta \to 1$.

These results suggest that when financiers are highly specialized in a way that both θ and γ are low, firms that depend on them for funds may exhibit substantial lumpiness without early financing—hence magnified dilution. On the flip side, even when shareholders have large bargaining power e.g. $\theta \lesssim 1$ and $\gamma \gg 0$, financial slack may still be considerable as long as financiers possess even just a little bargaining power $1 - \theta \gtrsim 0$ and $1/\gamma \gtrsim 0$.

Comparative statics in business parameters. In Figure 6, let us start with the contrast between future value Π (6a) and current profitability π (6b) previously hinted at. Figure 6a shows that even though Π is orthogonal to cash rundowns, shareholders pile up larger internal funds when Π is higher. This is because a higher future value of the business increases the size of financing rent, inducing greater financial slack to reduce it.

In contrast, π affects financial slack through two countervailing channels. The first is common with Π in increasing the equity value and hence the size of financing rent. Its effect is manifested in Figure 6b over the region of π where financial slack is upward-sloped. As π grows further, the second channel kicks in through the rising drift of the cash flow. Dilution becomes less likely to occur simply because cash rundowns are less likely. As an aside, note how 6b looks similar to 6d; λ in startups has the same two channels as π in operating firms.

Figure 6c also illustrates this contrast, perhaps more dramatically. The solid lines are where κ changes with all other parameters fixed, so that at $\kappa = 5$, the business is fundamentally valueless $\lambda\Pi - \kappa = 0$. The dashed lines co-vary $\Pi = \kappa/\lambda + 30$ so that business value is fixed. With vanishing business value (solid), financial slack eventually vanishes as cash burn rate increases, because the cost of dilution also vanishes. When business value is fixed (dashed) so that dilution cost does not vanish, slack increases with stronger cash rundowns.

The discussion on current cash flow and future value provides a nice segue into endogenizing the business cash flow with an element of intertemporal substitution. Specifically, investment is an act of reducing the current cash flow to increase the future value of the

[20] For the operating firm case, $\underline{\gamma} \approx 0.06$ is so small that it is barely visible in the plot.

[21] These propositions in the Appendix are stated for the case of $\gamma = 0$. But a sufficiently high $\theta \to 1$ gives $\underline{h} = 0$ so that $\underline{\gamma} > \gamma$, which gives equivalent outcomes as $\gamma = 0$. So the limits hold generally.

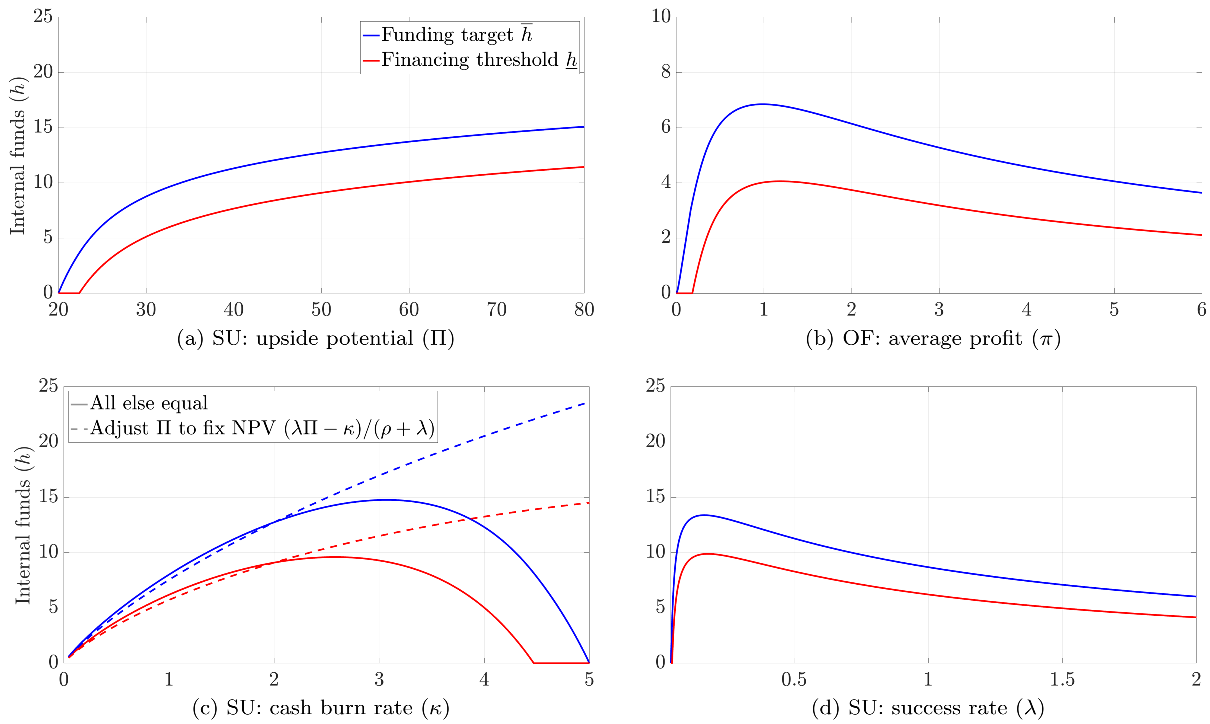


Figure 6: Comparative statics in business parameters

Horizontal axis: parameter being varied. Vertical line: equilibrium $(\underline{h}, \overline{h})$. SU: startup, OF: operating firm. Baseline parameters: $\rho = 0.05,\ r = 0,\ \theta = 0.5,\ \gamma = 1$; startups — $\lambda = 0.1,\ \Pi = 50,\ \kappa = 2$; operating firms — $\pi = 1,\ \sigma = 2$. In Figure 6c, the dashed lines co-vary Π along with κ such that $\frac{\lambda\Pi-\kappa}{\rho+\lambda} = 20$ is fixed.

business. The findings from the two stylized examples suggest that shareholders may face the largest incentive to preserve funds exactly when the return to investment is the highest. How does the incentive to reduce dilution of continuation value through financial slack shape the *shareholder-optimal* investment strategy?

5 Extension I: Lumpy Investment

To build upon the concluding insight from Section 4, I now introduce two models of investment under dilutive financing. The first one in the present section features stochastic arrival of opportunities to make lumpy investment. It highlights how the robustness of financing access γ may have a counterintuitive effect on the ways that firms fund investment.

5.1 Extended setup

Here, I continue to adopt the modeling setup from Section 2 and only modify the cash flow profile in 2.1 to add stochastic investment opportunity.

Cash flow profile. The business has 'normalized' running cash inflow $\pi\ dt + \sigma\ dB_t$, with $\pi, \sigma > 0$. Upon Poisson arrival at rate $\lambda > 0$, the business receives an opportunity to *scale* *up* by a factor of $\eta > 1$ at an *upfront investment cost* of $\xi > 0$; that is, if ξ is paid as a one-time cost, the business cash flow now becomes $\eta\pi\ dt + \eta\sigma\ dB_t$ from then on, with subsequent opportunities to further scale up by η arriving at the same rate of λ. To ensure that investment represents a positive net present value, assume the following.

Assumption 2. *$\pi > \lambda\xi$, $\rho > \lambda(\eta - 1)$, and $\frac{\pi}{\rho} \geq \frac{\xi}{\eta-1}$.*

The second inequality is simply to ensure that business value under the first-best is finite.

Investment choice. Upon receiving the opportunity at rate λ, shareholders with funds h may choose among the following options, each with its corresponding conditional value function:

1. *Fund* the investment internally, with value
$$\eta V\left(\frac{h-\xi}{\eta}\right).$$
2. *Forgo* the investment, with value $V(h)$.
3. *Finance* the investment externally, with value
$$V_o(h) + \theta\Big(\eta V(\overline{h}) - V_o(h) - \big(\eta\overline{h} + \xi - h\big)\Big).$$

A few comments on the setup of investment choice are due. First, the funding value utilizes the convenient features of homogeneity and stationarity in the setup. Second, the option to forgo implicitly assumes that each opportunity is *fleeting*. Third, by the cooperative nature of Nash bargaining, shareholders and financiers upon the third scenario agree to finance not only the investment expense ξ but up to the target funding capacity $\underline{h}$ post investment, so that the relevant financing amount is $\eta\overline{h} + \xi - h$. Fourth, the outside option at financing bargaining upon receiving the investment opportunity entails exclusion as well as loss of the opportunity. Lastly, investment choice upon the outside option is formulated in the identical manner, except that firms cannot finance the investment.

5.2 Illustration

Dynamics of financing and investment. Let us now see graphically how dilution affects shareholders' optimal investment strategy. For the baseline parameters, I let $\rho = 0.07,\ r = 0,\ \theta = 0.5,\ \gamma = 0.3,\ \pi = 1,\ \sigma = 2,\ \lambda = 0.5,\ \xi = 0.7$ and $\eta = 1.1$.

As Figure 7 illustrates, shareholders may sometimes forgo—represented as black square markers—investment opportunities that increases net present value. This occurs when (i) financing is highly dilutive due to weak backstop strategy (in the present case $\gamma = 0.3$), and (ii) funding capacity is somewhat low but sufficiently away from financing threshold. Shareholders in such a situation do not fully internalize the returns to investment because it will likely necessitate financing soon, diluting the returns with outside financiers.

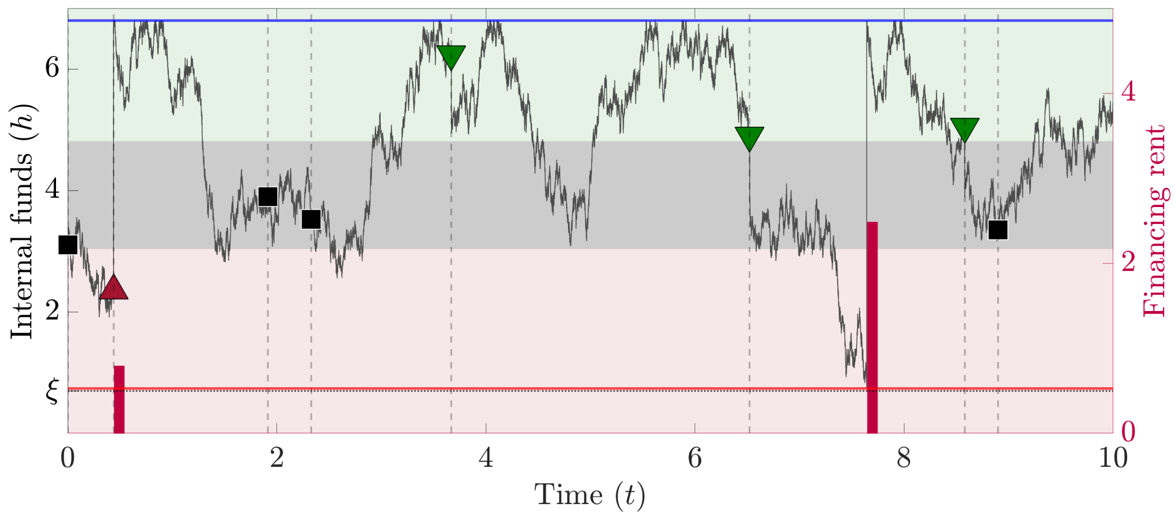


Figure 7: Three ways to handle investment opportunities

The left axis measures internal funds normalized against scale-ups. The horizontal lines in blue and red are, respectively, funding target $\overline{h}$ and financing threshold $\underline{h}$ (conditional on no λ arrival). The history of internal funds is the black curve. The vertical dashed lines in gray indicate the arrival of investment opportunities. The region in light green/gray/red is where such opportunities are internally funded/forgone/financed. The markers in each region indicate the choice made upon each opportunity in accordance with the investment policy described. Lastly, the purple bar is financing rent, against the right axis, at investment financing ($t \approx 0.44$), and non-investment financing ($t \approx 7.64$).

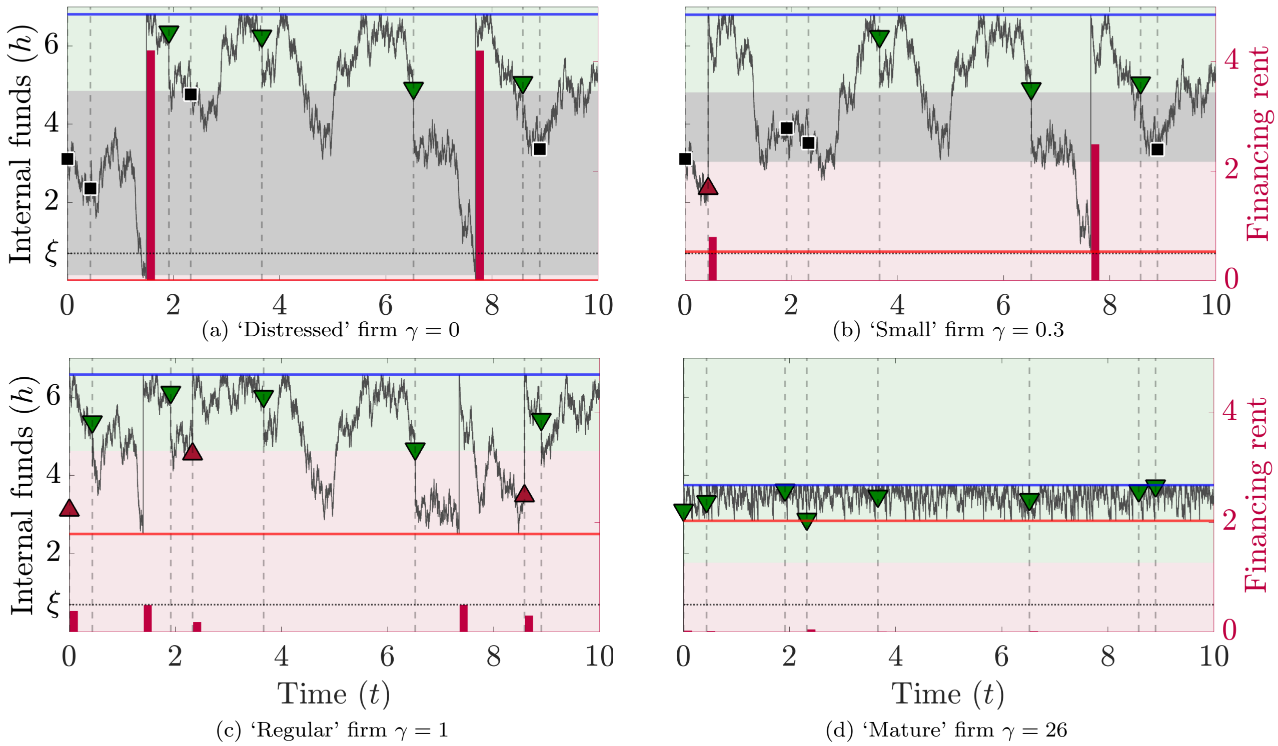
But if internal funds are close to financing threshold so that bargaining is already expected soon anyway, then the cooperative nature of Nash bargaining locally dominates. Shareholders now choose the locally efficient[22] strategy of executing the investment and simultaneously financing up to the funding target *post* the investment expense.

Comparative statics of financing access γ. Figure 8 explores how the strength of access to alternative financing influences firms' investment decisions. In 8a, shareholders face substantial dilution from financing due to the absence of backstop strategy $\gamma = 0$, and so do not fund investment when funding capacity is even moderately limited. They also choose to finance the investment extremely rarely—never, in the present simulation of timeframe $t \in [0, 10]$—and only when they have nearly run out of funding $h \leq 0.14 \ll 0.7 = \xi$.

The case of $\gamma = 0.3$ in Figure 8b is the same as Figure 7 discussed above. With $\gamma = 1$ in 8c, shareholders never forgo investment opportunities as dilution is now less severe. With higher funding capacity, they fund investment opportunities internally; with lower funding capacity, they finance investment opportunities. Notably, they finance investment even when internal funds are much more than sufficient to fund the expense $h \geq \underline{h} \approx 2.50 \gg 0.7 = \xi$.

Perhaps the most interesting case is Figure 8d, where shareholders can expect to find alternative financiers 'just in two weeks' $\gamma = 26$. They finance extremely frequently—53 times over $t \in [0, 10]$ in the present simulation—and in small lumps $\Delta h \approx 0.91$, thanks to the negligible size of dilution around 0.004 at financing threshold. Nevertheless, shareholders maintain sizable funding cushion $\underline{h} \approx 2.84$, well in excess of the contingent investment needs

[22]This efficiency is only 'local' in the sense that the strategy maximizes the joint surplus of the current shareholders and the *specific* group of financiers that they presently have chosen to bargain with.



(a) 'Distressed' firm $\gamma = 0$

(b) 'Small' firm $\gamma = 0.3$

(c) 'Regular' firm $\gamma = 1$

(d) 'Mature' firm $\gamma = 26$

Figure 8: How financing access influences investment choice

The axes and the exogenous shocks are identical across subplots. For explanation of graph components, see Figure 7.

$\xi = 0.7$. As discussed through the main model in Sections 3 and 4, this cushion is what compresses the dilution to such a negligible size in the first place.

Moreover, shareholders with such robust financing access never directly finance investment opportunities. They always internally fund investment, even though it often entails an immediate *post-investment* financing. On one hand, financing the investment when its opportunity is fleeting means that financing surplus encompasses the investment returns. Financing rent that shareholders must pay if they were to finance investment at $h = \underline{h}$ would, therefore, be the $1 - \theta$ fraction of the surplus enlarged due to this wedge of η in post-money value $V(\overline{h})$

$$\textit{Investment financing rent:} \quad (1-\theta)\Big(\eta V(\overline{h}) - V_o(\underline{h}) - \big(\eta\overline{h} + \xi - \underline{h}\big)\Big) \approx 0.617.$$

If, on the other hand, the lumpy investment is internally funded, then the funding capacity may drop well below financing threshold. Due to the extremely high γ, shareholders' reservation value at bargaining rises very steeply in h; and the closer h to 0, the even steeper the rise. Consequently, these shareholders may set financing threshold so high $\underline{h} \approx 2.84 \gg 0.7 = \xi$ that even if an investment opportunity arrives when $h_t \to \underline{h}^+$, the financing that immediately follows the internal funding, i.e. when $h = (\underline{h} - \xi)/\eta \approx 1.95$, still involves small dilution

$$\textit{Post-investment financing rent:} \quad \eta(1-\theta)\left(V\left(\overline{h}\right) - V_o\left(\frac{\underline{h}-\xi}{\eta}\right) - \left(\overline{h} - \frac{\underline{h}-\xi}{\eta}\right)\right) \approx 0.054,$$

higher only by a small difference than dilution at regular financing threshold

$$\textit{Regular financing rent:} \quad (1-\theta)\Big(V(\overline{h}) - V_o(\underline{h}) - \Delta h\Big) \approx 0.004.$$

As such, shareholders always fund investment internally and preserve enough funding capacity so that even after the funding of a lumpy investment, financing remains negligibly dilutive.[23]

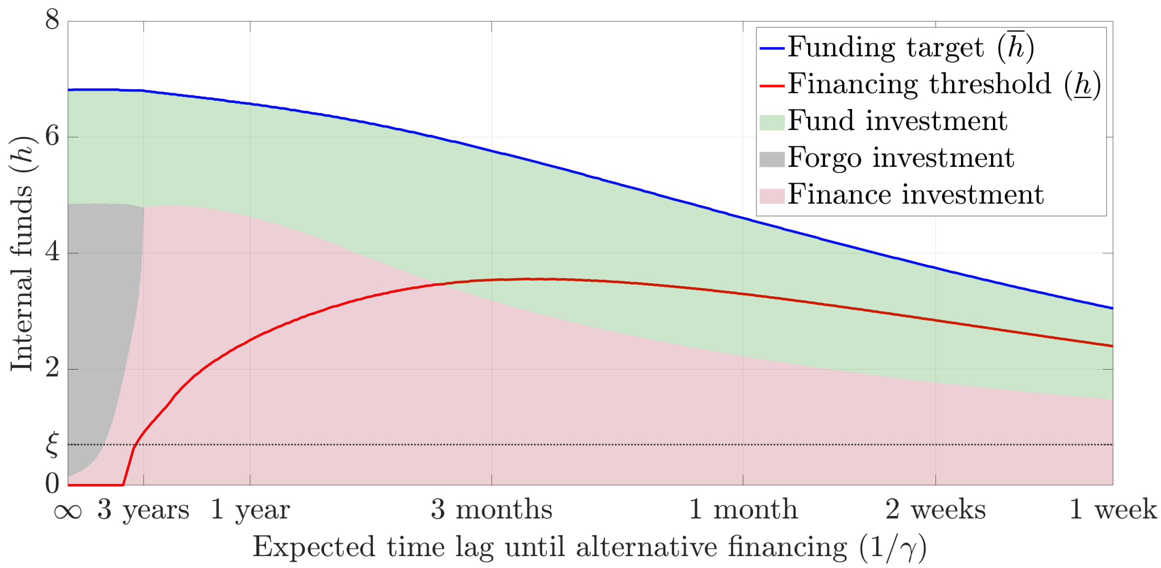


Figure 9: Comparative statics of financing and investment policy

The horizontal axis varies γ from 0 to 52. The labels report the expected time until finding alternative financiers $1/\gamma$, with the interpretation of a unit time period $[t, t+1)$ as a 'year'; in other words, $\gamma = 52$ gives 'one week,' $\gamma = 26$ 'two weeks,' $\gamma = 12$ 'one month,' and so forth.

Comparison to fixed cost. Let us briefly consider how this framework produces different predictions from a fixed-cost framework. As is well known, a fixed cost model, in its basic form, invariably predicts that $\underline{h} = 0$ and investment opportunities are financed only if $h \leq \xi$. If it is additionally assumed à la Froot, Scharfstein and Stein (1993), for instance, that investment must be paid 'out of pocket' first before financing, then the fixed-cost model should predict $\underline{h} \in \{0, \xi\}$, which still fails to effectively explain the inordinate sizes of *cash-holdings*, let alone internal *funds*, maintained by many of the largest companies in the United States.[24]

There is, admittedly, a way for this traditional framework to generate $\underline{h} \gg \xi$, by allowing the fixed cost to be exogenously amplified upon a foreseen regime switch; firms preserve internal funds for precautionary purposes during 'normal' times, so that they won't have to incur the magnified cost during 'financial crisis.' It is, however, difficult to harmonize this

[23] As an aside, with $\gamma = 1$ in 8c, these three values of financing rent become: 0.602 for investment financing at $h = \underline{h} \approx 2.50$; 1.196 for post-investment financing at $h = (\underline{h} - \xi)/\eta \approx 1.63$; and 0.490 for regular financing at $h = \underline{h}$. As such, they finance—rather than fund—investment with h_t near $\underline{h}$.

[24] Alphabet Inc. (Google), for instance, holds \$100.72 billion in cash and cash equivalents as of June 2024, far surpassing its largest-to-date acquisition deal for Motorola Mobility in 2012 at \$12.5 billion. For Apple Inc., the numbers are \$61.80 billion for cash and cash equivalents at June 2024, and \$3 billion for its largest-to-date acquisition for Beats Electronics in 2014.

modeling approach with the observation that those large firms holding a tremendous amount in cash tend to be much more robustly connected to the financial sector than others.

In stark contrast, the bargaining framework proposed in this present paper seamlessly rationalizes 'excessive' financing threshold $\underline{h} \gg \xi$ with robust—but imperfect—financing access $0 \ll \gamma < \infty$. As Figure 9 shows, even those firms that can find alternative financiers in a week on average[25] still employ sizable financial slack.

6 Extension II: Smooth Investment

Next, let us consider the second investment extension where investment is smoothed due to convex adjustment cost. The focus will be on the role of *di*vestment as another backstop strategy, and how business fundamentals matter critically in amplification of dilution when alternative financing becomes unavailable and investment perfectly *irreversible*.

Methodologically, here I consider stochastic Markov transition for parameters to enable more precise analysis. Notably, analytic transparency of the main model in Section 3 is largely preserved with a continuous Markov transition, which is maintained up until the analysis part in Section 6.2 to elucidate the role of *strategic* underinvestment in early financing. Discrete Markov transition is adopted for clear graphical illustration in the numerical parts of Sections 6.5 and 6.6—Sections 6.3 and 6.4 do not have a Markov transition.

6.1 Extended setup

There is a firm with standard 'AK' technology with stochastic TFP and convex investment adjustment cost. The firm produces cash using the only factor, called capital, that it owns.

Production technology. The firm has a linear expected revenue productivity of capital A_t such that capital K_t generates $\big(A_t\ dt + \sigma\ dB_t\big)\ K_t$ cash inflow. $A_t \geq 0$ evolves as

$$dA_t = \mu_A(A_t)\ dt + \sigma_A(A_t)\ dZ_t,$$

where Z_t is a standard Brownian motion, independent of B_t, that drives the evolution of A_t. The simplest case would be that $\mu_A(A) = \sigma_A(A) = 0$, in which case $A_t = A > 0$ is a constant. Cyclical productivity with mean reversion would be captured by an Ornstein-Uhlenbeck process, a continuous-time equivalent of stationary AR(1) processes:

$$d\log A_t = \frac{1}{\nu}\left(\mu_a - \log A_t\right)\ dt + \sqrt{2/\nu}\ \sigma_a\ dZ_t,$$

[25] As discussed in Section 2, this is a conservative estimate. In practice, if a firm can find the first alternative financier in, say, a week, then it will expect to find *successive* alternative financiers in more than a week, and they will have a higher required yield on financing—in the model's lingo, a higher discount rate ρ. In the current model, the expected time lag is fixed regardless of how many alternative financiers have already been accessed, and the financiers so accessed require the same level of yield ρ.

where $\log A_t \sim \mathcal{N}(\mu_a, \sigma_a^2)$, and $\nu > 0$ represents the persistence of its deviation from the mean. By Itô's lemma, the linear differential operator $\mathcal{A}$ on value function given as

$$\mathcal{A}(V) := \mu_A \cdot V_A + \frac{1}{2}\sigma_A^2 \cdot V_{AA}$$

fully characterizes the law of motion of the exogenous productivity. Since $\mathcal{A}$ is a linear operator reflecting the exogenous nature of the evolution of A_t, it will be entirely canceled out in the same derivations as in the proof of Lemma 2.

Cost of investment. Investment is subject to convex adjustment cost in the standard fashion. For the flow investment $I_t\, dt$, the firm incurs an additional flow cost in cash given by $\Psi(I_t/K_t)K_t\, dt$, where Ψ satisfies $\Psi(0) = \Psi'(0) = 0 < \Psi''$. When the firm is neither financing nor paying dividend, total internal funds H_t evolve as: writing $i_t := I_t/K_t$,

$$dH_t = \Big(A_t - i_t - \Psi(i_t)\Big)K_t\, dt + \sigma K_t\, dB_t.$$

For concreteness, assume a quadratic functional form for the adjustment cost: given $\psi > 0$,

$$\Psi(i) = \psi\frac{i^2}{2}.$$

Given depreciation rate $\delta > 0$, capital stock K_t evolves as $dK_t/K_t = \big(i_t - \delta\big)\, dt$.

I do not explicitly introduce capital trades into the model. The canonical construction of a centralized market for capital stock with instantaneous clearing is not well-suited for this paper's context. A sale of existing capital in the current framework is contemplated mainly as part of an outside option, following a bargaining failure occurring off equilibrium path. The presumed ability to *instantaneously* sell an arbitrary fraction of the existing capital upon the *unforeseen realization* of such a scenario is directly at odds with the core premise of this paper that funding cushion gives firms *time* to pursue a backstop strategy. As such, only sufficiently frictional models of capital trades would fit the present purpose. For now, I simply take Ψ as encompassing, in reduced form, any inefficiency in hastened trades of capital stock.

Homogeneity in capital and funds. The firm technology is modeled à la Hayashi (1982), so that its value is homogeneous with degree one in (K, H), greatly simplifying the analysis. Define W as the actual shareholder value function. Letting $h := H/K$ and $V(A, h) := W(A, 1, h)$, homogeneity gives

$$W(A, K, H) = KV(A, h).$$

It then follows that $W_K = V - hV_h,\ W_{KK} = -hV_{hh}$, and $W_H = V_h,\ W_{HH} = V_{hh}/K$.

Investment optimization. By homogeneity, the HJB equation during financial inactivity can be formulated in terms of V, not W, as follows: suppressing (A, h),

$$\rho V - rhV_h = \max_{i \in \mathbb{R}} \left\{ \big(A - i - \Psi(i)\big)V_h + \frac{1}{2}\sigma^2 V_{hh} + \big(i - \delta\big) \underbrace{\big(V - hV_h\big)}_{=W_K} + \mathcal{A}(V) \right\}$$
$$= \mathcal{H}(V) + \mathcal{K}(V) + \mathcal{A}(V), \tag{14}$$

where

$$\mathcal{H}(V) := -\left(\delta + \frac{1}{2\psi}\right) V + \left(A + \frac{1}{2\psi} + \left(\delta + \frac{1}{2\psi}\right) h\right) V_h + \frac{1}{2}\sigma^2 V_{hh},$$
$$\mathcal{K}(V) := \frac{1}{2\psi} \left(\frac{V}{V_h} - h - 1\right) \big(V - hV_h\big),$$
$$\mathcal{A}(V) := \mu_A V_A + \frac{1}{2}\sigma_A^2 V_{AA}$$

from the first-order condition on investment rate i which gives

$$i = \frac{1}{\psi} \left(\frac{V}{V_h} - h - 1\right). \tag{15}$$

All the HJB operators above are linear in V except for $\mathcal{K} = \frac{1}{2} i W_K$, which is optimal investment times marginal value of capital adjusted for the quadratic excess cost of investment $\Psi(i)$.[26] As will be seen shortly, this non-linearity that accounts for the optimized choice of investment i gives us a sharp analytic lens through which to understand how shareholders may strategically underinvest to mitigate the cost of dilution.

6.2 Analysis

The smooth investment cost in the present extended setup generates substantial tractability close in extent to that of the main model in Section 2. Here, I leverage it, similarly to Section 3, to obtain analytic insights on the interactions of financial slack and strategic underinvestment.

Underinvestment. Section 5 with lumpy investment illustrated that shareholders—in particular, those with weak access to alternative financing—may forgo efficient lumpy investment unless they have large internal funds. Let us presently analyze this theme of underinvestment. With convex investment cost, underinvestment relative to the first-best level arises *globally* for all levels of internal funds—more of it with low funds—and for all parameter values.

I will start by establishing that shareholders invest less when they have less internal funds. Equation (15) gives $\partial i(A, h)/\partial h = -\frac{1}{\psi} \frac{V(A,h)}{V_h(A,h)^2} V_{hh}(A, h)$. It therefore boils down to establishing the strict fund-concavity of V. The following lemma establishes that V is indeed strictly concave in internal funds below funding target.

[26] The quadratic Ψ makes non-linear terms more algebraically compact $\frac{1}{2\psi}\frac{V^2}{V_h}$ under full expansion, but I do not expand fully for better intuition. In general, non-linear terms are $iW_K - (i + \Psi(i))V_h$.

Lemma 3 (Funds-driven investment)**.** $V_{hh}(A,h) < 0$ *for* $h < \overline{h}(A) \equiv \inf\{h \mid V_h(A,h) = 1\}$.

Lemma 3, in particular its direct implication that $\partial i/\partial h > 0$, does not immediately validate the idea of global underinvestment, because one might ask whether at least the target funding capacity $\overline{h}(A)$ achieves first-best investment. The answer is a definitive no.

Proposition 4 (Underinvestment at the top)**.** *Let* $i^*(A)$ *denote the optimal investment with* $\theta^* = 1$*, which coincides with the first-best. Let* $\bar{\imath}(A)$ *denote the shareholder-optimal investment at productivity* A *when they hold* $\overline{h}(A)$*. Then,* $i^*(A) > \bar{\imath}(A)$ *for all* A*.*

Proof. In the first-best, optimal investment satisfies

$$i^*(A) = \frac{1}{\psi}\Big(V^*(A) - 1\Big),$$

where $V^*(A)$ is the corresponding per-capital value function. Whereas the highest shareholder-optimal investment satisfies: writing $\overline{V}(A) := V(A, \overline{h}(A))$,

$$\bar{\imath}(A) = \frac{1}{\psi}\Big(\overline{V}(A) - \overline{h}(A) - 1\Big),$$

since $\overline{V}_h = 1$. It must be that $V^*(A) > \overline{V}(A) - \overline{h}(A)$ so that $i^*(A) > \bar{\imath}(A)$. Basically, shareholders with perfect bargaining weight (i.e. represented by V^*) and somehow having just received a one-time windfall in funds $\overline{h}(A)$, all of which would optimally be paid out as dividends so that their value is $V^*(A) + \overline{h}(A)$, can instead deviate by mimicking the optimal strategy under $\theta < 1$ and achieve a strictly higher value than $\overline{V}(A)$, because financing involves no rent. Therefore, denoting the value of the $\theta^* = 1$ shareholders under this deviation strategy by $\widetilde{V}^*$, it follows that $\overline{V}(A) < \widetilde{V}^*(A, \overline{h}(A)) \leq V^*(A) + \overline{h}(A)$, as claimed. □

Investment is a funds-consuming activity. To the extent that dilution from financing incentivizes shareholders to preserve funding capacity as demonstrated by $\overline{h} > 0$, they must be investing strictly less than if they were not so incentivized due to absence of bargaining (e.g. $\theta^* = 1$ or $\gamma^* \to \infty$). With a lower h, they are more incentivized to add to internal funds as the strict concavity of V from Lemma 3 shows, and hence further reduce investment.

Strategic underinvestment and financial slack. Under the main model in Sections 2 through 4, shareholders only had financial slack as their strategic choice. Under the first extension of lumpy investment in Section 5, investment is enabled only upon an exogenous arrival of opportunities. Here, they can continually optimize with investment as well as financial slack. How does this added choice affect optimal financial slack against dilution?

First, let us ease notation: $\overline{V}(A) := V(A, \overline{h}(A))$, $\underline{V}(A) := V(A, \underline{h}(A))$, and $\underline{V}^o(A) := V^o(A, \underline{h}(A))$, where $\underline{h}(A)$ is the financing threshold given state A. Exclusion notation o is now in superscript for readability with partial derivatives. Let $\bar{\imath}(A)$, $\underline{i}(A)$, $\underline{i}^o(A)$ the optimal investment rates at the corresponding fund levels $\overline{h}(A)$, $\underline{h}(A)$, depending on access to financing. Also, let $\overline{W}_K(A) := W_K(A, K, \overline{h}(A)K) = \overline{V}(A) - \overline{h}(A)$, and $\underline{W}^o_K(A) := W^o_K(A, K, \underline{h}(A)K) =$

$\underline{V}^o(A) - \underline{h}(A)\underline{V}^o_h(A)$. Going forward, I will drop notational dependence on exogenous state variable (A). Accordingly, $i(h)$ denotes the optimal investment rate at h, with access to financing, which also depends on the notationally suppressed state A.

The same derivation as in Section 3, in particular Lemma 2, leads to key analytic insights. There are some technical subtleties in derivation due to state dependence, but the end result is identical in that all linear HJB terms cancel out, and different only in that there are now some non-linear terms that remain, giving us key insights for strategic underinvestment and financing. We obtain the following result mirroring Proposition 2.

Proposition 5 (Financial slack and underinvestment)**.** *For each A, and suppressing notation for dependence on A, $\underline{h} > 0$ if and only if*

$$(1-\theta)\gamma + \frac{1}{2}\underbrace{\left(\bar{\imath} - i(0)\right)}_{(a)} > \frac{(\rho - r)\overline{h}}{\overline{V} - \overline{h}}, \tag{16}$$

in which case

$$\begin{aligned}\theta\gamma\left(\overline{V} - \underline{V} - \Delta h\right) &+ \frac{1}{2}\theta\underbrace{\left(\bar{\imath} - \underline{i}\right)}_{(b)}\underbrace{\left(\overline{V} - \overline{h}\right)}_{=\overline{W}_K} \\ &= \theta(\rho - r)\Delta h + \frac{1}{2}(1-\theta)\underbrace{\left(\underline{i} - \underline{i}^o\right)}_{(c)}\underbrace{\left(\underline{V}^o - \underline{h}\underline{V}^o_h\right)}_{=\underline{W}^o_K}.\end{aligned} \tag{17}$$

Observe that with investment choice, $\gamma = 0$ does not necessarily imply $\underline{h} = 0$, as it did in Section 3 (see Corollary 1). The term (a) in Inequality (16) represents the maximum underinvestment across $h \in [0, \overline{h}]$ with access to financing. If it is too high compared to the total carry cost burden of financial slack $\frac{(\rho-r)\overline{h}}{\overline{V}-\overline{h}}$, then shareholders adopt a positive financing threshold $\underline{h} > 0$. One might conjecture that they do so because it pushes the more severe part of the underinvestment $\bar{\imath} - i(h)$ across $h \in [0, \underline{h})$ off equilibrium path, thereby curtailing investment inefficiency that cash rundowns induce.

But this efficiency reasoning fails to capture the core of shareholders' strategic incentives for early financing $\underline{h} > 0$. This is obvious from the fact that in the conventional fixed-cost framework, financing threshold is zero regardless of cash flow optimization; the 'fixed' cost is indeed held fixed, and so the same magnitude of underinvestment optimally arises on equilibrium path regardless of when financing occurs, such that zero financing threshold strictly dominates, as Bolton, Chen and Wang (2011) show. Therefore, any early financing $\underline{h} > 0$ in the current setup, even without alternative financing $\gamma = 0$, must still derive fundamentally from dilution being an endogenous financing cost due to strategic bargaining.

And it indeed does. To see why, let us investigate Equation (17). Note that it is, up to the first terms on either hand sides, identical to Equation (12) from Section 3 with exogenous cash flow. The second terms represent the additional effects of *strategic underinvestment*—both on equilibrium path (b) and as a backstop (c)—on dilution cost, as explained below.

The starting point for understanding early financing even without alternative financing is that on-path underinvestment (b) increases the *total* financing surplus. This is because upon financing, investment will optimally expand $\bar{\imath} > \underline{i}$ along with a greater marginal value of capital $\overline{W}_K = \overline{V} - \overline{h} > \underline{W}_K = \underline{V} - \underline{h}\underline{V}_h$. This increase in total value from higher investment returns hinges entirely on financing success, and therefore constitutes a part of financing surplus. Shareholders, therefore, may prefer to be *able* to limit on-path underinvestment, thereby reducing the surplus and hence the cost of dilution paid to financiers in bargaining.

But then, how can shareholders actually reduce the size of on-path underinvestment, which requires, *a priori*, a smaller financing cost? They can do so because of backstop underinvestment (c). In response to a bargaining failure and the ensuing exclusion, off equilibrium path, shareholders optimally delay business termination by reducing investment and saving cash. This ability to underinvest even further $\underline{i}^o < \underline{i}$ boosts shareholders' reservation value during the financing bargaining, and hence compresses the cost of dilution *on* equilibrium path. With the reduction in financing cost achieved by backstop underinvestment, shareholders can indeed lower on-path underinvestment.

The strategic complementarity between on-path underinvestment and financing cost—i.e. shareholders underinvest less on equilibrium path $h \in [\underline{h}, \overline{h}]$ if financing cost is less, and dilution is less costly if on-path underinvestment is less—amplifies the effect of backstop underinvestment in compressing dilution. That is, if backstop underinvestment reduces dilution, shareholders underinvest less on path, which further reduces dilution, and so on. Therefore, it is expected, as verified shortly in Section 6.3, that even without alternative financing, both the observed size of dilution and on-path underinvestment can be quite small.

Why can all these effects come into play only throug h early financing $\underline{h} > 0$? This is because of convex adjustment cost $\Psi(i)$. With very little internal funds, backstop underinvestment is useless upon exclusion because shareholders cannot generate additional cash quickly enough through underinvestment; in particular, the convex cost prevents swift *divestment* of capital. Put differently, positive funding cushion gives shareholders under exclusion a self-secured grace period before business termination to effectively substitute cash for capital. As an aside, this strategic role of divestment will be revisited in greater detail in Section 6.6.

Lastly, how is the exact early financing threshold $\underline{h} > 0$ then determined? Equation (17) compares the marginal costs and benefits of immediate financing at $h_t = \underline{h}$ relative to instantaneous delay h_{t+dt}. The first terms on both sides are, respectively, the reduction in rent at t due to the chance of instantaneous access to alternative financing and shareholders' share of the instantaneous carry cost; this is the exact same comparison as in Proposition 2. The term $\frac{1}{2}\theta(\bar{\imath} - \underline{i})\overline{W}_K$ on the left-hand side captures the gain in total surplus from higher instantaneous investment returns, net of adjustment cost, of which shareholders retain a θ fraction. On the right-hand side, $\frac{1}{2}(1-\theta)(\underline{i} - \underline{i}^o)\underline{W}_K^o$ shows that with immediate financing, shareholders' outside option involves a backstop investment rate of $\underline{i}^o\, dt$, which is lower than $\underline{i}\, dt$ if they delayed financing by the dt instant. In other words, shareholders, by financing immediately, choose to bargain with less capital as part of their outside option, i.e. they accept a reservation value that is lower by $\frac{1}{2}(\underline{i}-\underline{i}^o)\underline{W}^o_K\, dt$ than under instantaneous delay, worsening their bargaining outcome on equilibrium path by a factor of $1-\theta$. Optimal threshold for early financing, then, equalizes these marginal costs and benefits, giving Equation (17).

6.3 Numerical analysis 1: strategic underinvestment

I now transition to numerical analysis of the extended model. After discussing a benchmark equilibrium in detail to illustrate Proposition 5 and also briefly addressing comparative statics, I explore two applications. One involves fluctuating investment returns to highlight why the so-called 'growth' firms that invest heavily exhibit large financial slack in general, reinforcing the findings from the main model's comparative statics in Section 4.2. The other application explores the importance of business fundamentals in amplification of dilution when both alternative financing and divestment—two key backstop strategies—become unavailable. Appendix B.1 explains the computational algorithm in detail.

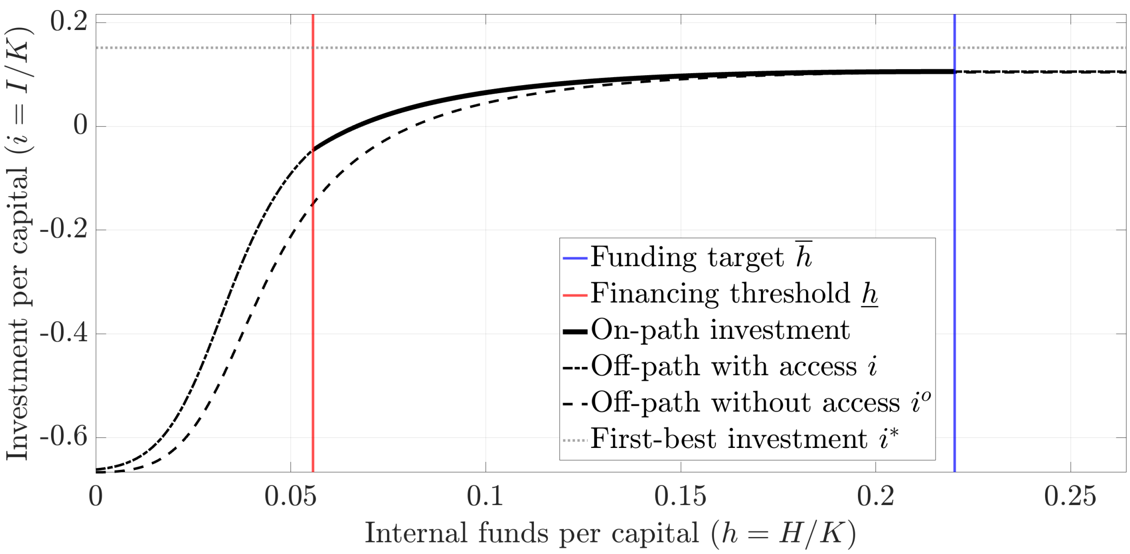


Figure 10: Investment and financial slack

Parameters are: $\rho = 0.06$, $r = 0.05$, $\theta = 0.5$, $\gamma = 0$, $A = 0.18$, $\delta = 0.1007$, $\sigma = 0.09$ and $\psi = 1.5$. The vertical lines in blue and red represent funding target $\overline{h}$ and financing threshold (i.e. funding cushion) $\underline{h}$, respectively. Note the absence of alternative financing $\gamma = 0$.

The parameters are as follows: $\rho = 0.06$, $r = 0.05$, $\theta = 0.5$, $\gamma = 0$, $A = 0.18$, $\delta = 0.1007$, $\sigma = 0.09$, and $\psi = 1.5$. The non-strategic parameters $(\rho, r, A, \delta, \sigma, \psi)$ are directly adopted from Bolton, Chen and Wang (2011). I temporarily turn off alternative financing $\gamma = 0$ to isolate the effect of underinvestment on funding cushion and the size of dilution.

Figure 10 illustrates the firm' optimal policy on financing and investment. The vertical lines in blue and red represent the funding target $\overline{h} \approx 0.2202$ and the financing threshold (i.e. funding cushion) $\underline{h} \approx 0.0558$, respectively. As expected from Proposition 4, investment is lower than under the first-best, and by a noticeable difference even when the firm has sufficient funds: $\bar{\imath} \approx 0.1057 < 0.1512 \approx i^*$.

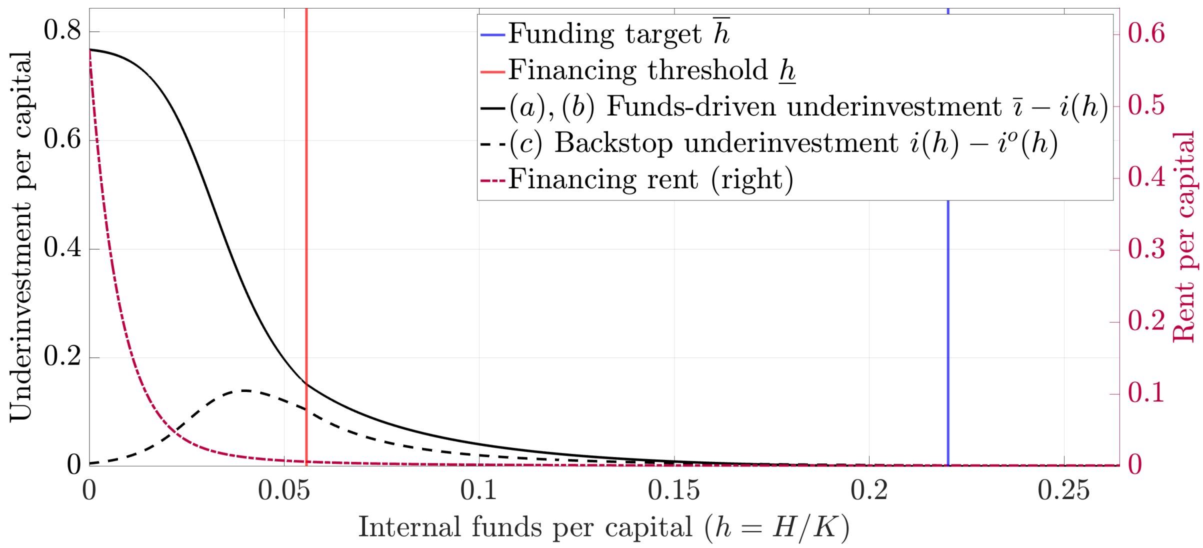


Figure 11: Strategic underinvestment and dilution

The labels $(a), (b), (c)$ are in reference to Proposition 5. Financing rent is computed under a one-shot deviation of immediate financing at each h, which is counterfactual for $h > \underline{h}$. Due to exclusion, this counterfactual financing rent does not completely vanish $(1-\theta)\big(\overline{V} - V^o(\overline{h})\big) \approx 0.001 > 0$.

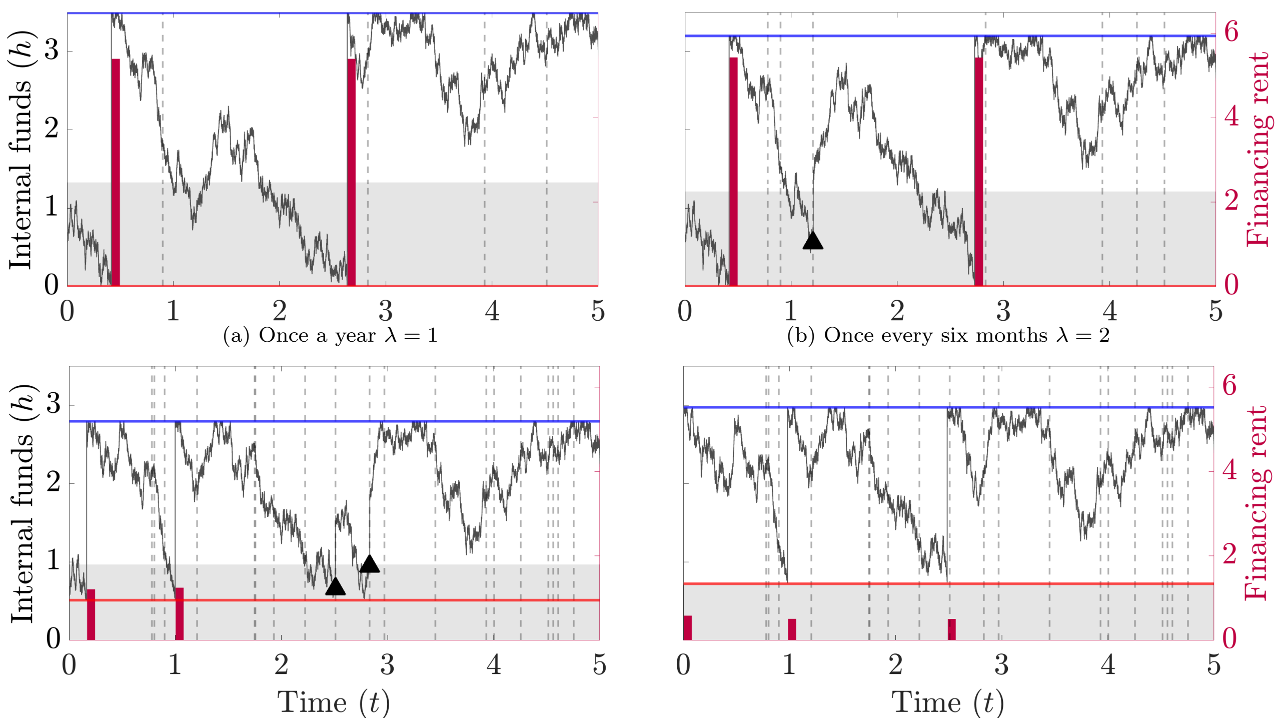
Note that funding cushion is positive despite the absence of alternative financing $\gamma = 0$. As discussed in depth through Proposition 5, backstop underinvestment enabled by this funding cushion compresses the dilution cost, so that on-path underinvestment is reduced, which further reduces the dilution cost, and so on. In sum, on-path underinvestment is greatly curtailed by funding cushion, from $(a) := \bar{\imath} - i(0) \approx 0.767$ down to $(b) := \bar{\imath} - \underline{i} \approx 0.151$. And as Figure 11 shows, the combination of reduced on-path underinvestment and sizable backstop underinvestment[27] ends up tremendously compressing the size of dilution, from 0.579 at $h = 0$ down to 0.006 at $h = \underline{h}$, despite the absence of alternative financing $\gamma = 0$.[28]

Lumpy divestment. As an auxiliary exercise, let us revisit the lumpy investment opportunity setup from Section 5, which I presently modify into a model of *divestment*. The business has 'normalized' running cash inflow $\pi\, dt + \sigma\, dB_t$, and with a Poisson arrival rate of $\lambda > 0$, the business receives an opportunity to *downsize* for cash proceeds. That is, once the opportunity is grasped, the business receives cash $-\xi > 0$ but the future cash flow scales down by $\eta < 1$. I let the parameters $\rho = 0.07,\ r = 0,\ \pi = 1,\ \sigma = 1,\ \xi = -0.7,\ \eta = 0.9$ so that divestment is inefficient under the first-best. As for bargaining parameters, $\theta = 0.5,$ *and* $\gamma = 0$.

As Figure 12 shows, the ability to swiftly underinvest—in this case, divest—incentivizes firms to finance early for a better outside option and hence lower financing rent. Consequently,

[27] In Figure 11, the relevant part of the funds-driven underinvestment $\bar{\imath} - i$ in solid curve is on $[\underline{h}, \overline{h}]$, because this tracks the gains in investment returns from successful financing. Whereas the entire domain of backstop underinvestment $i - i^o$ in dashed curve is relevant, and particularly so for the lower part on $[0, \underline{h}]$, because it compresses the cost of exclusion $\underline{V} - \underline{V}^o$ by delaying termination.

[28] Of course, the total effect of dilution must also account for the frequency of dilution—see Section 3.3. The large reduction of underinvestment on equilibrium path makes dilution more frequent. But this heightened frequency has a small impact exactly because the size of each dilution is now negligible.



(a) Once a year $\lambda = 1$

(b) Once every six months $\lambda = 2$

(c) Once every two months $\lambda = 6$

(d) Once a month $\lambda = 12$

Figure 12: Stochastic divestment opportunities and financing threshold given $\gamma = 0$

Plots track internal funds, financing rent and divestment given stochastic arrivals of inefficient divestment opportunities (vertical dashed lines in gray). In the gray area, optimal policy is to divest; the black marker indicates divestment.

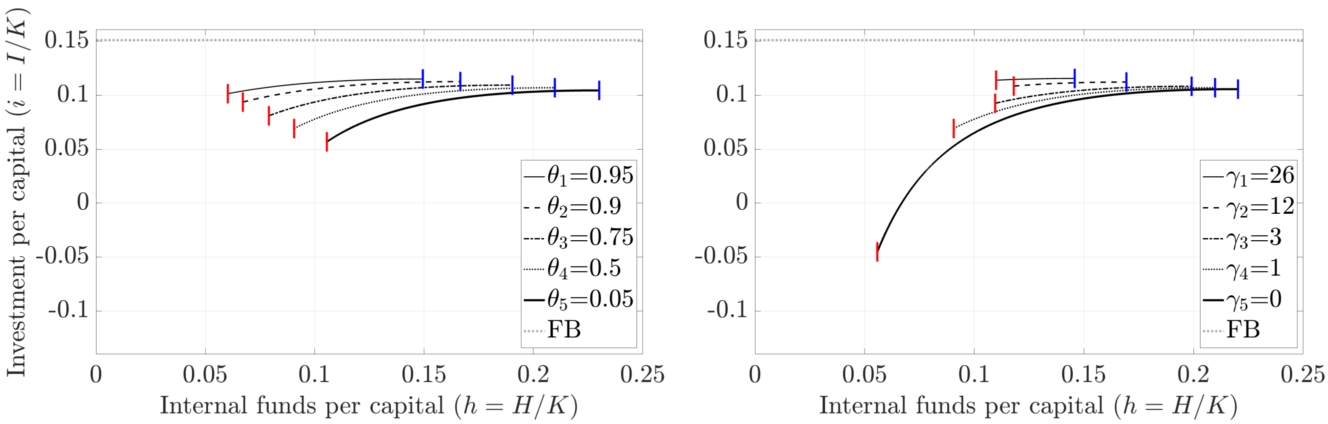
capital trade liquidity λ may reduce inefficient divestment on equilibrium: with high λ, the divestment area in gray falls mostly—entirely with $\lambda = 12$—below financing threshold $\underline{h}$.

6.4 Numerical analysis 2: comparative statics

For comparative statics, I maintain all baseline parameters and only revert γ from 0 to 1. Let us first vary strategic parameters θ, γ and then business parameters A, σ.

Strategic parameters (θ, γ)**.** Figure 13a revisits the result from Section 4 (Figures 5a and 5b in particular) that even minimal bargaining power of financiers substantially affects shareholders' strategy. A mere $1 - \theta = 0.05$ still induces substantial financial slack of $\overline{h} \approx 0.149$, $\underline{h} \approx 0.06$, and underinvestment on equilibrium path ranging between $i^* - \underline{i} \approx 4.97\%$ and $i^* - \overline{\imath} \approx 3.61\%$. Obviously, $1 - \theta = 0$ achieves the first-best $\overline{h} = \underline{h} = 0$, $\overline{\imath} = \underline{i} = i^*$.

Figure 13b also delivers a broadly similar result to Section 4 (Figures 5c and 5d), in addition to Section 5 (Figure 9) with regards to financing threshold $\underline{h}$. There are two additional features. First, with investment choice available, firms may still maintain positive funding cushion even without any access to alternative financing. This is because (i) the ability to reduce investment and boost cash inflow contingent on exclusion may improve shareholders' reservation value and hence bargaining outcome, but (ii) a lower investment and the resulting increase in cash inflow can delay fund depletion only if funds are not already depleted.



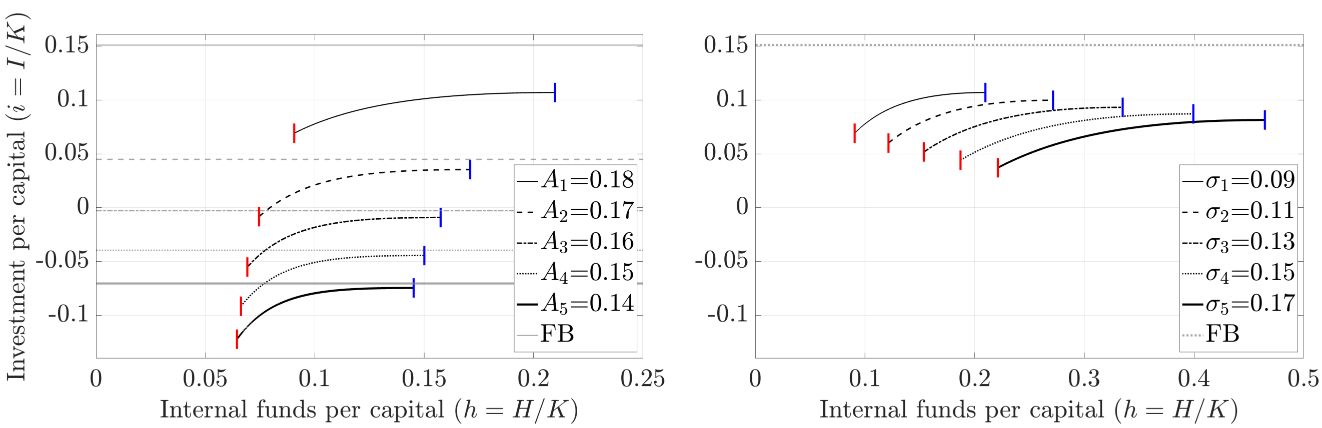
(a) Bargaining weight θ

(b) Accessibility of alternative financing γ

Figure 13: Comparative statics in strategic parameters

The black curves, in different line styles, represent optimal investment policy given internal funds that occur on equilibrium path, bounded in domain by funding target $\overline{h}$ in blue line segment and financing threshold $\underline{h}$ in red line segment. The flat line in gray is the first-best investment. In each plot, θ_4 and γ_4 correspond to the baseline parameter.

Second, with intensive margin on investment choice unlike in Section 5, firms exhibit sizable underinvestment even with robust financing access $\gamma = 26$ (i.e. finding other financiers in two weeks[25]). Across $h \in [\underline{h}, \overline{h}]$, firms underinvest between $i^* - \underline{i} \approx 3.61\%$ and $i^* - \overline{\imath} \approx 3.56\%$.



(a) Productivity A

(b) Cash flow volatility σ

Figure 14: Comparative statics in business parameters

The black curves, in different line styles, represent optimal investment policy given internal funds that occur on equilibrium path, bounded in domain by funding target $\overline{h}$ in blue line segment and financing threshold $\underline{h}$ in red line segment. The flat lines in gray are the first-best investment. In each plot, A_1 and σ_1 correspond to the baseline parameter. Note that with 14b, the horizontal axis has doubled in scale.

Business parameters (A, σ)**.** Higher capital productivity A means that more of the firm's value derives from building up capital stock for the future. Shareholders of such a firm, therefore, have more to lose from dilution of equity value. Financial slack and underinvestment relative to the first-best, therefore, increase in A, as Figure 14a displays.

Figure 14b shows how cash flow volatility increases slack and underinvestment. The key driver is the differential likelihoods of cash rundowns, hence of dilution, which prompt shareholders to respond to a higher volatility with greater slack and weakened investment.

6.5 Numerical analysis 3: stochastic investment returns

Let us now consider stochastic returns to investment. Concretely, consider a firm fluctuating through a discrete Markov chain $s \in \{1, 2, 3, 4\}$ across normal times $A^2 = A^3 = 0.18$, a boom $A^1 = 1.2A^2$ and a bust $A^4 = 0.8A^3$. The two normal times are distinguished in terms of their prospects. At A^2, it is likelier to enter a boom soon, whereas at A^3, it is likelier to enter a bust. The following matrix summarizes the Markov chain:

From\To	A^1	A^2	A^3	A^4
$A^1 = 0.216$	$\cdot$	0.3	0	0
$A^2 = 0.180$	0.3	$\cdot$	0.3	0
$A^3 = 0.180$	0	0.3	$\cdot$	0.3
$A^4 = 0.144$	0	0	0.3	$\cdot$

where the entries are Poisson rates of exiting a state (in row) and entering another (in column). This chain implies a uniform stationary distribution over the Markov state space $\{1, 2, 3, 4\}$. The above setup is designed to isolate the effect of *future* investment returns. Both A^2 and A^3 have the same current revenue, but A^2 still merits increased investment due to greater expected returns.

To have a well-defined first-best solution, I increase ψ from 1.5 to 2 so that investment is harder to scale.[29] I maintain the other baseline parameters, including $\gamma = 1$.

Figure 15 plots shareholder-optimal financing and investment strategies the states $s \in \{1, 2, 3, 4\}$. Table 1 compares the average per capital dynamics across the states and against the first-best allocation using the ergodic distribution on $h \in [\underline{h}^s, \overline{h}^s]$ for each $s = 1, 2, 3, 4$, conditional on no Markov shift having occurred in the past—i.e. 'timeless.' I proceed by discussing underinvestment, then demonstrating how it is mainly fluctuations in *expected*—rather than realized—investment returns that shift financial slack.

(1) Underinvestment. As the very last row of Table 1 shows, average cutback in investment at A^2 relative to the first-best is around 2.06%, higher than that at A^3 around 1.53% despite the same current revenue. With lucrative expected returns, shareholders underinvest more to avoid dilution exactly because their continuation value is higher and so is the cost of dilution. With lower expected returns, they underinvest less because continuation value is lower and so is the cost of dilution. Dilution concerns induce more underinvestment when firms expect a higher capital productivity in a near future, despite the convex adjustment cost that incentivizes anticipatory investment smoothing.

[29] Compared to the baseline case of constant $A = 0.18$, there is much greater upward growth potential $A^1 = 0.216$. If ψ is low, then A^1 is an inordinately great time to substantially scale up investment with small inefficiency, so that first-best continuation value blows up.

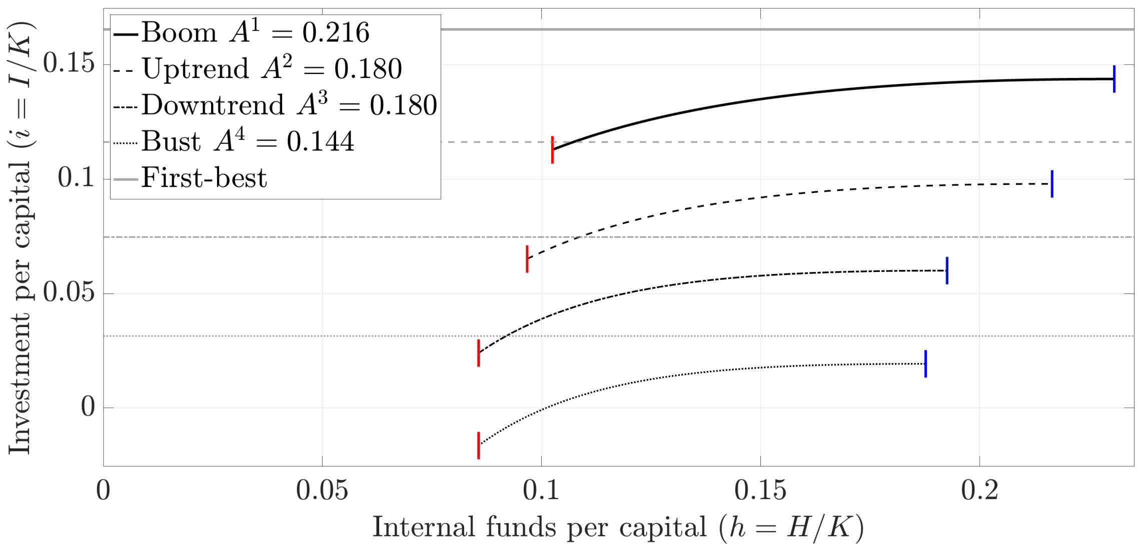


Figure 15: Expected investment returns increase financial slack the most

The black curves, in different line styles, represent optimal investment policy given internal funds in each state that occur on equilibrium path, bounded in domain by funding target $\overline{h}^s$ in blue line segment and financing threshold $\underline{h}^s$ in red line segment. The flat lines in gray are the first-best investment under each state. Other parameters are $\rho = 0.06,\ r = 0.05,\ \theta = 0.5,\ \gamma = 1,\ \delta = 0.1007,\ \sigma = 0.09,\ \psi = 2$.

(2) Financial slack. Both financing threshold and funding target $(\underline{h}^s, \overline{h}^s)$ expand most noticeably as *future* investment returns fluctuate. Shareholders' continuation value rises—and so does the size of dilution—in anticipation of a boom, even as the current net cash inflow $\big(A^s - i_t - \Psi(i_t)\big)K_t\ dt$ drops due to greater investment. This is exactly analogous to the combination, from the startup example in Section 4.2, of a higher upside potential Π and a higher cash burn rate κ such that the flow business value $\lambda\Pi - \kappa$ has not decreased—see Figures 6a and 6c (dashed lines). Each of the two changes unambiguously increases optimal financial slack.

In contrast, the *realized* changes in productivity, i.e. $A^2 \to A^1$ and $A^3 \to A^4$, lead to relatively small adjustments in financial slack. With $A^2 \to A^1$, continuation value per capital rises but so does the *current* gross cash inflow $A^s\ dt$, so that net cash inflow does not change much despite the increased investment. In addition, the rise in continuation value is dampened because there is no more upside potential in the future, only downside. The combined effect is that shareholders fear dilution somewhat more, but they still have similar net cash inflow. With $A^3 \to A^4$, similar effects are at play but in the opposite directions.

To sum up, with fluctuations in realized productivity, the effect of changing continuation value on financial slack gets partially offset by the variation in cash revenue and additionally dampened by mean reversion. With fluctuations in *expected* investment returns, net cash flow varies directly and in the opposite direction due to variation in investment, and mean reversion has less bite. Financial slack, therefore, is most sensitive to fluctuations in expected investment returns.

	Boom	Uptrend	Downtrend	Bust
Markov Chain				
Productivity (A^s)	$A^1 = 0.216$	$A^2 = 0.180$	$A^3 = 0.180$	$A^4 = 0.144$
Upward jump rate ($A^s \to A^{s-1}$)	-	0.3	0.3	0.3
Downward jump rate ($A^s \to A^{s+1}$)	0.3	0.3	0.3	-
Stationary distribution	0.25	0.25	0.25	0.25
Financial Slack				
Funding target ($\overline{h}^s := \overline{H}^s/K$)	0.2307	0.2165	0.1925	0.1876
Financing threshold ($\underline{h}^s := \underline{H}^s/K$)	0.1025	0.0967	0.0856	0.0856
Financing frequency	0.2311	0.192	0.1385	0.1361
Value, Dividend, Investment				
Gross value ($V^s := W^s/K$)	1.4815	1.3813	1.289	1.2046
Net value ($V^s - h$)	1.2872	1.1957	1.12	1.0386
Dividend ratio ($d^s := D^s/K$)	0.0783	0.0997	0.1403	0.1536
Investment ($i^s := I^s/K$)	0.1416	0.0956	0.0593	0.0213
Avg. dividend ($\rho d^s/(\rho + \delta - i^s)$)	0.2464	0.0919	0.083	0.0661
First-best				
FB value ($V^{*s} := W^{*s}/K$)	1.3307	1.2324	1.1492	1.0631
FB dividend ratio ($d^{*s} := D^{*s}/K$)	0.0233	0.0503	0.0998	0.1115
FB investment ($i^{*s} := I^{*s}/K$)	0.1654	0.1162	0.0746	0.0315
Underinvestment ($i^{*s} - i^s$)	0.0237	0.0206	0.0153	0.0103

Table 1: Financial slack and underinvestment across business cycle

Financing frequency, gross/net value, and dividend ratio are computed as arithmetic mean across the conditional ergodic distribution, and investment as ergodic geometric mean of gross capital growth $1 + i_t - \delta$, subtracted by $1 - \delta$. 'Average dividend' is flow average of the total discounted amount of expected lifetime dividend per *today*'s capital, conditional on no Markov jump going forward.

6.6 Numerical analysis 4: investment irreversibility

As the last analysis, let us revisit Proposition 5 in Section 6.2, in particular Inequality (16). It shows, as illustrated in Section 6.3, that funding cushion allows shareholders upon financing failure to delay termination—thereby improving the bargaining outcome on equilibrium path—as they seek to not only find alternative financiers but *also* to cut down on investment to boost cash inflow. Such reduction in investment can manifest itself in the form of *divestment*, whereby shareholders without financing access seek to dispose of capital to obtain cash. As long as firms cannot do so swiftly e.g. due to the convex adjustment cost, funding cushion is necessary for divestment to be employed as a backstop strategy to boost shareholders' outside option. Funding cushion, therefore, has two strategic uses—one in alternative financing and the other in cash generation through underinvestment, including divestment.

Divestment, however, can often be costlier to implement than cutting down on positive investment expenses; salvage value of capital after scrapping is typically much lower than book value, high user specificity of custom-built plants and equipments can compress resale prices, and there may also be the problem of lemons' market. With this asymmetry in investment adjustment cost—positive investment versus divestment—in place, it is easily conjectured that financing threshold would respond to persistent shocks differentially depending on whether (i) shareholders can find alternative financiers relatively easily, and (ii) backstop underinvestment involves divestment.

To explore the interaction tractably, I introduce a reversibility parameter $\phi \geq 0$ to the present extension such that investment adjustment cost is: for $\phi > 0$

$$\Psi_\phi(i) := \begin{cases} \psi \frac{i^2}{2}, & i \geq 0, \\ \frac{\psi}{\phi} \frac{i^2}{2}, & i < 0, \end{cases}$$

and $\phi = 0$ is implemented as the domain restriction on investment choice $i \geq 0$. With $\phi < 1$, $i < 0$ involves a greater adjustment cost than when $i \geq 0$. Note that this modified Ψ_ϕ is still continuously differentiable on the entire real line for any $\phi > 0$ by construction, and as such, the solution method remains largely unchanged. $\phi = 1$ nests the basic investment setup where the adjustment cost is symmetric, while $\phi = 0$ is when capital cannot be converted back into cash. For clarity of insight, I abstract from other considerations such as fixed costs of investment and divestment.

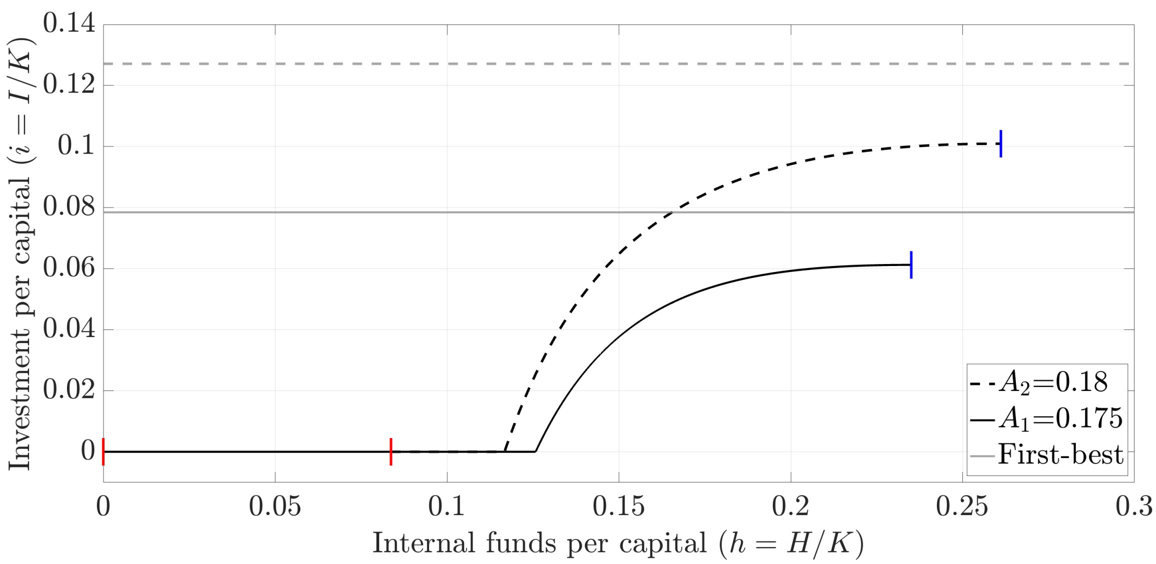
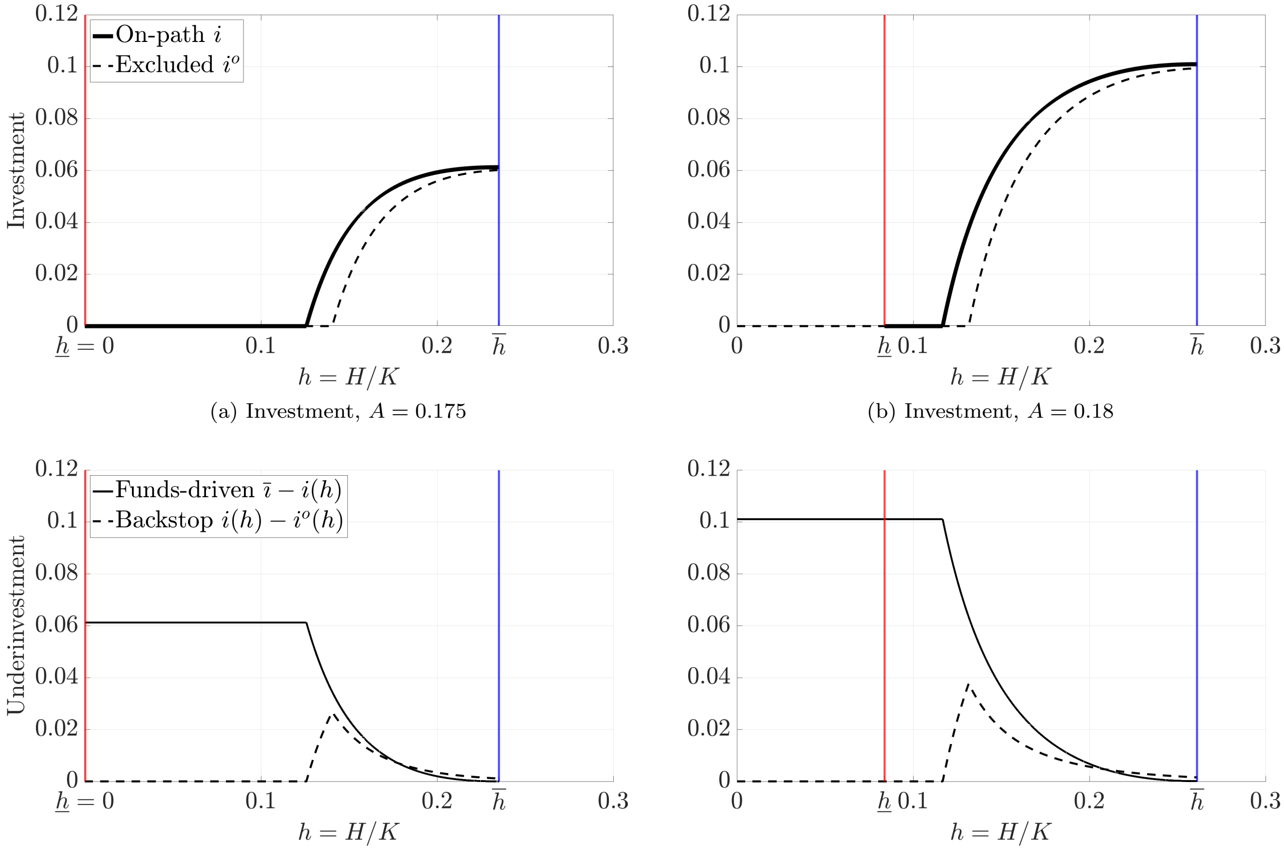


Figure 16: Comparative statics given $\gamma = \phi = 0$

Other parameters are $\rho = 0.06,\ r = 0.05,\ \theta = 0.5,\ \gamma = 0,\ \delta = 0.1007,\ \sigma = 0.09,\ \psi = 1.5,\ \phi = 0.$

Revenue and financing threshold. In Figure 16, I let $\gamma = \phi = 0$ so that there is no explicit backstop strategy, but positive investment is allowed $i \geq 0$, subject to convex adjustment cost $\frac{i^2}{2\psi}$. I compare the two equilibria across $A \in \{0.175, 0.18\}$. First, consider only $A_1 = 0.175$, represented by the solid black curve bounded by its respective funding target $\overline{h}_1$ in blue segment and financing threshold $\underline{h}_1$ in red one. The fact that there is no early financing $\underline{h}_1$ is easily rationalized by the insight from Section 3.2, specifically Corollary 1, that early financing strengthens outside option only when there is a viable backstop strategy to pursue upon bargaining failure. With $\gamma = \phi = 0$, there seems to be no such strategy to pursue, and hence shareholders do not choose to finance early.

But then why does this reasoning fail to hold for a higher productivity $A_2 = 0.18$, represented in dashed curve, where $\underline{h}_2 > 0$ even though $\gamma = \phi = 0$? One might guess that it is due to the higher continuation value under $A = 0.18$ that increases the cost of dilution. The conjecture gives only a partial explanation at best, because in the startup example in the model without investment choice for instance, financing threshold is invariably zero no matter how large the future value Π is, as long as there is no backstop strategy $\gamma = 0$.



(a) Investment, $A = 0.175$
(b) Investment, $A = 0.18$
(c) Underinvestment, $A = 0.175$
(d) Underinvestment, $A = 0.18$

Figure 17: Productivity and funding cushion

Other parameters are $\rho = 0.06,\ r = 0.05,\ \theta = 0.5,\ \gamma = 0,\ \delta = 0.1007,\ \sigma = 0.09,\ \psi = 1.5,\ \phi = 0$. In particular, both alternative financing and divestment are precluded.

Two observations from Figure 17 regarding *backstop* underinvestment $i - i^o$, i.e. the black dashed curves on the bottom two subplots (17c and 17d), point to the core economics behind the stark difference in financing threshold. First, due to the greater investment with access to financing i under higher $A = 0.18$, i.e. (17b versus 17a), there is more backstop underinvestment overall with $A = 0.18$ than with $A = 0.175$. Second and more importantly, due to the irreversibility constraint $i \geq 0$, the entirety of backstop underinvestment for either case $A \in \{0.175, 0.18\}$ is located on the region of high internal funds. A strong cash drift, i.e. high A, is therefore needed for the upward-concentrated backstop underinvestment to become sufficiently *relevant* for the reservation value at a given financing threshold. Otherwise, the odds of remaining constrained by irreversibility until fund depletion will dominate, resulting

	Normal $s = 0$	Crisis $s = 1$	Crisis $s = 2$
Markov Chain			
State (γ^s, ϕ^s, A^s)	$(1, 0.5, 0.18)$	$(0, 0, 0.17)$	$(0, 0, 0.16)$
Arrival rate $(0 \to s)$	-	0.1	0.1
Exit rate $(s \to 0)$	-	0.5	0.5
Financing & Investment			
Funding target $(\overline{h}^s := \overline{H}^s/K)$	0.2105	0.2358	0.2378
Financing threshold $(\underline{h}^s := \underline{H}^s/K)$	0.0992	0.0956	0
Buffer stock $(\Delta h_s := \overline{h}_s - \underline{h}_s)$	0.1113	0.1401	0.2378
Financing frequency	0.1366	0.0395	0.0005
Investment $(i^s := I^s/K)$	0.0696	0.061	0.0498
Dilution			
Ownership retention $(\underline{x}^s \in [0, 1])$	0.9144	0.8944	0.4116
Financing rent $((1 - \underline{x}^s)\overline{V}^s - \Delta h^s)$	0.0015	0.0017	0.5534

Table 2: Crisis with different productivity

Financing frequency and investment are computed as arithmetic and geometric mean over the timeless ergodic distribution at each state. In normal times, $A^0 = 0.18$, $\phi^0 = 0.5$ and $\gamma^0 = 1$. During Crisis $s \in \{1, 2\}$, $\phi^s = \gamma^s = 0$ and $A^1 = 0.17 > 0.16 = A^2$. Other parameters are unchanged: $\rho = 0.06$, $r = 0.05$, $\theta = 0.5$, $\delta = 0.1007$, $\sigma = 0.09$, $\psi = 1.5$.

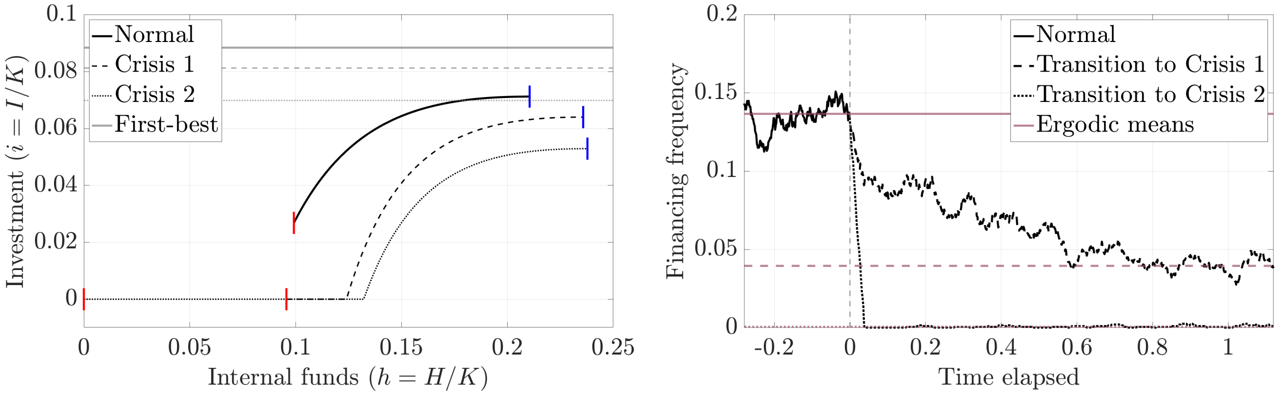
in zero financing threshold.

Business fundamentals and amplification of dilution. Motivated by the above discussions, let us conduct a simple business fluctuation exercise where I allow (γ, ϕ, A) to co-vary stochastically. There are three states $s \in \{0, 1, 2\}$ with the following Markov chain in Poisson arrival rates:

From\To	(γ^s, ϕ^s, A^s)	$s = 0$	$s = 1$	$s = 2$
Normal: $s = 0$	$(1, 0.5, 0.18)$	$\cdot$	0.1	0.1
Crisis 1: $s = 1$	$(0, 0, 0.17)$	0.5	$\cdot$	0
Crisis 2: $s = 2$	$(0, 0, 0.16)$	0.5	0	$\cdot$

The exogenous state variables during the 'normal' time $s = 0$ are given as $(\gamma^0, \phi^0, A^0) = (1, 0.5, 0.18)$. Both Crises $s \in \{1, 2\}$ exhibit perfect irreversibility and absence of alternative financing $\gamma^s = \phi^s = 0$ as well as a drop in productivity $A^1, A^2 < A^0$. A crisis episode occurs with Poisson rate 0.2, half as Crisis 1 and half as Crisis 2, and ends with Poisson rate 0.5. The two Crises are different only in that the first has a higher productivity than the second $A^1 = 0.17 > 0.16 = A^2$. Table 2 and Figure 18 describe the equilibrium, with the latter also reporting the aggregate transitional dynamics of financing upon the arrival of each crisis, initiated from the ergodic distribution during the normal time $s = 0$.

Table 2 indeed shows, as illustrated by Figure 18a, that a correlated crisis of investment irreversibility and lack of alternative financing has a vastly different impact on financing threshold depending on the size of the accompanying drop in productivity. Given a larger drop $A^2 = 0.16 < 0.18 = A^0$, backstop underinvestment is unlikely to become relevant for



(a) Financial slack and investment

(b) Transitional financing

Figure 18: Equilibrium and crisis transition

shareholders' outside option scenario due to the weak upward drift in cash, and hence firms simply delay financing until funds are depleted. As a result, they now face an endogenously magnified cost of dilution $(1 - \underline{x}^2)\overline{V}^2 - \Delta h^2 \approx 0.553 > 0.001 \approx (1 - \underline{x}^0)\overline{V}^0 - \Delta h^0$ due to zero funding cushion $\underline{h}^2 = 0$ and so avoid financing by substantially delaying dividend payout relative to the financing threshold $\Delta h^2 = \overline{h}^2 \approx 0.238 > 0.111 \approx \Delta h^0$. Consequently, financing dries up almost entirely, with one in every 2,000 firms raising funds in a unit period.

In contrast, a crisis that is otherwise identical but accompanied by a somewhat higher level of crisis productivity $A^1 = 0.17$ leads to a completely different pattern. This is because the upward-concentrated backstop underinvestment becomes 'closer,' making the irreversibility constraint *dynamically* less binding and thereby allowing funding cushion to retain its strategic value $\underline{h}^1 \approx 0.096$. The cost of dilution, therefore, remains almost invariably negligible $(1 - \underline{x}^1)\overline{V}^1 - \Delta h^1 \approx 0.0017 > 0.0015 \approx (1 - \underline{x}^0)\overline{V}^0 - \Delta h^0$, and firms expand buffer stock only moderately in response $\Delta h^1 \approx 0.140 > 0.111 \approx \Delta h^0$. While considerably rarer than during normal times, financing still occurs with decent frequency; on average, one in every 25 firms—down from one in every seven—finances in a unit period.

Expectedly, transitional dynamics differ starkly, as Figure 18b shows. Crisis 2 induces an abrupt freeze in financing, because of a sharp plummet in financing threshold $\underline{h}^2 = 0 < 0.099 \approx \underline{h}^0$. Since the conditional long-term mean frequency as shown in Table 2 is nearly zero, the freeze essentially spans the entire duration of the episode. Crisis 1, in contrast, involves a negligible drop in funding cushion $\underline{h}^1 \approx 0.096 < 0.099 \approx \underline{h}^0$, inducing a gradual decline in financing frequency to its conditional mean that is meaningfully away from zero.

In sum, small variation in business fundamentals, such as revenue and internal investment, may induce drastically different financing costs and dynamics when financial market depth dries up and investment irreversibility peaks.

7 Conclusion

In this paper, I propose a tractable model of an endogenous friction in financing that arises solely from the presence of strategic bargaining in the financial markets. Without fixed transaction cost or uncertainty in financing such as search frictions or regime switching, the model predicts not only lumpy financing that indicates the emergence of a friction, but also in general strictly positive financing threshold. The paper offers a novel understanding of this positive funding cushion at financing, in particular with respect to its strategic (rather than precautionary) role of dynamic bargaining; shareholders preserve funding cushion to improve their outside option at bargaining as it allows time upon off-equilibrium financing failure to pursue backstop strategies such as finding other financiers and reducing investment. As funding cushion, when accompanied by viable backstop strategies, compresses the surplus from financing, the model predicts that observed size of dilution is typically small. At the same time, it offers a novel and coherent explanation of why financing cost is magnified when firms finance with little internal funds remaining.

Notably, the framework gives rise to a direct channel through which firms' equity value directly and positively intensifies financial slack. Consequently, firms hold more internal funds and underinvest more when they expect a higher upside potential. Furthermore, higher *expected* investment returns exacerbate financial slack. Overall, the framework seamlessly rationalizes why the so-called 'growth' firms exhibit large financial slack in general.

Additionally, the paper presents a counterintuitive yet empirically relevant prediction that firms with robust financing access may preserve substantial funding capacity in sizable excess of contingent investment needs—such as time-sensitive mergers and acquisitions—that they always fund internally. Whereas firms that have limited access to alternative financiers may finance investment opportunities when their funding capacity is low but forgo investment when it is moderate. Over a reasonably broad parameter range, financing strategy changes in a seemingly opposite direction to the strength of firms' financing access.

Lastly, business fundamentals are shown to have critical role in amplification of dilution when financing and capital market environments drastically deteriorate. When it becomes infeasible to find alternative financiers or sell off existing capital stock, firms that can maintain robust revenue streams and internal investment continue financing early, thereby incurring negligible dilution. In contrast, firms that experience a tangible drop in revenue and thus must cut down on investment choose to delay financing as much as possible, amplifying dilution significantly when they have indeed run out of funding.

Overall, this paper suggests that bargaining in the financing markets may be at the heart of financial slack and its dynamics.

References

Admati, Anat R, Peter M DeMarzo, Martin F Hellwig, and Paul Pfleiderer (2018) "The Leverage Ratchet Effect," *The Journal of finance*, 73 (1), 145–198.

Bates, Thomas W, Kathleen M Kahle, and René M Stulz (2009) "Why Do U.S. Firms Hold so Much More Cash than They Used To?," *The Journal of finance*, 64 (5), 1985–2021.

Baumol, William J (1952) "The Transactions Demand for Cash: An Inventory Theoretic Approach," *The quarterly journal of economics*, 66 (4), 545–556.

Bolton, P and D Scharfstein (1996) "Optimal debt structure and the number of creditors," *The journal of political economy*, 104 (1), 1–25.

Bolton, Patrick, Hui Chen, and Neng Wang (2011) "A Unified Theory of Tobin's q, Corporate Investment, Financing, and Risk Management," *The Journal of finance*, 66 (5), 1545–1578.

——— (2013) "Market timing, investment, and risk management," *Journal of financial economics*, 109 (1), 40–62.

Caggese, Andrea (2007) "Financing constraints, irreversibility, and investment dynamics," *Journal of monetary economics*, 54 (7), 2102–2130.

Cui, Wei (2022) "Macroeconomic Effects of Delayed Capital Liquidation," *Journal of the European Economic Association*, 20 (4), 1683–1742.

Dangl, Thomas and Josef Zechner (2021) "Debt Maturity and the Dynamics of Leverage," *The review of financial studies*, 34 (12), 5796–5840.

Décamps, Jean-Paul, Thomas Mariotti, Jean-Charles Rochet, and Stéphane Villeneuve (2011) "Free Cash Flow, Issuance Costs, and Stock Prices," *The Journal of finance*, 66 (5), 1501–1544.

DeMarzo, Peter M and Michael J Fishman (2007a) "Agency and Optimal Investment Dynamics," *The review of financial studies*, 20 (1), 151–188.

——— (2007b) "Optimal Long-Term Financial Contracting," *The review of financial studies*, 20 (6), 2079–2128.

DeMarzo, Peter M, Michael J Fishman, Zhiguo He, and Neng Wang (2012) "Dynamic agency and the q theory of investment," *The Journal of finance*, 67 (6), 2295–2340.

DeMarzo, Peter M and Zhiguo He (2021) "Leverage dynamics without commitment," *The Journal of finance*, 76 (3), 1195–1250.

Donaldson, Jason Roderick, Denis Gromb, and Giorgia Piacentino (2020) "The paradox of pledgeability," *Journal of financial economics*, 137 (3), 591–605.

Duffie, Darrell, Nicolae Gârleanu, and Lasse Heje Pedersen (2005) "Over-the-counter markets," *Econometrica: journal of the Econometric Society*, 73 (6), 1815–1847.

Farboodi, Maryam, Gregor Jarosch, Guido Menzio, and Ursula Wiriadinata (2019) "Intermediation as Rent Extraction."

Froot, Kenneth A, David S Scharfstein, and Jeremy C Stein (1993) "Risk Management: Coordinating Corporate Investment and Financing Policies," *The Journal of finance*, 48 (5), 1629–1658.

Graham, John R (2022) "Presidential address: Corporate finance and reality," *The journal of finance*, 77 (4), 1975–2049.

Graham, John R and Mark T Leary (2018) "The Evolution of Corporate Cash," *The review of financial studies*, 31 (11), 4288–4344.

Hart, Oliver and John Moore (1998) "Default and Renegotiation: A Dynamic Model of Debt," *The Quarterly Journal of Economics*, 113 (1), 1–41.

Hayashi, Fumio (1982) "Tobin's Marginal q and Average q: A Neoclassical Interpretation," *Econometrica: journal of the Econometric Society*, 50 (1), 213–224.

Hendershott, Terrence, Dan Li, Dmitry Livdan, and Norman Schürhoff (2020) "Relationship Trading in Over-the-Counter Markets," *The Journal of finance*, 75 (2), 683–734.

Hugonnier, Julien, Semyon Malamud, and Erwan Morellec (2014) "Capital Supply Uncertainty, Cash Holdings, and Investment," *The review of financial studies*, 28 (2), 391–445.

Jensen, Michael C (1986) "Agency costs of free cash flow, corporate finance, and takeovers," *American Economic Review*, 76 (2), 323–329.

Kurlat, Pablo (2013) "Lemons Markets and the Transmission of Aggregate Shocks," *The American economic review*, 103 (4), 1463–1489.

Lanteri, Andrea (2018) "The Market for Used Capital: Endogenous Irreversibility and Reallocation over the Business Cycle," *The American economic review*, 108 (9), 2383–2419.

McClellan, Andrew (2024) "Dynamic outside options and optimal negotiation strategies," *American Economic Review*, 114 (10), 3284–3313.

Mian, Atif and João A C Santos (2018) "Liquidity risk and maturity management over the credit cycle," *Journal of financial economics*, 127 (2), 264–284.

Myers, Stewart C (1977) "Determinants of corporate borrowing," *Journal of financial economics*, 5 (2), 147–175.

Nash, John F (1950) "The Bargaining Problem," *Econometrica: journal of the Econometric Society*, 18 (2), 155–162.

Opler, Tim, Lee Pinkowitz, René Stulz, and Rohan Williamson (1999) "The determinants and implications of corporate cash holdings," *Journal of financial economics*, 52 (1), 3–46.

Rajan, R G (1992) "Insiders and Outsiders: The Choice between Informed and Arm's-Length Debt," *The Journal of finance*, 47 (4), 1367–1400.

Rampini, Adriano A and S Viswanathan (2010) "Collateral, risk management, and the distribution of debt capacity," *The Journal of finance*, 65 (6), 2293–2322.

Tobin, James (1956) "The Interest-Elasticity of Transactions Demand For Cash," *The review of economics and statistics*, 38 (3), 241–247.

Üslü, Semih (2019) "Pricing and Liquidity in Decentralized Asset Markets," *Econometrica: journal of the Econometric Society*, 87 (6), 2079–2140.

Appendix A Proofs

A.1 Section 3 (Analysis)

Lemma 1 (Unique existence). *There uniquely exist V and V_o, the equilibrium value function of shareholders and their reservation value function.*

Proof. Let $\mathcal{W}$ denote the space of all continuous bounded functions on $[0, H]$ with an arbitrarily large $H > 0$. Define three self-maps T_o, T, $T^* : \mathcal{W} \to \mathcal{W}$ as follows: for given $W \in \mathcal{W}$,

1. $T_o W$ is the value function of *excluded* shareholders who, *upon re-inclusion* which arrives at Poisson rate $\gamma \geq 0$, transitions to W as their new value function,
2. TW is the value function of *non-excluded* shareholders whose *outside option* upon bargaining is given by W, and
3. $T^* W := (T \circ T_o)W$.

The continuity restriction ensures that the above mappings are well-defined. Endow $\mathcal{W}$ with the L^∞ supremum metric, so that it is complete. An equilibrium value function V, if it exists, is a fixed point of T^*. If T^* is a contraction in $\mathcal{W}$, Contraction Mapping Theorem states that there exists a unique fixed point $V \in \mathcal{W}$ such that $V = T^* V = \lim_{n\to\infty} (T^*)^n W \in$ for some(any) $W \in \mathcal{W}$, in which case $V_o := T_o V$. Uniqueness follows because the model setup with cash inflow and reserve ensures continuity and boundedness of value functions.

The claim on T^* immediately follows from Blackwell's Lemma. First, obviously T^* is monotone increasing. Next, for any $W \in \mathcal{W}$ and $w > 0$,

$$T^*(W + w) < T^* W + (1 - \theta)w.$$

Basically, a w bonus in value that shareholders would receive only if they became excluded (by walking out from bargaining) and then re-included is worth less than a $(1 - \theta)w$ bonus in value that they could enjoy immediately. The $(1 - \theta)$ factor is due to the coefficient on V_o in Equation (4). By Blackwell's Lemma, therefore, T^* is a contraction. □

Lemma 2 (Monotone financing strategy). *If $h \in B$, then $[0, h] \subset B$.*

Proof. The claim is trivial if $B = \{0\}$. Suppose otherwise. On B, V satisfies Equation (4), i.e. $V(h) = \theta(V(\bar{h}) - \bar{h} + h) + (1 - \theta)V_o(h)$ $\forall h \in B$. Since immediate financing is optimal, it is better than instantaneously postponed financing: writing $\mathcal{H}(V)(h) := \lambda\big(\Pi + h - V(h)\big) + \mu V'(h) + \frac{1}{2}\sigma^2 V''(h)$,

$$\begin{aligned} &\rho V(h) - rhV'(h) \geq \mathcal{H}(V)(h) \\ \iff &\theta\Big(\big(\rho V(\bar{h}) - r\bar{h} \cdot V'(\bar{h})\big) - (\rho - r)(\bar{h} - h)\Big) + (1 - \theta)\Big(\rho V_o(h) - rh \cdot V_o'(h)\Big) \\ &\qquad \geq \theta \mathcal{H}(V)(\bar{h}) + (1 - \theta)\mathcal{H}(V_o)(h), \end{aligned} \tag{A.1}$$

on the interior of B. Equation (4) is being substituted on both hand sides of the inequality. In particular, because Equation (4) is an identity on B, the following obtain on B[30]:

$$V'(h) = \theta + (1-\theta)V_o'(h) = \theta V'(\overline{h}) + (1-\theta)V_o'(h), \tag{A.2}$$
$$V''(h) = (1-\theta)V_o''(h) = \theta V''(\overline{h}) + (1-\theta)V_o''(h), \tag{A.3}$$

because $V'(\overline{h}) = 1$ and $V''(\overline{h}) = 0$ by smooth pasting and super contact[8] conditions. This, in combination with the linearity of the operator $\mathcal{H}$, gives:

$$\mathcal{H}(V)(h) = \theta\mathcal{H}(V)(\overline{h}) + (1-\theta)\mathcal{H}(V_o)(h),$$

which is being substituted on the right-hand side. Next, note that:

$$\rho V(\overline{h}) - r\overline{h}V'(\overline{h}) = \mathcal{H}(V)(\overline{h}), \tag{A.4}$$
$$\rho V_o(h) - rhV_o'(h) = \mathcal{H}(V_o)(h) + \gamma\Big(V(h) - V_o(h)\Big). \tag{A.5}$$

Substituting (A.4) and (A.5) into the left-hand side of (A.1) cancels out all $\mathcal{H}$ terms, giving

$$G(h) := (1-\theta)\gamma\Big(V(h) - V_o(h)\Big) - \theta(\rho - r)\Big(\overline{h} - h\Big) \geq 0. \tag{A.6}$$

Note that from $V'(h) = \theta + (1-\theta)V_o'(h)$,

$$G'(h) = \theta\Big((\rho - r) - (1-\theta)\gamma\big(V_o'(h) - 1\big)\Big). \tag{A.7}$$

Since $\underline{h} := \sup B > 0$, shareholders with $h_t = \underline{h}$ are indifferent between financing and instantaneous waiting, i.e. $G(\underline{h}) = 0$. Since G is non-negative on B but not on the outside, it must be that $G'(\underline{h}) \leq 0$. If V_o is strictly concave in h on $[0, \overline{h}]$, then Equation (A.7) implies that $G'(h) \leq 0$ for all $h < \underline{h}$ so that $G(h) \geq 0$ for all $h < \underline{h}$. Therefore, $[0, \underline{h}] \subset B$.

Regarding the strict concavity of V_o, consider the reservation value V_o and the associated funding target $\overline{h}_o > 0$. The proof proceeds as (i) $\overline{h}_o > \overline{h}$, (ii) $V_o'' < 0$ on $[\underline{h}, \overline{h}_o)$, and (iii) $V_o'' < 0$ on $[0, \underline{h})$. As an aside, it will be proven at the end that $V'' < 0$ on $[0, \overline{h})$.

(i) $\overline{h}_o > \overline{h}$. Suppose not. Smooth pasting and super contact at $h = \overline{h}_o$ gives

$$\begin{aligned}\rho V_o(\overline{h}_o) - r\overline{h}_o &= \mu + \lambda\Big(\Pi + \overline{h}_o - V_o(\overline{h}_o)\Big) + \gamma\Big(V(\overline{h}_o) - V_o(\overline{h}_o)\Big)\\ &\geq \mu + \lambda\Big(\Pi + \overline{h}_o - V_o(\overline{h}_o)\Big),\end{aligned}$$

because $\gamma \geq 0$ and $V(\overline{h}_o) > V_o(\overline{h}_o)$ from exclusion. Solve for $V_o(\overline{h}_o)$ to obtain

$$V_o(\overline{h}_o) \geq \frac{1}{\rho + \lambda}\Big(\mu + \lambda\Pi + (r + \lambda)\overline{h}_o\Big).$$

[30] They obtain on the interior from the identity itself. They also obtain on the boundary due to smooth pasting and super contact.

Since $\overline{h}_o$ is the funding target under exclusion and $\overline{h} \geq \overline{h}_o$ by assumption,

$$V_o(\overline{h}) = V_o(\overline{h}_o) + (\overline{h} - \overline{h}_o) \geq \frac{1}{\rho + \lambda}\Big(\mu + \lambda\Pi + (r + \lambda)\overline{h}_o\Big) + (\overline{h} - \overline{h}_o)$$
$$\geq \frac{1}{\rho + \lambda}\Big(\mu + \lambda\Pi + (r + \lambda)\overline{h}\Big) = V(\overline{h}),$$

where the last inequality is because $\rho > r$. This contradicts Proposition 1.

(ii) $V_o'' < 0$ on $[\underline{h}, \overline{h}_o)$. First, suppose $\sigma > 0$. Differentiate (A.5) at $h = \overline{h}_o$ while substituting $V_o'(\overline{h}_o) = 1,\ V_o''(\overline{h}_o) = 0$ to obtain a third (left) derivative as

$$V_o'''(\overline{h}_o) = \frac{2}{\sigma^2}\Big(\rho - r - \gamma\big(V'(\overline{h}_o) - 1\big)\Big) = \frac{2}{\sigma^2}(\rho - r) > 0,$$

because $\overline{h}_o > \overline{h}$ and so $V'(\overline{h}_o) = 1$. Therefore, there must exist some neighborhood below $\overline{h}_o$ on which $V_o'' < 0$. Suppose by way of contradiction that there exists $\widehat{h} \in [\underline{h}, \overline{h}_o)$ such that $V_o'' < 0$ on $(\widehat{h}, \overline{h}_o)$ but $V_o''(\widehat{h}) = 0$. Then, it must hold that $V_o'''(\widehat{h}) \leq 0$. Differentiating the HJB at $h = \widehat{h}$ gives a third (left) derivative as

$$0 \geq V_o'''(\widehat{h}) = \frac{2}{\sigma^2}\Big((\rho - r)V_o'(\widehat{h}) + \lambda\big(V_o'(\widehat{h}) - 1\big) + \gamma\big(V_o'(\widehat{h}) - V'(\widehat{h})\big)\Big).$$

Since $V_o'(\widehat{h}) > 1$ from $V_o'(\overline{h}_o) = 1$ and $V_o'' < 0$ on $(\widehat{h}, \overline{h}_o)$, the above implies that $\gamma > 0$ (that is, a contradiction is already reached if $\gamma = 0$) and $V_o'(\widehat{h}) - V'(\widehat{h}) < 0$. Because $V'(h) = 1$ for $h \geq \overline{h}$, it must be that $\widehat{h} < \overline{h}$. Since $V_o'(\overline{h}) - V'(\overline{h}) > 0$ from $\overline{h} \in (\widehat{h}, \overline{h}_o)$, the intermediate value theorem implies that there exists $\widetilde{h} \in (\widehat{h}, \overline{h})$ such that $V_o'(\widetilde{h}) - V'(\widetilde{h}) = 0$. But

$$V_o''(\widetilde{h}) - V''(\widetilde{h}) = \frac{2}{\sigma^2}\Bigg(\Big((\rho + \lambda + \gamma)V_o(\widetilde{h}) - r\widetilde{h}V'(\widetilde{h}) - \lambda(\Pi + \widetilde{h}) - \mu V'(\widetilde{h}) - \gamma V(\widetilde{h})\Big)$$
$$- \Big((\rho + \lambda)V(\widetilde{h}) - r\widetilde{h}V'(\widetilde{h}) - \lambda(\Pi + \widetilde{h}) - \mu V'(\widetilde{h})\Big)\Bigg)$$
$$= \frac{2}{\sigma^2}(\rho + \lambda + \gamma)\Big(V_o(\widetilde{h}) - V(\widetilde{h})\Big) < 0.$$

In other words, the graph of $V_o' - V'$ can never cross zero *from below* as necessitated by $V_o'(\widehat{h}) - V'(\widehat{h}) < 0 < V_o'(\overline{h}) - V'(\overline{h})$ and $\widehat{h} < \overline{h}$, a contradiction.

Second, suppose $\sigma = 0$. Assumption 1 requires that $\mu < 0$, $\lambda > 0$ and $\Pi > -\frac{\mu}{\lambda} > 0$. Since this is essentially the startup example from Section 2.1, relabel parameters $\kappa \equiv -\mu > 0$ and $\Pi \equiv \Pi > \frac{\kappa}{\lambda}$. Differentiating (A.5) at $h = \overline{h}_o$ gives a second (left) derivative as

$$V_o''(\overline{h}_o) = -\frac{\rho - r}{\kappa - r\overline{h}_o},$$

because of smooth pasting $V_o'(\overline{h}_o) = 1$ and $\overline{h}_o > \overline{h}$ giving $V'(\overline{h}_o) = 1$. Since the last part of Assumption 1 ensures that $\kappa > r\overline{h}_o$, it follows that $V_o''(\overline{h}_o) < 0$. Again, suppose by way of contradiction that there exists $\widehat{h} \in [\underline{h}, \overline{h}_o)$ such that $V_o'' < 0$ on $(\widehat{h}, \overline{h}_o]$ and $V_o''(\widehat{h}) = 0$. Differentiating (A.5) at $\widehat{h}$ gives

$$(\rho - r)V_o'(\widehat{h}) + \lambda\Big(V_o'(\widehat{h}) - 1\Big) + \gamma\Big(V_o'(\widehat{h}) - V'(\widehat{h})\Big) = 0.$$

Since $V_o'(\widehat{h}) > 1$, this again implies that $\gamma > 0$ and $V_o'(\widehat{h}) - V'(\widehat{h}) < 0$. The same reasoning as with $\sigma > 0$ implies $\widehat{h} < \overline{h}$. Since $V_o'(\overline{h}) - V'(\overline{h}) > 0$, the intermediate value theorem again implies that there exists $\widetilde{h} \in (\widehat{h}, \overline{h})$ such that $V_o'(\widetilde{h}) = V'(\widetilde{h})$. Then,

$$\begin{aligned}
\rho V_o(\widetilde{h}) - r\widetilde{h}V'(\widetilde{h}) &= \lambda\Big(\Pi + \widetilde{h} - V_o(\widetilde{h})\Big) - \kappa V'(\widetilde{h}) + \gamma\Big(V(\widetilde{h}) - V_o(\widetilde{h})\Big), \\
\rho V(\widetilde{h}) - r\widetilde{h}V'(\widetilde{h}) &= \lambda\Big(\Pi + \widetilde{h} - V(\widetilde{h})\Big) - \kappa V'(\widetilde{h}),
\end{aligned}$$

and therefore, $V_o(\widetilde{h}) = V(\widetilde{h})$, contradicting exclusion $\gamma < \infty$.

(iii) $V_o'' < 0$ on $[0, \underline{h})$. First, suppose $\sigma > 0$. Because $V_o''(\underline{h}) < 0$, there exists a neighborhood below $\underline{h}$ on which $V_o'' < 0$. The aforementioned observation on G' implies that if $V_o'' < 0$ on the interior of a neighborhood below $\underline{h}$, then the neighborhood, including its closure, is a subset of B. Substituting (5) that holds on B, the HJB for V_o on this interval is written as

$$\rho V_o(h) - rhV_o'(h) = \lambda\big(\Pi + h - V_o(h)\big) + \mu V_o'(h) + \frac{1}{2}\sigma^2 V_o''(h) + \theta\gamma\Big(\big(V(\overline{h}) - \overline{h}\big) + h - V_o(h)\Big).$$

Suppose by way of contradiction that there is $\widetilde{h} \in [0, \underline{h})$ such that $V_o'' < 0$ on $(\widetilde{h}, \underline{h})$ but $V_o''(\widetilde{h}) = 0$. Note that $V_o'(\widetilde{h}) > 1$, because $V_o'(\overline{h}_o) = 1$ and $V_o'' < 0$ on $(\widetilde{h}, \overline{h}_o)$. Differentiating the above HJB at $\widetilde{h}$ gives a third derivative as

$$V_o'''(\widetilde{h}) = \frac{2}{\sigma^2}\Big((\rho - r + \lambda + \theta\gamma)V_o'(\widetilde{h}) - (\lambda + \theta\gamma)\Big) > \frac{2}{\sigma^2}(\rho - r) > 0,$$

which contradicts $V_o''(\widetilde{h}) = 0$ and $V_o'' < 0$ on $(\widetilde{h}, \underline{h}]$.

Next, suppose $\sigma = 0$ and use the startup relabeling $\kappa \equiv -\mu > 0$, $\Pi \equiv \Pi > 0$. Since $V_o''(\underline{h}) < 0$, there is a neighborhood below $\underline{h}$ which is a subset of B and on which $V_o'' < 0$. Differentiating the HJB gives

$$(\rho - r)V_o'(h) + (\lambda + \theta\gamma)\Big(V_o'(h) - 1\Big) + (k - rh)V_o''(h) = 0.$$

Suppose by way of contradiction that there exists $\widehat{h} < \overline{h}$ such that $V_o'' < 0$ on $(\widehat{h}, \overline{h}]$ but $V_o''(\widehat{h}) = 0$. Then, $[\widehat{h}, \overline{h}] \subset B$ by the property of G' and so the above equality must hold at $h = \widehat{h}$. But it contradicts $V_o'(\widehat{h}) > 1$, which is implied by $V_o'(\overline{h}_o) = 1$ and $V_o'' < 0$ on $(\widehat{h}, \overline{h}_o]$.

Side claim: $V'' < 0$ on $[0, \overline{h})$. First, establish the claim on $[\underline{h}, \overline{h})$. If $\sigma > 0$, then strict concavity is established by the same reasoning (ii) above, but with $\gamma = 0$ so that a third (left) derivative at $\widehat{h}$ already gives a contradiction. Suppose $\sigma = 0$ and adopt the startup relabeling. Differentiating the HJB (3) gives a second (left for $h = \overline{h}$, right for $h = \underline{h}$) derivative on $[\underline{h}, \overline{h}]$

as

$$V''(h) = -\frac{(\rho - r)V'(h) + \lambda\big(V'(h) - 1\big)}{\kappa - rh}.$$

Assumption 1 ensures that $r\overline{h} < \kappa$. By smooth pasting $V'(\overline{h}) = 1$, $V''(\overline{h}) < 0$. For any $\widehat{h} \in [\underline{h}, \overline{h})$ such that $V'' < 0$ on $(\widehat{h}, \overline{h}]$, $V'(\widehat{h}) > 1$ and so $V''(\widehat{h}) < 0$. As such, $V'' < 0$ on $[\underline{h}, \overline{h}]$.

Strict concavity of V on $[0, \underline{h}]$ is immediate because $V'' = (1-\theta)V_o'' < 0$ on it. □

Corollary 1. Given other parameters, there exists $\underline{\gamma} \in (0, \infty)$ such that $\underline{h} = 0$ if and only if $\gamma \leq \underline{\gamma}$. In particular, $\gamma = 0$ always implies $\underline{h} = 0$

Proof. With $\gamma = 0$, $G(0) = -(\rho - r) \cdot \overline{h} < 0$ from Equation (A.6). Since the inequality is strict, it must be that $G(0) < 0$ for γ in some neighborhood above zero. By proof of Lemma 2, $G(0) < 0$ implies that $\underline{h} = 0$. Also, the result $\gamma \to \infty \implies \overline{h} \to 0$ in Part 4 of Proposition 3 means[31] that $\underline{h}$ cannot be globally zero for all $\gamma \in [0, \infty)$ since otherwise γ would be irrelevant in equilibrium and thus $\overline{h}$ could not be affected. Lastly, in Inequality (11), the right-hand side is non-increasing and the right-hand side strictly increasing in γ. Therefore, once $\underline{h} > 0$, then it remains strictly positive for any higher γ. $\underline{\gamma} > 0$ thus exists (uniquely). □

Proposition 3 (Comparative statics in θ and γ).

1. $\overline{h}$ decreases[18] in θ. $\overline{h}$ is constant in γ when $\gamma < \underline{\gamma}$ and decreasing otherwise.
2. $\underline{h}$ decreases in θ when $\underline{h} > 0$. $\underline{h} = 0$ is constant in θ above some $\underline{\theta} < 1$.
3. When $\underline{h} > 0$, Δh is constant in θ if $r = 0$ and increasing if $r \in (0, \rho)$. When $\underline{h} = 0$, $\Delta h = \overline{h}$ decreases in θ. Δh is constant in γ when $\gamma < \underline{\gamma}$. When $\gamma \geq \underline{\gamma}$, Δh is decreasing in γ if $r = 0$.
4. $\overline{h} \to 0$ as either $\theta \to 1$ or $\gamma \to +\infty$.

Proof. **Part 1**. Constancy of $\overline{h}$ in $\gamma \in [0, \underline{\gamma}]$ is obvious since shareholders bargain only at $h_t = 0$, where re-inclusion is simply irrelevant. Take $\gamma_2 > \gamma_1 \geq \underline{\gamma}$ and consider the equilibrium with $\gamma = \gamma_2$. When shareholders bargain with financiers, they choose $\overline{h}_2$ to maximize $V(\overline{h}_2; \gamma_2) - \overline{h}_2$. Suppose that they agree, as a one-shot deviation, to choose $\overline{h}_1$ instead and then mimic the optimal financing strategy under $\gamma = \gamma_1$ (i.e. refinance at $\underline{h}_1$, pay out above $\overline{h}_1$) until next financing. Denote the payoff function associated with this strategy as $\widetilde{V}$. Note that $\widetilde{V}(\overline{h}_1; \gamma_2) > V(\overline{h}_1; \gamma_1)$, as the reservation value at $h_t = \underline{h}_1 > 0$ is strictly higher with $\gamma = \gamma_2 > \gamma_1$. Since $\overline{h}_2$ without the one-shot deviation is optimal, $V(\overline{h}_2; \gamma_2) - \overline{h}_2 \geq \widetilde{V}(\overline{h}_1; \gamma_2) - \overline{h}_1 > V(\overline{h}_1; \gamma_1) - \overline{h}_1$. Finally, since

$$V(\overline{h}; \gamma) - \overline{h} = \frac{1}{\rho + \lambda}\Big(\mu + \lambda\Pi - (\rho - r) \cdot \overline{h}\Big), \tag{A.8}$$

from evaluating Equation (3) at $h = \overline{h}$ with $V'(\overline{h}) = 1$, $V''(\overline{h}) = 0$, we have $\overline{h}_2 < \overline{h}_1$. Global strict monotonicity in θ is established by a similar reasoning.

[31]There is no circular reasoning since Part 4 does not rely on Corollary 1 for its proof.

Part 2. Strict monotonicity of $\underline{h} = \overline{h} - \Delta h > 0$ in θ is immediate from the decreasing $\overline{h}$ in Part 1 and the non-decreasing Δh in Part 3. The claim on the existence of such a $\underline{\theta}$ is from Parts 3 and 4 since $\Delta h > 0$ is non-decreasing in θ when $\underline{h} > 0$ but $\Delta h \to 0$ when $\theta \to 1$.

Part 3. Constancy of Δh in $\gamma \in [0, \underline{\gamma}]$ is obvious since $\gamma \leq \underline{\gamma} \implies \Delta h = \overline{h}$. Strict monotonicity of Δh in θ when $\underline{h} = 0$ is implied by strict monotonicity of $\overline{h}$ in θ in Part 1. Now suppose that $\underline{h} > 0$. By Proposition 2,

$$\frac{V(\overline{h}) - V(\underline{h})}{\Delta h} = 1 + \frac{\rho - r}{\gamma}. \tag{A.9}$$

Three important observations on V and Equation (A.9) can be drawn. First, Equation (A.9) stipulates an *average* rate of change in V over $[\underline{h}, \overline{h}]$. The required rate is decreasing in γ and independent of θ. Second, recall that V on $[\underline{h}, \overline{h}]$ satisfies Equation (3) along with the two boundary conditions $V'(\overline{h}) = 1,\ V''(\overline{h}) = 0$ determined by dividend payout optimality.[8] Equation (3), to be solved *downward* from $h = \overline{h}$, does not directly depend on either γ or θ; the only indirect channel through which the differential equation might depend on them is the $rhV'(h)$ term, since the starting point $\overline{h}$ changes with $\gamma \geq \underline{\gamma}$ and θ. Third, Δh is determined as the extent of descent in h from $\overline{h}$ over which the average of the marginal rates of change in V, as obtained by (3) and the payout optimality, equals the right-hand side of (A.9).

If $r = 0$, then, the evolution of V' below $\overline{h}$ is independent of $\gamma \geq \underline{\gamma}$ and θ, and the required average is decreasing in γ and independent of θ. Therefore, Δh is independent of θ (as long as $\underline{h} > 0$), and decreasing in $\gamma \geq \underline{\gamma}$ since γ compresses the required *excess* rate of change above one while $V'(\overline{h}) = 1$ and V' rises as h falls. This establishes the two claims for $r = 0$.

If $r \in (0, \rho)$, the aforementioned subtlety is introduced since the level of $\overline{h}$ changes how V' evolves below it. Consider two equilibria $(\underline{h}_1, \overline{h}_1)$, $(\underline{h}_2, \overline{h}_2)$ with $\underline{h}_1, \underline{h}_2 > 0$ and $\overline{h}_1 < \overline{h}_2$ and sharing the same parameters other than γ, θ. Write Equation (3) as

$$\rho V(h) = rhV'(h) + \mathcal{M}(V)(h) + \frac{1}{2}\sigma^2 V''(h),$$

where $\mathcal{M}$ subsumes all terms of Λ, $\mathcal{H}$ except the second order one. For $i \in \{1, 2\}$, consider solving V_i downward starting from $\overline{h}_i$. With every n^{th} step of descent $dh > 0$ in h_i below $\overline{h}_i$ (i.e. $h_i^0 = \overline{h}_i,\ h_i^n = h_i^{(n-1)} - dh$), the left-hand side $\rho V_i(h_i^n)$ falls (approximately) by $\rho V_i'(h_i^{(n-1)}) \cdot dh$. On the right-hand side, the rise in V_i', i.e. $dV_i'^n := V_i'(h_i^n) - V_i'(h_i^{(n-1)})$, has greater positive contribution with h_2^n than with h_1^n since $h_2^n > h_1^n$ and $dV_i'^n > 0$. If $\sigma^2 > 0$, then it must be that $dV_2''^n < dV_1''^n$ to restore the equality, so that $dV_2'^{(n+1)} > dV_2'^{(n+1)} > 0$. If $\sigma^2 = 0$ (implying $a_1 < 0$ by Part 3 of Assumption 1), then the coefficient on V' in the above equation is $-(-\mu - rh)$. Since $-\mu > r\overline{h}$ by Part 4 of Assumption 1, a higher $h_2^n > h_1^n$ means that $dV_2'^n > 0$ must be greater than $dV_1'^n > 0$ to restore the inequality.

In conclusion, a higher $\overline{h}$ corresponds, ceteris paribus and with $\underline{h} > 0$, to a *steeper* V below $\overline{h}$. Therefore, it takes *less* extent of descent Δh to achieve the required average rate of change in V. Since $\overline{h}$ decreases in $\gamma \geq \underline{\gamma}$ and θ by Part 1 and the required rate is decreasing in γ and independent of θ, Δh increases in θ (when $\underline{h} > 0$). Whereas a higher $\gamma \geq \underline{\gamma}$ has two opposing effects of making V less steep and lowering the required rate, making comparative statics ambiguous; it is clear, though, that the latter effect dominates at least asymptotically, since $\overline{h} \geq 0$ is bounded from below.

Part 4. Since $\overline{h} \geq 0$ decreases (strictly) in θ, it converges as $\theta \to 1^-$ by monotone convergence theorem. Suppose by way of contradiction that $\overline{h} \to \widetilde{h} > 0$ as $\theta \to 1^-$. Then, Inequality (11) implies that there exists[32] some $\underline{\theta} \in (0, 1)$ such that $\underline{h} = 0$ for any $\theta \in [\underline{\theta}, 1)$, because $\overline{h} \geq \widetilde{h} > 0$ and $V(\overline{h}) - \overline{h}$ is bounded above by the frictionless net present value.

By zero liquidation value, $V(0) = \theta\big(V(\overline{h}) - \overline{h}\big) = V(\overline{h}) - \overline{h} - (1-\theta)\big(V(\overline{h}) - \overline{h}\big)$. Consider $\theta > \underline{\theta}$. Since $\underline{h} = 0$, the equilibrium financing cost is $(1-\theta)\big(V(\overline{h}) - \overline{h}\big)$, which vanishes as $\theta \to 1^-$ because $V(\overline{h}) - \overline{h}$ is bounded above. At the same time, the buffer interval $[\underline{h}, \overline{h}]$ converges, in a two-dimensional sense, to $[0, \widetilde{h}]$ with a strictly positive length. For any $\theta \in [\underline{\theta}, 1)$, shareholders with $h = \widetilde{h} < \overline{h}$ incur a carry cost of delayed dividend, and its size does not vanish with $\theta \to 1^-$ exactly because the buffer interval does not fully shrink in length. But the benefit of internal funds $h = \widetilde{h}$ in delaying financing cost fully vanishes because the financing cost does. As such, given a sufficiently high $\theta \in [\overline{\theta}, 1)$ and $h = \widetilde{h}$, it strictly profits to deviate by receiving an immediate dividend payout of $\widetilde{h}$, a contradiction.

Next, Inequality (11) implies that $\underline{h}$ is positive for γ sufficiently high, since net value $V(\overline{h}) - \overline{h}$ is increasing in γ and $\overline{h}$ decreasing. Then, (12) gives $\Delta h \to 0$ as $\gamma \to \infty$ by the same reasoning as $\theta \to 1^- \implies \overline{h} \to 0$. Therefore, it suffices to show that $\underline{h} = \overline{h} - \Delta h \to 0$ as well. Suppose by way of contradiction that $\underline{h} \to \widetilde{h} > 0$.[33] For any fixed small $\varepsilon \in (0, \widetilde{h})$, $V - V_o \to 0$ uniformly on $[\varepsilon, \widetilde{h}]$ as $\gamma \to \infty$, and so does $V' - V_o' \to 0$. Nash bargaining (4) then implies that $V_o'(h) \to 1$ uniformly on $[\varepsilon, \widetilde{h}]$. Therefore, across $h \in [\varepsilon, \widetilde{h}]$, the marginal reduction in the financing rent $(1-\theta)\big(V_o'(h) - 1\big)$ for a marginal increase in h vanishes. Hence, the marginal benefit of $\underline{h}$, taken as a single-dimensional Markov strategy, vanishes—at least with respect to the excess of the arbitrarily small ε. But its marginal cost is constant at $\frac{\rho - r}{\rho + \lambda} > 0$, a contradiction. □

A.2 Section 6 (Extension II: Smooth Investment)

Lemma 3 (Funds-driven investment). *$V_{hh}(A, h) < 0$ for $h < \overline{h}(A) \equiv \inf\{h \mid V_h(A, h) = 1\}$.*

Proof. For this lemma I only consider the case of ordinary differential equations, by removing the state dependence on A. While not rigorously proven, concavity of value functions in internal funds for the general case with exogenous state variable appears heuristically plausible.

[32] This claim actually holds in itself, as Part 2 shows. But since Part 2 relies on Part 4, here I independently reason from the contradictory assumption to tentatively establish the existence of $\underline{\theta}$.

[33] In the logically possible scenario of no convergence, construct an increasing sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ such that $\gamma_n \to +\infty$ and $\underline{h}_n \to \widetilde{h} > 0$ as $n \to +\infty$. Bolzano Weierstrass theorem would guarantee the existence of such a sequence because $\underline{h} \in [0, \overline{h})$ and $\overline{h}$ is decreasing in γ.

For reasons to become clear soon, I use a slightly different notation for this proof only. Let $\mathbf{B}$ (not B) be the set of internal fund levels $h \leq \overline{h}$ where shareholders find it optimal to finance and let $\underline{h} = \sup \mathbf{B}$. Similar to the previous proof of Lemma 2, the argument proceeds as (i) $\overline{h}^o > \overline{h}$, (ii) $V^o_{hh}(h),\ V_{hh}(h) < 0$ on $[0, \overline{h}) \setminus \mathbf{B}$, and (iii) $V^o_{hh}(h),\ V_{hh}(h) < 0$ on $\mathbf{B}$. Unlike in the main model, non-linearity from the investment choice makes it elusive to prove that $\mathbf{B}$, which must contain zero, is an interval from zero. As such, I do not impose the structure of $\mathbf{B}$, except that $\underline{h} < \overline{h}$ for which Proposition 1 still applies, in proving the strict concavity of V. Therefore, Parts (ii) and (iii) must be jointly and iteratively proven. As an aside, numerical algorithm in Appendix B.1 does not rely on $\mathbf{B}$ being an interval, so that the analyses in Sections 6.3 through 6.6 are still valid; and they all generate intervals for $\mathbf{B}$.

(i) $\overline{h}^o > \overline{h}$. Suppose not. First, rearrange the main HJB as

$$\left(\rho + \delta + \frac{1}{\psi}\right) V - \left(A + \frac{1}{2\psi} + \left(r + \delta + \frac{1}{\psi}\right) h\right) V_h - \frac{1}{2\psi}\left(\frac{V^2}{V_h} - 2hV - h^2 V_h\right) - \frac{1}{2}\sigma^2 V_{hh}$$
$$=: \widetilde{\rho} V - \left(\widetilde{\mu} + \widetilde{r} h\right) V_h - \frac{1}{2\psi}\left(\frac{V^2}{V_h} - 2hV - h^2 V_h\right) - \frac{1}{2}\sigma^2 V_{hh} = 0.$$

Impose smooth pasting and super contact for V at $h = \overline{h}$ and V^o at $h = \overline{h}^o$ to obtain

$$\frac{1}{2\psi}\overline{V}^2 - \left(\widetilde{\rho} + \frac{\overline{h}}{\psi}\right)\overline{V} + \left(\widetilde{\mu} + \widetilde{r}\overline{h} - \frac{\overline{h}^2}{2\psi}\right) = 0,$$
$$\frac{1}{2\psi}\overline{V}^{o2} - \left(\widetilde{\rho} + \frac{\overline{h}^o}{\psi}\right)\overline{V}^o + \left(\widetilde{\mu} + \widetilde{r}\overline{h}^o - \frac{\overline{h}^{o2}}{2\psi}\right) + \gamma\left(V(\overline{h}^o) - \overline{V}^o\right) = 0.$$

Define $\Phi(v|h) := \frac{1}{2\psi}v^2 - \left(\widetilde{\rho} + \frac{h}{\psi}\right)v + \left(\widetilde{\mu} + \widetilde{r}h - \frac{h^2}{2\psi}\right)$. Then, it must be that $\Phi_v(\overline{V}|\overline{h}) < 0$. This is because, letting V^* denote the first-best value, both roots that solve $\Phi(v^*|0) = 0$ are strictly between $v^- - \overline{h}$ and $v^+ - \overline{h}$, where $v^\pm$ are the two roots of $\Phi(v|\overline{h}) = 0$, even as $\overline{V} - \overline{h} < V^*$ by the virtue of first-best. Since $V^o(\overline{h}) = \overline{V}^o + (\overline{h} - \overline{h}^o) < \overline{V}$ from $\overline{h} \geq \overline{h}^o$ as assumed, it must be that $\Phi(V^o(\overline{h})|\overline{h}) > 0$, or

$$0 < \frac{1}{2\psi}\left(\overline{V}^{o2} + 2(\overline{h} - \overline{h}^o)\overline{V}^o + (\overline{h} - \overline{h}^o)^2\right) - \left(\widetilde{\rho} + \frac{\overline{h}}{\psi}\right)\left(\overline{V}^o + (\overline{h} - \overline{h}^o)\right) + \left(\widetilde{\mu} + \widetilde{r}\overline{h} - \frac{\overline{h}^2}{2\psi}\right)$$
$$= \frac{1}{2\psi}\overline{V}^{o2} - \left(\widetilde{\rho} + \frac{\overline{h}^o}{\psi}\right)\overline{V}^o + \left(\widetilde{\mu} + \widetilde{r}\overline{h}^o\right) - (\rho - r)(\overline{h} - \overline{h}^o) + \frac{\overline{h}^{o2}}{2\psi} - \frac{\overline{h}^2}{\psi}$$
$$\leq \frac{1}{2\psi}\overline{V}^{o2} - \left(\widetilde{\rho} + \frac{\overline{h}^o}{\psi}\right)\overline{V}^o + \left(\widetilde{\mu} + \widetilde{r}\overline{h}^o - \frac{\overline{h}^{o2}}{2\psi}\right) = -\gamma\left(V(\overline{h}^o) - \overline{V}^o\right),$$

where the second inequality is from $\overline{h} \geq \overline{h}^o$. This contradicts exclusion $V(\overline{h}^o) > \overline{V}^o$.

Before proceeding to Parts (ii) and (iii), it helps to define the following collections of intervals. Construct $\widetilde{\mathbf{B}}$ by removing from $\mathbf{B}$ any isolated points[34]; for example, if $\mathbf{B} = \{0\}$, then $\widetilde{\mathbf{B}}$ is empty. Let $\mathcal{B}$ be a collection of all disjoint and closed intervals with nonempty interiors whose union gives $\widetilde{\mathbf{B}}$. Similarly, let $\mathcal{C}$ be a collection of all disjoint intervals with nonempty interiors whose union gives the closure of $[0, \overline{h}) \setminus \widetilde{\mathbf{B}}$ but excluding $\overline{h}$.[35] Index each interval in each collection by its supremum, i.e.

$$
\begin{aligned}
&\mathcal{B} =: \{B_{h_j} \mid j \in \mathcal{J}_B\}, \quad \mathcal{C} =: \{C_{h_j} \mid j \in \mathcal{J}_C\}, \quad \text{where} \\
&\{h_j \mid j \in \mathcal{J}_B\} := \{\sup B \mid B \in \mathcal{B}\} \text{ and } \{h_j \mid j \in \mathcal{J}_C\} := \{\sup C \mid C \in \mathcal{C}\}.
\end{aligned}
$$

Given the requirement of nonempty interiors, both index sets $\mathcal{J}_B,\ \mathcal{J}_C$ are at most countable.

(ii) $V^o_{hh},\ V_{hh} < 0$ on C_{h_j} for all $j \in \mathcal{J}_C$. First, consider $C_{\overline{h}} = [\underline{h}, \overline{h})$. I will establish the claim for V^o first. Differentiate the HJB for V^o at $h = \overline{h}^o > \overline{h}$ for a third (right) derivative:

$$
\overline{V}^o_{hhh} = \frac{2}{\sigma^2}\left(\rho - r + \frac{2}{\psi}\overline{h}^o\right) > 0,
$$

where $\overline{h}^o > \overline{h}$ gives $V_h(\overline{h}^o) = \overline{V}^o_h$. Therefore, $V^o_{hh} < 0$ on some neighborhood below $\overline{h}^o$. Suppose by way of contradiction that there exists $\widehat{h} \in [\underline{h}, \overline{h}^o)$ such that $V^o_{hh} < 0$ on $(\widehat{h}, \overline{h}^o)$ but $V^o_{hh}(\widehat{h}) = 0$. Then,

$$
0 \geq V^o_{hhh}(\widehat{h}) = \frac{2}{\sigma^2}\left((\rho - r)V^o_h(\widehat{h}) + \frac{2}{\psi}\widehat{h}V^o_h(\widehat{h}) - \gamma\Big(V_h(\widehat{h}) - V^o_h(\widehat{h})\Big)\right).
$$

Therefore, it must be that $\gamma > 0$ and $V_h(\widehat{h}) - V^o_h(\widehat{h}) > 0$. Since $V_h(\overline{h}) - V^o_h(\overline{h}) < 0$ from $\overline{h}^o > \overline{h}$, intermediate value theorem implies there is $\widetilde{h} \in (\widehat{h}, \overline{h})$ such that $V_h(\widetilde{h}) - V^o_h(\widetilde{h}) = 0$.

$$
\begin{aligned}
V_{hh}(\widetilde{h}) - V^o_{hh}(\widetilde{h}) &= \frac{2}{\sigma^2}\left(\left(\widetilde{\rho} + \gamma + \frac{\widetilde{h}}{\psi}\right)\Big(V(\widetilde{h}) - V^o(\widetilde{h})\Big) - \frac{1}{2\psi}\frac{V(\widetilde{h})^2 - V^o(\widetilde{h})^2}{V_h(\widetilde{h})}\right) \\
&= \frac{2}{\sigma^2}\Bigg(\widetilde{\rho} + \gamma - \frac{1}{\psi}\underbrace{\left(\frac{V(\widetilde{h}) + V^o(\widetilde{h})}{2V_h(\widetilde{h})} - \widetilde{h}\right)}_{=:(a)}\Bigg)\underbrace{\Big(V(\widetilde{h}) - V^o(\widetilde{h})\Big)}_{>0}.
\end{aligned}
$$

Note that $(a) < \psi\widetilde{\rho}$. This is established by the following reasoning. First,

$$
\frac{V(\widetilde{h}) + V^o(\widetilde{h})}{2V_h(\widetilde{h})} - \widetilde{h} < \frac{V(\widetilde{h})}{V_h(\widetilde{h})} - \widetilde{h} < \overline{V} - \overline{h},
$$

because $V_{hh} < 0$ on $[\underline{h}, \overline{h})$—which holds by the present reasoning (specifically, up to imposing

[34]For purely technical reasons (to address an analytically pathological situation that is highly implausible anyway given the setup), here I use a slightly modified definition of isolated points: a point $x \in X$ is isolated in X if for any $\varepsilon > 0$, neither $(x - \varepsilon, x)$ nor $(x, x + \varepsilon)$ is a subset of X. Barring unlikely pathology, the standard definition of isolated points will also do.

[35]The wording ensures that all the intervals in $\mathcal{C}$ are closed except $[\underline{h}, \overline{h})$.

non-positive third derivative of V at $h = V_{hh}^{(-1)}(0)$) but with $\gamma = 0$—and so

$$\frac{\partial}{\partial h}\left(\frac{V(h)}{V_h(h)} - h\right) = 1 - \frac{V(h)}{V_h(h)^2}V_{hh}(h) - 1 > 0.$$

Next, $\overline{V} - \overline{h} < \psi\widetilde{\rho}$, because $\overline{V}$ is, as shown when proving (i), the lower root of $\Phi(v|\overline{h}) = 0$, i.e.

$$\overline{V} = \overline{h} + \psi\left(\widetilde{\rho} - \sqrt{\left(\widetilde{\rho} + \frac{\overline{h}}{\psi}\right)^2 - 2\frac{\widetilde{\mu} + \widetilde{r}\overline{h}}{\psi} + \frac{\overline{h}^2}{\psi^2}}\right) < \overline{h} + \psi\widetilde{\rho}.$$

As such, $V_{hh}(\widetilde{h}) - V_{hh}^o(\widetilde{h}) > 0$. That is, $V_h - V_h^o$ cannot cross zero from above, as required by $\overline{h} > \widehat{h}$ and $V_h(\widehat{h}) - V_h^o(\widehat{h}) > 0 > V_h(\overline{h}) - V_h^o(\overline{h})$, a contradiction.

As for V on $C_{\overline{h}}$, the same reasoning as above but with $\gamma = 0$ gives a contradiction more directly since $0 < V_{hhh}(\widehat{h})$. Next, for other C_{h_j}'s with $h_j < \overline{h}$, first assume that $V_{hh}^o < 0$ on $[h_j, \overline{h}^o)$ and $V_{hh} < 0$ on $[h_j, \overline{h})$[36] and proceed with the above reasoning starting at the contradiction-inducing assumption of the existence of $\widehat{h} \in C_{h_j}$ where the second derivative becomes zero for the first time (with h going down). When it comes to the application of intermediate value theorem towards the existence of $\widetilde{h}$, which is relevant only for V^o, it needs to be invoked that such a $\widetilde{h}$ can only exist on one of the C_{h_j}'s; on any $B_h \in \mathcal{B}$, $V_h = \theta + (1-\theta)V_h^o < V_h^o$ since $V_h^o > 1$ by the assumed strict concavity of V^o and $\overline{V}_h^o = 1$. Jointly with Part (iii), then, the initial assumption of strict concavity above h_j will hold.

(iii) V_{hh}^o, $V_{hh} < 0$ on B_{h_j} for all $j \in \mathcal{J}_B$. Start by assuming that $V_{hh}^o < 0$ on $[h_j, \overline{h}^o)$, which will be iteratively validated jointly with Part (ii). Suppose by way of contradiction that there exists $\widehat{h} \in B_{h_j} \setminus \{h_j\}$ such that $V_{hh}^o < 0$ on $(\widehat{h}, h_j)$ but $V_{hh}^o(\widehat{h}) = 0$. Since $B_{h_j} \subset \mathbf{B}$, substitute the identity $V = \theta\big(\overline{V} - (\overline{h} - h)\big) + (1-\theta)V^o$ on B_{h_j} and differentiate the HJB for V^o to obtain the third derivative at $h = \widehat{h}$ as

$$V_{hhh}^o(\widehat{h}) = \frac{2}{\sigma^2}\left(\left(\rho - r + 2\frac{\widehat{h}}{\psi}\right)V_h^o(\widehat{h}) + \theta\gamma\Big(V_h^o(\widehat{h}) - 1\Big)\right) > 0,$$

because $V_h^o(\widehat{h}) > 1$ from $V_{hh}^o < 0$ on $(\widehat{h}, \overline{h}^o)$ and $\overline{V}_h^o = 1$. This contradicts $V_{hh}^o(\widehat{h}) = 0$ and $V_{hh}^o < 0$ on a neighborhood above $\widehat{h}$. Hence, $V_{hh}^o < 0$ and also $V_{hh} = (1-\theta)V_{hh}^o < 0$ on B_{h_j}.

Parts (ii) and (iii) jointly—and iteratively[37] from above—prove that $V_{hh} < 0$ on $[0, \overline{h})$. □

[36] Because super contact is not invoked at the boundaries of $B_h \in \mathcal{B}$, it is in principle possible that the second derivative of V is not continuous, which in principle should allow $V_{hh}^-(h_j) \geq 0 > V_{hh}^+(h_j)$. But continuity of V_{hh} actually holds, because the HJB equation $\rho V - rhV_h = \mathcal{H}(V) + \mathcal{K}(V)$, which includes the term $\frac{1}{2}\sigma^2 V_{hh}$, should hold at the boundary h_j using the right derivatives as well—due to the absence of underlying payoff irregularities such as kinks at the boundary, shareholders at it must be indifferent between instantaneous waiting and immediate financing. Since V and V_h are continuous by value matching and smooth pasting, it must follow that V_{hh} is also continuous.

[37] This 'iteration' is well-defined because of the countability of $\mathcal{J}_B$ and $\mathcal{J}_C$, as well as the continuity of V_{hh}^o that follows from the single HJB equation that holds on $[0, \overline{h}]$ across which both V and V_h are

Proposition 5 (Financial slack and underinvestment). *For each A, and suppressing notation for dependence on A, $\underline{h} > 0$ if and only if*

$$(1-\theta)\gamma + \frac{1}{2}\underbrace{\left(\bar{\imath} - i(0)\right)}_{(a)} > \frac{(\rho - r)\overline{h}}{\overline{V} - \overline{h}}, \tag{A.10}$$

in which case

$$\begin{aligned}\theta\gamma\left(\overline{V} - \underline{V} - \Delta h\right) + \frac{1}{2}\theta \underbrace{\left(\bar{\imath} - \underline{i}\right)}_{(b)} \underbrace{\left(\overline{V} - \overline{h}\right)}_{=\overline{W}_K} \\ = \theta(\rho - r)\Delta h + \frac{1}{2}(1-\theta) \underbrace{\left(\underline{i} - \underline{i}^o\right)}_{(c)} \underbrace{\left(\underline{V}^o - \underline{h}\underline{V}^o_h\right)}_{=\underline{W}^o_K}.\end{aligned} \tag{A.11}$$

Proof. Denote $B \subset \mathbb{R}_{++} \times \mathbb{R}_+$ as the set of states (A, h) at which shareholders optimally refinance, and $B(A) \subset \mathbb{R}_+$ as its projection onto the A dimension. As in Lemma 2, by optimality of immediate financing relative to instantaneously postponed financing, for all $(A, h) \in B$,

$$\rho V - rhV_h \geq \mathcal{H}(V) + \mathcal{K}(V) + \mathcal{A}(V), \tag{A.12}$$

where

$$\begin{aligned}\mathcal{H}(V) &:= \left(A + \frac{1}{2\psi}\right)V_h + \frac{1}{2}\sigma^2 \cdot V_{hh} - \left(\delta + \frac{1}{2\psi}\right)\left(V - hV_h\right), \\ \mathcal{K}(V) &:= \frac{1}{2\psi}\left(\frac{V}{V_h} - h - 1\right)\left(V - h \cdot V_h\right), \\ \mathcal{A}(V) &:= \mu_A(A)V_A + \frac{1}{2}\sigma_A^2 V_{AA}.\end{aligned}$$

At the same time, Section 2.3 still holds, so that, letting V^o denote the excluded value,

$$\begin{aligned}V(A,h) &= \theta\big(V(A,\overline{h}(A)) - \overline{h}(A) + h\big) + (1-\theta) \cdot V^o(A,h) \\ \implies (1-\theta)\big(V(A,\overline{h}(A)) - V^o(A,h)\big) &= V(A,\overline{h}(A)) - V(A,h) - \theta(\overline{h}(A) - h).\end{aligned}$$

Lastly, V^o on B satisfies

$$\rho V^o - rhV^o_h = \mathcal{H}(V^o) + \mathcal{K}(V^o) + \mathcal{A}(V^o) + \gamma\Big(V(A,h) - V^o(A,h)\Big).$$

continuous due to value matching and smooth pasting.

Let us establish that $\mathcal{H}$ and $\mathcal{A}$ are linear operators in the appropriate sense. First,

$$\begin{aligned}
V_h(A,h) &= \theta \underbrace{V_h(A,\overline{h}(A))}_{=1} + (1-\theta)V_h^o(A,h),\\
V_{hh}(A,h) &= \theta \underbrace{V_{hh}(A,\overline{h}(A))}_{=0} + (1-\theta)V_{hh}^o(A,h),\\
V(A,h) - hV_h(A,h) &= \theta\Big(V(A,\overline{h}(A)) - \overline{h}(A)\underbrace{V_h(A,\overline{h}(A))}_{=1}\Big) + (1-\theta)\Big(V^o(A,h) - hV_h^o(A,h)\Big).
\end{aligned}$$

Consequently, $\mathcal{H}(V)(A,h) = \theta\mathcal{H}(V)(A,\overline{h}(A)) + (1-\theta)\mathcal{H}(V^o)(A,h)$.

Second, $\overline{h}(A)$ satisfies both $V_h(A,\overline{h}(A)) = 1$ and $V_{hh}(A,\overline{h}(A)) = 0$. Implicit function theorem on the second identity gives a well-defined $\overline{h}'(A)$. Then, differentiating the first identity in A gives $0 = V_{Ah}(A,\overline{h}(A)) + V_{hh}(A,\overline{h}(A))\cdot\overline{h}'(A) = V_{Ah}(A,\overline{h}(A))$. Therefore,

$$\begin{aligned}
\frac{d}{dA}\Big(V(A,\overline{h}(A)) - \overline{h}(A)\Big) &= V_A(A,\overline{h}(A)) + V_h(A,\overline{h}(A))\overline{h}'(A) - \overline{h}'(A) = V_A(A,\overline{h}(A)),\\
\frac{d^2}{dA^2}\Big(V(A,\overline{h}(A)) - \overline{h}(A)\Big) &= V_{AA}(A,\overline{h}(A)) + V_{Ah}(A,\overline{h}(A))\overline{h}'(A) = V_{AA}(A,\overline{h}(A)).
\end{aligned}$$

As such, $\mathcal{A}(V)(A,h) = \theta\mathcal{A}(V)(A,\overline{h}(A)) + (1-\theta)\mathcal{A}(V^o)(A,h)$.

Therefore, the same derivation as in the proof of Lemma 2 leads to the cancellation of the linear $\mathcal{H}$ and $\mathcal{A}$ while $\mathcal{K}$ survives, giving the rearrangement of Inequality (A.12) as

$$\begin{aligned}
(1-\theta)\gamma\Big(V(A,h) - V^o(A,h)\Big) &\geq \theta(\rho - r)\big(\overline{h}(A) - h\big)\\
&\quad + \mathcal{K}(V)(A,h) - \theta\mathcal{K}(V)(A,\overline{h}(A)) - (1-\theta)\mathcal{K}(V^o)(A,h). \qquad \text{(A.13)}
\end{aligned}$$

Note, as an aside, that the first line is exactly Inequality (A.6).

Using the two identities

$$\begin{aligned}
(1-\theta)\Big(V(A,h) - V^o(A,h)\Big) &= \theta\Big(V(A,\overline{h}(A)) - V(A,h) - \big(\overline{h}(A) - h\big)\Big),\\
V(A,h) - hV_h(A,h) &= \theta\Big(V(A,\overline{h}(A)) - \overline{h}(A)V_h(A,\overline{h}(A))\Big) + (1-\theta)\Big(V^o(A,h) - hV_h^o(A,h)\Big),
\end{aligned}$$

which hold for $(A,h) \in B$, evaluate (A.13) at $h = 0$ to obtain (A.10), and enforce equality for (A.13) to obtain (A.11). □

A.3 Section C.1 (Startups)

Proposition C.1.1 (Comparative statics — Startups without re-inclusion).

$$\frac{\partial \overline{h}}{\partial \theta} = -\frac{1}{\rho} \cdot \frac{\lambda \Pi - \kappa - \rho \overline{h}}{\exp\left(\frac{\rho+\lambda}{\kappa} \cdot \overline{h}\right) - \theta} < 0, \ \lim_{\theta \to 0} \overline{h} < \frac{\lambda \Pi - \kappa}{\rho}, \ \lim_{\theta \to 1} \overline{h} = 0, \quad \textit{and} \ \lim_{\theta \to 1} \frac{\partial \overline{h}}{\partial \theta} = -\infty, \tag{A.14}$$

$$\frac{\partial \overline{h}}{\partial \Pi} = \frac{\lambda}{\rho} \cdot \frac{1-\theta}{\exp\left(\frac{\rho+\lambda}{\kappa} \cdot \overline{h}\right) - \theta} > 0, \tag{A.15}$$

$$\lim_{\lambda \to \kappa/\Pi} \overline{h} = \lim_{\lambda \to \infty} \overline{h} = 0. \tag{A.16}$$

Proof. Most results above are straightforward. For the first inequality in (A.14), it is sufficient to show that $\lambda \Pi - \kappa - \rho \overline{h}(0) > 0$ where $\overline{h}(\theta)$ is $\overline{h}$ expressed as a function of $\theta \in [0, 1]$.

To prove the claim, note that

$$\begin{aligned} & \lambda \Pi - \kappa > \rho \overline{h}(0) \\ \iff \exp\left(\frac{\rho+\lambda}{\kappa} \cdot \frac{\lambda \Pi - \kappa}{\rho}\right) & > \exp\left(\frac{\rho+\lambda}{\kappa} \cdot \overline{h}(0)\right) \\ & = \frac{\lambda}{\rho}\left((\rho+\lambda)\frac{\Pi}{\kappa} - 1\right) \quad \because \text{ Equation } (C.3) \text{ with } \theta \equiv 0 \\ & = 1 + \frac{\rho+\lambda}{\kappa} \cdot \frac{\lambda \Pi - \kappa}{\rho} \\ \iff \frac{\rho+\lambda}{\kappa} \cdot \frac{\lambda \Pi - \kappa}{\rho} & > 0, \end{aligned}$$

which is equivalent to Assumption C.1.1. □

Proposition C.1.2 (Startup financing). *Denote*

$$\eta := \frac{(1-\theta)\gamma}{\rho + \lambda + (1-\theta)\gamma}, \quad \xi := \frac{\lambda \Pi - \kappa}{\rho}.$$

The equilibrium is characterized by $\underline{h} = 0$ and $\overline{h}$ implicitly defined by Equation (C.3) if

$$\theta \rho \big((\rho+\lambda)\eta\xi + \kappa\big) + (1-\theta)\lambda\big((\rho+\lambda)\Pi - \kappa\big) \geq \rho\kappa \cdot \exp\left(\frac{\rho+\lambda}{\kappa} \cdot \eta\xi\right). \tag{A.17}$$

If the inequality is strictly reversed, then $\underline{h} = \overline{h} - \Delta h > 0$ and $\overline{h} > \Delta h$ is implicitly defined by

$$\begin{aligned} & \frac{1-\theta}{\rho+\lambda}\left((\rho+\lambda+\theta\gamma)\lambda\Pi - \theta\rho\gamma\left(\overline{h} + \frac{\kappa}{\rho+\lambda+\theta\gamma}\right) - (\lambda+\theta\gamma)\kappa\right) \\ & \qquad = \rho\left(\frac{\rho+\lambda+\gamma}{\gamma}\Delta h + (1-\theta)\frac{\kappa}{\rho+\lambda+\theta\gamma}\right) \cdot \exp\left(\frac{\rho+\lambda+\theta\gamma}{\kappa} \cdot (\overline{h} - \Delta h)\right), \end{aligned} \tag{A.18}$$

where $\Delta h = \bar{h} - \underline{h} > 0$ *is given by*

$$1 + \frac{\rho+\lambda}{\kappa} \cdot \left(1 + \frac{\rho+\lambda}{\gamma}\right) \cdot \Delta h = \exp\left(\frac{\rho+\lambda}{\kappa} \cdot \Delta h\right). \tag{A.19}$$

Proof. If (A.17) holds, then Inequality (C.7) also holds with $\bar{h}$ defined by (C.3). Therefore, $\underline{h} = 0$. Now consider the case where (A.17) fails. Let us use Smooth Pasting and Stationary Recursion to determine $(\underline{h}, \bar{h})$.

Smooth Pasting. For ease of notation, denote $\underline{V}_o := V_o(\underline{h})$ and $\underline{V} := \theta(V(\bar{h}) - \bar{h} + \underline{h}) + (1 - \theta)\underline{V}_o$. The shareholders' non-excluded and excluded value functions for $h \in [\underline{h}, \bar{h}]$ are given by

$$\begin{aligned} V(h) &= \int_0^{(h-\underline{h})/\kappa} \lambda e^{-(\rho+\lambda)t} \big(\Pi + h - \kappa t\big)\, dt + e^{-\frac{\rho+\lambda}{\kappa}(h-\underline{h})} \cdot \underline{V} \\ &= \frac{\lambda}{\rho+\lambda}\left(\Pi + h - \frac{\kappa}{\rho+\lambda}\right) - \left(\frac{\lambda}{\rho+\lambda}\left(\Pi + \underline{h} - \frac{\kappa}{\rho+\lambda}\right) - \underline{V}\right) \cdot e^{-\frac{\rho+\lambda}{\kappa}(h-\underline{h})}, \\ V_o(h) &= V(h) - e^{-\frac{\rho+\lambda+\gamma}{\kappa}(h-\underline{h})} \cdot \left(\underline{V} - \underline{V}_o\right). \end{aligned}$$

$V_o(h)$ is derived based on the observation that, given the strategy of waiting on $(\underline{h}, h)$ regardless of market access, the only difference that exclusion creates is that you find yourself with $\underline{V}_o$ instead of $\underline{V}$ at $h = \underline{h}$ if neither success nor re-inclusion occurs while the internal funds h run down to $\underline{h}$.

Note that

$$\begin{aligned} V(\bar{h}) &= \frac{1}{\rho+\lambda}\big(\lambda(\Pi + \bar{h}) - \kappa\big) \\ \implies \frac{\rho\kappa}{(\rho+\lambda)^2} &= \left(\frac{\lambda}{\rho+\lambda}\left(\Pi + \underline{h} - \frac{\kappa}{\rho+\lambda}\right) - \underline{V}\right) \cdot \exp\left(-\frac{\rho+\lambda}{\kappa}(\bar{h} - \underline{h})\right) \\ \implies V(h) &= \frac{\lambda}{\rho+\lambda}\left(\Pi + h - \frac{\kappa}{\rho+\lambda}\right) - \frac{\rho\kappa}{(\rho+\lambda)^2} \cdot \exp\left(\frac{\rho+\lambda}{\kappa} \cdot (\bar{h} - h)\right), \text{ and} \\ \underline{V} - \underline{V}_o &= \frac{\theta}{1-\theta}\left(-\frac{\rho}{\rho+\lambda}\left(\bar{h} - \underline{h} + \frac{1}{\rho+\lambda}\kappa\right) + \frac{\rho\kappa}{(\rho+\lambda)^2} \cdot \exp\left(\frac{\rho+\lambda}{\kappa} \cdot (\bar{h} - \underline{h})\right)\right). \end{aligned}$$

Next, denote by V_d a payoff function on $(\underline{h}, \bar{h}]$ for the deviation strategy of immediate financing. That is, for $h \in (\underline{h}, \bar{h}]$,

$$V_d(h) := \theta\big(V(\bar{h}) - \bar{h} + h\big) + (1-\theta)V_o(h).$$

Smooth pasting condition is

$$V'(\underline{h}) = V_d'(\underline{h}) = \theta + (1-\theta)\left(V'(\underline{h}) + \frac{\rho+\lambda+\gamma}{\kappa}(\underline{V} - \underline{V}_o)\right),$$

which, after some algebra, is equivalent to Equation (A.19).

Stationary Recursion. This time, start by deriving V_o on $[0, \underline{h}]$, which satisfies:

$$\begin{aligned}
&\rho V_o(h) = \gamma\Big(\theta\big(V(\overline{h}) - \overline{h} + h\big) + (1-\theta)V_o(h) - V_o(h)\Big) + \lambda\big(\Pi + h - V_o(h)\big) - \kappa V_o'(h),\\
&V_o(0) = 0\\
\implies\ & V_o(h) = \frac{1}{\rho+\lambda+\theta\gamma}\Bigg[\lambda\Pi + \theta\gamma\big(V(\overline{h}) - \overline{h}\big) - \frac{\lambda+\theta\gamma}{\rho+\lambda+\theta\gamma}\kappa + (\lambda+\theta\gamma)h\\
&\qquad - \left(\lambda\Pi + \theta\gamma\big(V(\overline{h}) - \overline{h}\big) - \frac{\lambda+\theta\gamma}{\rho+\lambda+\theta\gamma}\kappa\right)\cdot \exp\left(-\frac{\rho+\lambda+\theta\gamma}{\kappa}\cdot h\right)\Bigg].
\end{aligned}$$

Since $\Delta h := \overline{h} - \underline{h} > 0$ has been determined by Equation (A.19), $\overline{h}$ is obtained by the recursion:

$$V(\overline{h} - \Delta h) = \theta\Big(V(\overline{h}) - \Delta h\Big) + (1-\theta)V_o(\overline{h} - \Delta h).$$

Simplifying and substituting (A.19) give Equation (A.18). □

Proposition C.1.4 (Breakeven re-inclusion — Startups). *$\underline{\gamma}$ strictly decreases in Π, and converges to zero as Π goes to ∞. It goes to ∞ as Π goes down to κ/λ, the lower bound in Assumption C.1.1.*

Proof. $\underline{\gamma}$ is defined by $\underline{\eta}\xi = \overline{h}^*$ where $(\underline{h}^*, \overline{h}^*)$ is the equilibrium associated with $\gamma = \underline{\gamma}$ and $(\underline{\eta}, \xi)$ given by Proposition C.1.2. Since $\gamma = \underline{\gamma}$ implies $\underline{h}^* = 0$, Proposition C.1.1, in particular Equation (C.5) holds with $\overline{h}$ replaced with $\underline{\eta}\xi$. Note that

$$\begin{aligned}
&\frac{\lambda}{\rho}\cdot\frac{1-\theta}{\exp\left(\frac{\rho+\lambda}{\kappa}\cdot\underline{\eta}\xi\right) - \theta} = \frac{\partial(\underline{\eta}\xi)}{\partial\Pi} = \underline{\eta}\frac{\partial\xi}{\partial\Pi} + \xi\frac{\partial\underline{\eta}}{\partial\Pi}, \quad \frac{\partial\xi}{\partial\Pi} = \frac{\lambda}{\rho}\\
\implies & \frac{\partial\underline{\eta}}{\partial\Pi} = \frac{\lambda}{\lambda\Pi - \kappa}\left(\frac{1-\theta}{\exp\left(\frac{\rho+\lambda}{\kappa}\cdot\underline{\eta}\xi\right) - \theta} - \frac{1-\theta}{1 + \frac{\rho+\lambda}{\underline{\gamma}} - \theta}\right).
\end{aligned}$$

Since $\overline{h}^* = \underline{\eta}\xi$, smooth pasting holds at $\underline{h}^* = 0$. Therefore, from Equation (C.10),

$$\exp\left(\frac{\rho+\lambda}{\kappa}\cdot\underline{\eta}\xi\right) = 1 + \frac{\rho+\lambda}{\underline{\gamma}}\cdot\frac{\rho+\lambda+\underline{\gamma}}{\kappa}\cdot\underline{\eta}\xi.$$

Next, assume for now that $\underline{\eta}\xi = \overline{h}^* > \frac{\kappa}{\rho+\lambda+\underline{\gamma}}$, which will be established at the end. Then,

$$\exp\left(\frac{\rho+\lambda}{\kappa}\cdot\underline{\eta}\xi\right) > 1 + \frac{\rho+\lambda}{\underline{\gamma}},$$

And so $\partial\underline{\eta}/\partial\Pi < 0$. Since $\underline{\eta} = \frac{(1-\theta)\underline{\gamma}}{\rho+\lambda+(1-\theta)\underline{\gamma}}$, we have $\partial\underline{\gamma}/\partial\Pi < 0$.

Next is the convergence claim. Since $\overline{h}^*$ satisfies Equation (C.3), it goes to ∞ as Π does. Note that $\Delta h^* := \overline{h}^* - \underline{h}^* = \overline{h}^*$ satisfies Equation (C.10) with $\gamma = \underline{\gamma}$. Since ρ, λ, κ are fixed, the only way for the solution of Equation (C.10) to be satisfied by a Δh that diverges to infinity is by having the linear coefficient on LHS also diverge to infinity. This can only be achieved if $\underline{\gamma}$ goes to zero, as claimed.

The divergence claim is straightforward from $\underline{\eta}\xi > \frac{\kappa}{\rho+\lambda+\underline{\gamma}}$. Since $\xi = (\lambda\Pi - \kappa)/\rho$ and $\underline{\eta}$ is in the unit interval, LHS vanishes as Π goes down to κ/λ. Therefore, RHS also vanishes, i.e. $\underline{\gamma} \to \infty$.

Finally, as for the intermediate claim on the strict lower bound on $\underline{\eta}\xi$, first rearrange the Smooth Pasting condition—i.e. Equation (C.10)—into the following:

$$\frac{\gamma}{\rho+\lambda} \cdot \left(\exp\left(\frac{\rho+\lambda}{\kappa} \cdot \Delta h \right) - 1 \right) = \frac{\rho+\lambda+\gamma}{\kappa} \cdot \Delta h.$$

Denote LHS and RHS above as functions of Δh. Note that LHS(0) = RHS(0) and LHS′(0) < RHS′(0). Therefore, LHS crosses RHS only once and from below on $\mathbb{R}_{++}$. Note that

$$\text{LHS}\left(\frac{\kappa}{\rho+\lambda+\gamma} \right) = \frac{\gamma}{\rho+\lambda} \left(\exp\left(\frac{\rho+\lambda}{\rho+\lambda+\gamma} \right) - 1 \right) < 1 = \text{RHS}\left(\frac{\kappa}{\rho+\lambda+\gamma} \right).$$

This holds for any set of parameters because, letting $f(x) := x \cdot \left(\exp\left(\frac{1}{1+x} \right) - 1 \right)$, we have

$$\forall\, x > 0,\ f'(x) > 0, \text{ and } \lim_{x\to\infty} f(x) = 1.$$

Therefore, LHS(Δh) < RHS(Δh) for any $\Delta h \in (0, \frac{\kappa}{\rho+\lambda+\gamma}]$. That is, if Smooth Pasting holds at $\underline{h}$, then it must be that $\Delta h = \overline{h} - \underline{h} > \frac{\kappa}{\rho+\lambda+\gamma}$. Since $\gamma = \underline{\gamma}$ means that Smooth Pasting holds at $\underline{h}^* = 0$, it must be that $\underline{\gamma}\xi = \overline{h}^* = \Delta h^* > \frac{\kappa}{\rho+\lambda+\underline{\gamma}}$, as claimed. □

A.4 Section C.2 (Operating firms)

Proposition C.2.1 (Comparative statics — Operating firms without re-inclusion).

$$\frac{\partial\overline{h}}{\partial\theta} = -\frac{1}{\rho} \cdot \frac{\pi - \rho\overline{h}}{\frac{\Phi}{\Phi+\phi} \cdot e^{-\phi\cdot\overline{h}} + \frac{\phi}{\Phi+\phi} \cdot e^{\Phi\cdot\overline{h}} - \theta} < 0,$$

$$\lim_{\theta\to 0} \overline{h} < \frac{\pi}{\rho},\ \lim_{\theta\to 1} \overline{h} = 0,\ \textit{and}\ \lim_{\theta\to 1} \frac{\partial\overline{h}}{\partial\theta} = -\infty, \tag{A.20}$$

$$\frac{\partial\overline{h}}{\partial\sigma^2} = \frac{\rho}{\pi^2 + 2\rho\sigma^2} \cdot \frac{\overline{h}\sqrt{\pi^2+2\rho\sigma^2}\left(e^{\Phi\cdot\overline{h}} + e^{-\phi\cdot\overline{h}}\right) - \sigma^2\left(e^{\Phi\cdot\overline{h}} - e^{-\phi\cdot\overline{h}}\right)}{\sqrt{\pi^2+2\rho\sigma^2}\left(e^{\Phi\cdot\overline{h}} + e^{-\phi\cdot\overline{h}} - 2\theta\right) - \pi\left(e^{\Phi\cdot\overline{h}} - e^{-\phi\cdot\overline{h}}\right)} > 0,$$

$$\lim_{\sigma^2\to 0} \overline{h} = 0,\ \lim_{\sigma^2\to\infty} \overline{h} = \frac{\pi}{\rho},\ \lim_{\sigma^2\to 0} \frac{\partial\overline{h}}{\partial\sigma^2} = \infty,\ \textit{and}\ \lim_{\sigma^2\to\infty} \frac{\partial\overline{h}}{\partial\sigma^2} = 0, \tag{A.21}$$

$$\lim_{\pi\to 0} \overline{h} = \lim_{\pi\to\infty} \overline{h} = 0. \tag{A.22}$$

Proof. Most results are straightforward. As for the sign of $\partial\overline{h}/\partial\sigma^2$, first write the denominator of the second fraction as $e^{-\phi\cdot\overline{h}}\cdot DN(\overline{h})$ where

$$DN(z) := \sqrt{\pi^2+2\rho\sigma^2}\left(e^{(\Phi+\phi)\cdot z}+1-2\theta\cdot e^{\phi\cdot z}\right)-\pi\left(e^{(\Phi+\phi)\cdot z}-1\right).$$

Then, it is easily verified that $DN(0)>0$, $DN'(z)>0$. Therefore, the denominator is positive. Next, write the numerator as $e^{-\phi\cdot\overline{h}}\cdot NM(\overline{h})$ where

$$NM(z) := z\sqrt{\pi^2+2\rho\sigma^2}\left(e^{(\Phi+\phi)\cdot z}+1\right)-\sigma^2\left(e^{(\Phi+\phi)\cdot z}-1\right).$$

Then, it is easily verified that $NM(0)=NM'(0)=NM''(0)=0<NM'''(z)$ for all $z\geq 0$. Therefore, for any positive z, NM is positive as well. Since $\overline{h}>0$, positivity is established.

Lastly, the limit of $\partial\overline{h}/\partial\sigma^2$ as $\sigma^2\to 0$ is established as follows. First, note that as $\sigma^2\to 0$,

$$\frac{\Phi/\phi}{\Phi+\rho}\to\frac{\pi}{\rho},\quad \frac{\phi/\Phi}{\Phi+\phi}\to 0.$$

Since $\phi\to\rho/\pi$ and $\overline{h}\to 0$, the first term on the right-hand side of (C.13) goes to π/ρ. Therefore,

$$\frac{\phi/\Phi}{\Phi+\phi}\cdot e^{\Phi\overline{h}}\to(1-\theta)\frac{\pi}{\rho}>0,$$

implying that $e^{\Phi\overline{h}}\to+\infty$. Since $\Phi\overline{h}=\left(\sqrt{\pi^2+2\rho\sigma^2}+\pi\right)\frac{\overline{h}}{\sigma^2}$, it follows that $\overline{h}/\sigma^2\to+\infty$. Since $\overline{h}\to 0$ as $\sigma^2\to 0$, L'hospital's rule establishes that

$$+\infty=\lim_{\sigma^2\to 0}\frac{\overline{h}}{\sigma^2}=\lim_{\sigma^2}\frac{\partial\overline{h}/\partial\sigma^2}{\partial\sigma^2/\partial\sigma^2}=\lim_{\sigma^2\to 0}\frac{\partial\overline{h}}{\partial\sigma^2},$$

as claimed. □

Proposition C.2.2 (Operating firm financing). *Denote*

$$\eta := \frac{(1-\theta)\gamma}{\rho+(1-\theta)\gamma},\quad \xi := \frac{\pi}{\rho}.$$

The equilibrium is characterized by $\underline{h}=0$ *and* $\overline{h}$ *implicitly defined by Equation (C.13) if*

$$\theta(1-\eta)\xi\leq\frac{1}{\Phi+\phi}\left(\frac{\Phi}{\phi}\cdot\exp\left(-\phi\cdot\eta\xi\right)-\frac{\phi}{\Phi}\cdot\exp\left(\Phi\cdot\eta\xi\right)\right). \tag{A.23}$$

If the inequality is strictly reversed, then $\underline{h} = \overline{h} - \Delta h > 0$ *and* $\overline{h} > \Delta h$ *is implicitly defined by*

$$\left(\frac{\pi}{\rho+\theta\gamma} - \frac{1}{1-\theta}\left(1+\frac{\rho}{\gamma}\right)\cdot\Delta h\right)\cdot\frac{\phi_o\exp\big(\phi_o(\overline{h}-\Delta h)\big)+\Phi_o\exp\big(-\Phi_o(\overline{h}-\Delta h)\big)}{\exp\big(\phi_o(\overline{h}-\Delta h)\big)-\exp\big(-\Phi_o(\overline{h}-\Delta h)\big)}$$
$$+\,\theta\left(\Phi_o+\phi_o\right)\frac{\gamma}{\rho}\left(\left(1+\frac{\rho}{\rho+\theta\gamma}\right)\frac{\pi}{\rho}-\overline{h}\right)\cdot\frac{\exp\left(-2\pi(\overline{h}-\Delta h)/\sigma^2\right)}{\exp\big(\phi_o(\overline{h}-\Delta h)\big)-\exp\big(-\Phi_o(\overline{h}-\Delta h)\big)}$$
$$=\frac{\rho+\theta\gamma}{(1-\theta)\rho}\left(\phi\left(\frac{\pi}{\rho}-\left(1+\frac{\rho}{\gamma}\right)\cdot\Delta h\right)+\frac{\phi}{\Phi}\cdot\exp\big(\Phi\cdot\Delta h\big)\right)-\frac{\theta}{1-\theta}\left(1+\frac{\gamma}{\rho}\right), \quad \text{(A.24)}$$

where

$$\Phi_o := (\sqrt{\pi^2+2(\rho+\theta\gamma)\sigma^2}+\pi)/\sigma^2, \quad \phi_o := (\sqrt{\pi^2+2(\rho+\theta\gamma)\sigma^2}-\pi)/\sigma^2$$

and $\Delta h = \overline{h} - \underline{h} > 0$ *is given by*

$$\frac{\pi}{\rho}-\left(1+\frac{\rho}{\gamma}\right)\cdot\Delta h=\frac{1}{\Phi+\phi}\left(\frac{\Phi}{\phi}\cdot e^{-\phi\cdot\Delta h}-\frac{\phi}{\Phi}\cdot e^{\Phi\cdot\Delta h}\right). \quad \text{(A.25)}$$

Proof. First, Inequality (A.23) is simply Inequality (C.17) reformulated through Equation (C.13). Therefore, the equilibrium claim when $\underline{h} = 0$ is straightforward. Suppose now that Inequality (A.23) fails.

Threshold Indifference. V_o on $[0, \underline{h}]$ satisfies

$$\rho V_o(h) = \gamma\theta\left(\frac{\pi}{\rho}-\overline{h}+h-V_o(h)\right)+\pi V_o'(h)+\frac{1}{2}\sigma^2 V_o''(h),$$
$$V_o(0)=0, \quad V_o(\underline{h})=\frac{\pi}{\rho}-\left(1+\frac{\rho}{(1-\theta)\gamma}\right)\cdot\Delta h\equiv\underline{V}_o$$
$$\Longrightarrow V_o(h)=\frac{\theta\gamma}{\rho+\theta\gamma}\cdot\left[\left(1+\frac{\rho}{\rho+\theta\gamma}\right)\cdot\frac{\pi}{\rho}+h-\overline{h}\right]$$
$$+\frac{\rho}{\rho+\theta\gamma}\left[\left\{\frac{\pi}{\rho+\theta\gamma}-\frac{1}{1-\theta}\left(1+\frac{\rho}{\gamma}\right)\cdot\Delta h\right.\right.$$
$$\left.+\frac{\theta\gamma}{\rho}\left(\left(1+\frac{\rho}{\rho+\theta\gamma}\right)\cdot\frac{\pi}{\rho}-\overline{h}\right)\cdot e^{-\Phi_o\cdot\underline{h}}\right\}\cdot\frac{e^{\phi_o\cdot h}-e^{-\Phi_o\cdot h}}{e^{\phi_o\cdot\underline{h}}-e^{-\Phi_o\cdot\underline{h}}}$$
$$\left.-\frac{\theta\gamma}{\rho}\left(\left(1+\frac{\rho}{\rho+\theta\gamma}\right)\cdot\frac{\pi}{\rho}-\overline{h}\right)\cdot e^{-\Phi_o\cdot h}\right]. \quad \text{(A.26)}$$

The boundary condition at $\underline{h}$ is given by Threshold Indifference ($G(\underline{h}) = 0$).

Stationary Recursion. As h goes down to $\underline{h}$ from above, V on $[\underline{h}, \overline{h}]$ must converge to the financing value based on $\underline{V}_o$. Substituting $\underline{h}$ into Equation (C.11), denoting $\Delta h \equiv \overline{h} - \underline{h}$ and equating it to $\theta\left(V(\overline{h}) - \Delta h\right) + (1-\theta)\underline{V}_o$ give (A.25).

Smooth Pasting. On $[0, \underline{h}]$, V is characterized by immediate financing. Therefore, for

$h \in [0, \underline{h}]$,

$$V(h) = \theta \left(\frac{\pi}{\rho} - \overline{h} + h \right) + (1 - \theta) V_o(h).$$

Its derivative at $h = \underline{h} = \overline{h} - \Delta h$, with V_o given by Equation (A.26), must agree with the derivative of Equation (C.11) evaluated at the same point. Some algebra with substituting (A.25) gives (A.24). $\square$

Proposition C.2.3 (Comparative statics — Operating firms). *Both $\overline{h}$ and Δh strictly increase in σ^2. There exists $\overline{\sigma}^2 > 0$ such that $\sigma^2 \geq \overline{\sigma}^2$ if and only if $\underline{h} = 0$. Above it, $\overline{h} = \Delta h$ converge to $\frac{\pi}{\rho}$ as $\sigma^2 \to \infty$. $\overline{h}$, $\underline{h}$, Δh converge to zero as either σ^2 goes to zero or π goes to zero. Δh converges to zero as π goes to ∞. Lastly, there exists $\underline{\pi} > 0$ such that $\pi \leq \underline{\pi}$ if and only if $\underline{h} = 0$.*

Proof. First, on σ^2. A higher σ^2 is less desirable due to forcing more frequent dilution, hence $V(\overline{h}) - \overline{h} = \frac{\pi}{\rho} - \overline{h}$ must be decreasing in σ^2. Monotonicity of Δh and existence of $\overline{\sigma}^2$ are since

$$\frac{\partial}{\partial \sigma^2} \left[\frac{1}{\Phi + \phi} \left(\frac{\Phi}{\phi} \cdot e^{-\phi \cdot z} - \frac{\phi}{\Phi} \cdot e^{\Phi \cdot z} \right) \right] < 0, \quad \lim_{\sigma^2 \to 0} \left[\frac{1}{\Phi + \phi} \left(\frac{\Phi}{\phi} \cdot e^{-\phi \cdot z} - \frac{\phi}{\Phi} \cdot e^{\Phi \cdot z} \right) \right] = -\infty.$$

When $\sigma^2 > \overline{\sigma}^2$, Proposition C.2.1 applies. As $\sigma^2 \to 0$, the business becomes a constant perpetuity stream. Hence, $\overline{h} \to 0$ and so do Δh, $\underline{h}$ since they add up to $\overline{h}$.

Next, on π. Since $V(\overline{h}) - \overline{h} = \frac{\pi}{\rho} - \overline{h} \geq 0$, $\pi \to 0$ implies $\overline{h} \to 0$. The existence of $\underline{\pi}$ is immediate from that of $\overline{\sigma}^2$ since an equilibrium with (π, σ) is isomorphic to that with $(b\pi, b\sigma)$ for any $b > 0$. As $\pi \to \infty$, the left and right-hand sides of (C.20) go to $+\infty$, $-\infty$ with any fixed $\Delta h > 0$. Therefore, $\Delta h \to 0$. $\square$

Proposition C.2.4 (Breakeven re-inclusion — Operating firms). *$\underline{\gamma}$ is strictly increasing in σ^2 and strictly decreasing in π, and diverges to ∞ as either σ^2 goes to ∞ or π goes to zero. It converges to zero as either σ^2 goes to zero or π goes to ∞.*

Proof. $\underline{\gamma}$ is defined by $\underline{\eta}\xi = \overline{h}^*$ where $(\underline{h}^*, \overline{h}^*)$ is the equilibrium associated with $\gamma = \underline{\gamma}$ and $(\underline{\eta}, \xi)$ defined by Proposition C.2.2. Since $\gamma = \underline{\gamma}$ implies $\underline{h}^* = 0$, Proposition C.2.3, in particular Equation (C.15), holds with $\overline{h}$ replaced with $\underline{\eta}\xi$, that is, $\partial \underline{\eta}\xi / \partial \sigma^2 > 0$. Since $\partial \eta / \partial \gamma > 0$, $\partial \xi / \partial \gamma = 0$, it follows that $\partial \underline{\gamma} / \partial \sigma^2 > 0$. The limit claims follow from the existence of $\overline{\sigma}^2$ for any γ in Proposition C.2.3. The remaining claims on π follow from the isomorphism stated in the proof of Proposition C.2.3. $\square$

Appendix B Solution Algorithms

B.1 Numerical algorithm for smooth investment model

I explain numerical algorithm for the setup in Section 6—the algorithm for Section 5 is a simpler adaptation of it. I first set up the main algorithm for the case without business fluctuations in Section B.1.1. I then introduce Markov chains, both discrete and continuous, in Section B.1.2. Lastly, I briefly explain in Section B.1.3 how to solve the model under investment irreversibility, as introduced in Section 6.6.

B.1.1 Main algorithm

Formulation. Start by setting some $H > 0$. It should be higher than $\overline{h}^o$, the funding target under exclusion, which is higher than $\overline{h}$ but typically by a slight margin. V on $[0, H]$ satisfies:

$$h \geq \overline{h} \implies 0 = V_{hh}(h) \quad (\because V_h = 1 \text{ on } [\overline{h}, \infty)) \tag{B.1}$$

$$\begin{aligned} h \in [\underline{h}, \overline{h}] \implies \rho V(h) - rhV_h(h) &= \max_i \big(A - i - \Psi(i)\big)V_h + \big(i - \delta\big)\big(V - hV_h\big) + \frac{1}{2}\sigma^2 V_{hh} \\ &= \left(A + \frac{1}{2\psi} + \left(\delta + \frac{1}{\psi}\right)h\right)V_h - \left(\delta + \frac{1}{\psi}\right)V + \frac{1}{2\psi}\frac{(V - hV_h)^2}{V_h} + \frac{1}{2}\sigma^2 V_{hh} \end{aligned} \tag{B.2}$$

$$h \leq \underline{h} \implies V(h) = \theta\Big(V(H) - H + h\Big) + (1 - \theta)V^o(h). \tag{B.3}$$

Note that (B.1) implies $V(H) - H = V(\overline{h}) - \overline{h}$, which is being substituted in (B.3). Next, V^o on $[0, H]$ satisfies:

$$h \geq \overline{h}^o \implies 0 = V^o_{hh}(h) \quad (\because V^o_h = 1 \text{ on } [\overline{h}^o, \infty)) \tag{B.4}$$

$$\begin{aligned} h \in [0, \overline{h}^o] \implies & \rho V^o(h) - rhV^o_h(h) \\ &= \left(A + \frac{1}{2\psi} + \left(\delta + \frac{1}{\psi}\right)h\right)V^o_h - \left(\delta + \frac{1}{\psi}\right)V^o + \frac{1}{2\psi}\frac{(V^o - hV^o_h)^2}{V^o_h} + \frac{1}{2}\sigma^2 V^o_{hh} \\ &\quad + \gamma\big(V(h) - V^o(h)\big). \end{aligned} \tag{B.5}$$

For ease of notation, define

$$\alpha := \rho + \delta + \frac{1}{\psi}, \quad \beta(h) := A + \frac{1}{2\psi} + \left(r + \delta + \frac{1}{\psi}\right)h, \quad \xi(v, v_h, h) := \frac{1}{2\psi}\frac{(v - hv_h)^2}{v_h}.$$

The five piecewise equalities above—(B.1) through (B.5)—switch to strict inequalities when evaluated outside the respective intervals, with left-hand sides being higher. Therefore, these can be summarized as: for $h \in [0, H]$,

$$\alpha V(h) - \beta(h)V_h(h) - \frac{1}{2}\sigma^2 V_{hh}(h) = \mathrm{NL}(V, V^o, h)$$
$$:= \max\Big\{\alpha V(h) - \beta(h)V_h(h), \xi\big(V(h), V_h(h), h\big),$$
$$\alpha\Big(\theta\big(V(H) - H + h\big) + (1-\theta)V^o(h)\Big) - \beta(h)V_h(h) - \frac{1}{2}\sigma^2 V_{hh}(h)\Big\}, \quad \text{(B.6)}$$

$$(\alpha + \gamma)V^o(h) - \beta(h)V_h^o(h) - \frac{1}{2}\sigma^2 V_{hh}^o(h) = \mathrm{NL}^o(V, V^o, h)$$
$$:= \max\Big\{(\alpha+\gamma)V^o(h) - \beta(h)V_h^o(h), \xi\big(V^o(h), V_h^o(h), h\big) + \gamma V(h)\Big\}. \quad \text{(B.7)}$$

Both NL and NL^o capture the nonlinear components of the pair of differential equations. In the expression for NL, the first element gives the maximum on $[\overline{h}, H]$, the second on $[\underline{h}, \overline{h}]$ and the last on $[0, \underline{h}]$ (if $\underline{h} > 0$), and similarly in NL^o given $\underline{h}^o = 0$.[38] Lastly, the boundary conditions are:

$$V(0) = \theta\Big(V(H) - H\Big), \; V_h(H) = 1 \quad \text{(B.8)}$$

$$V^o(0) = 0, \; V_h^o(H) = 1. \quad \text{(B.9)}$$

Discretization and linearization. Let us discretize the fund space $[0, H]$ into N_h evenly spaced grids and let $\Delta H := \frac{H}{N_h - 1}$ the grid size. For now, let $i \in \{1, 2, \ldots, N_h\}$ index $[0, H]$ increasingly and denote $h \in [0, H]^{N_h}$ as the column vector discretizing $[0, H]$, such that $h(0) = 0, \; h(N_h) = H$. Posit $V_0, V_0^o \in \mathbb{R}^{N_h}$ as column vectors representing conjectured approximate value functions under inclusion and exclusion, respectively. Let $W_0 \in \mathbb{R}^{2N_h}$ with

$$W_0 := \begin{pmatrix} V_0 \\ V_0^o \end{pmatrix},$$

represent the stacked value functions. $i \in \{N_h + 1, \ldots, 2N_h\}$ represents funds under exclusion.

The core of the algorithm is to summarize the left-hand sides of the combined HJB equations (B.6), (B.7) as well as the boundary conditions (B.8), (B.9) into a single $2N_h \times 2N_h$ sparse matrix $M(W)$, which depends on the true stacked value function W, such that $M(W) \cdot W = \mathbf{NL}(W)$, where $\mathbf{NL}$ is essentially a stack of NL and NL^o, but with some additional adjustments at the respective end rows, to be specified soon. M should depend on W exclusively due to the upwinding method described shortly. I will start with some initial

[38] The first elements of NL and NL^o make use of the fact that $V_{hh}, V_{hh}^o \leq 0$ with the equality if and only if $h \geq \overline{h}, \; h \geq \overline{h}^o$, respectively. The second elements are the standard HJB equations. The third element in NL derives from $V \geq \theta\big(V(H) - H + h\big) + (1-\theta)V^o$, with the equality if and only if $h \leq \underline{h}$.

W_0 and use a linear solver to obtain W_1 such that

$$\frac{W_1 - W_0}{\Delta t} + M(W_0) \cdot W_1 = \mathbf{NL}(W_0).$$

Above, the first term on the left-hand side is the pseudo-time derivative to enable efficient convergence, with $\Delta t \in \mathbb{R}_{++}$. Until W_1 is sufficiently close to W_0, update W_0 and repeat.

In principle, $M(\cdot)$ is a mapping from $\mathbb{R}^{2N_h}$ to $\mathbb{R}^{2N_h \times 2N_h}$, although the sparsity of the target matrix substantially reduces its rank. This mapping should capture everything linear in both the combined HJB equations, including the linear differential terms, and the boundary conditions. More concretely, given V_0 and V_0^o, it should specify the linear coefficient on each grid point of the newly obtained V_1 and V_1^o such that $M(W_0) \cdot W_1$ captures all the linear requirements in terms of V_1 and V_1^o.

Upwinding the derivatives. I will approximate the combined HJB equations (B.6), (B.7) for the interior rows of h only, i.e. $i = 2, 3, \ldots, N_h - 1$. The endpoints $i = 1, N_h$ will be reserved for the boundary conditions (B.8), (B.9). For each of the interior rows $i \in \{2, 3, \ldots, N_h - 1\}$, I approximate the first and second derivatives of a given approximated function V as follows: letting $\widehat{V}(i) := \Big(V(i-1), V(i), V(i+1)\Big)'$,

$$\begin{aligned}
\Delta_h^f \cdot \widehat{V}(i) &= \frac{V(i+1) - V(i)}{\Delta H}, \\
\Delta_h^b \cdot \widehat{V}(i) &= \frac{V(i) - V(i-1)}{\Delta H}, \\
\Delta_h^c \cdot \widehat{V}(i) &= \frac{1}{2}\frac{V(i+1) - V(i-1)}{\Delta h}, \\
\Delta_h^2 \cdot \widehat{V}(i) &= \frac{V(i+1) - 2V(i) + V(i-1)}{\Delta H^2}.
\end{aligned}$$

Here, Δ_h^f denotes forward first difference, Δ_h^b backward first difference, Δ_h^c centered first difference, and Δ_h^2 second difference, in h. As can be seen, each of Δ_h^f, Δ_h^b, Δ_h^c and Δ_h^2 can be thought of as a three-dimensional *row* vector given as:

$$\begin{aligned}
\Delta_h^f &:= \left(0, -\frac{1}{\Delta H}, \frac{1}{\Delta H}\right), \\
\Delta_h^b &:= \left(-\frac{1}{\Delta H}, \frac{1}{\Delta H}, 0\right), \\
\Delta_h^c &:= \left(-\frac{1}{2\Delta H}, 0, \frac{1}{2\Delta H}\right), \\
\Delta_h^2 &:= \left(\frac{1}{\Delta H^2}, -\frac{2}{\Delta H^2}, \frac{1}{\Delta H^2}\right).
\end{aligned}$$

I follow the standard numerical method of 'upwinding' where the forward/backward difference is used in approximating the first derivative with a positive/negative drift. Determining the sign of the drift in cash flow, however, is somewhat tricky since it depends on which region—$[0, \underline{h}]$, $(\underline{h}, \overline{h}]$ or $(\overline{h}, H]$ under inclusion and $[0, \overline{h}^o]$ or $(\overline{h}^o, H]$ under exclusion—the current internal funds level belongs to. Determining the region is equivalent to determining which element is the maximum in NL and NL^o in Equations (B.6) and (B.7), whose elements all involve first derivatives.

Therefore, I employ the *centered* first difference to determine the regions and then use them to implement the upwinding. Given V_0 and V_0^o, define ten *row* indicator vectors—$f_m, b_m \in \{0,1\}^{N_h}$ for $m = 1,2,3$ and $f_m^o, b_m^o \in \{0,1\}^{N_h}$ for $m = 1,2$—such that $i \in \{1, N_h\} \implies \forall m,\ f_m(i) = b_m(i) = f_m^o(i) = b_m^o(i) = 0$ and for $i \in \{2,3,\ldots,N_h-1\}$, $f_1(i) = f_1^o(i) = 1$,

$$f_2(i) := \mathbb{1}\left(A - \mathrm{E}\Big(V_0(i), \Delta_h^c \cdot \widehat{V}_0(i), h(i)\Big) + \Big(r + \delta - \mathrm{I}\Big(V_0(i), \Delta_h^c \cdot \widehat{V}_0(i), h(i)\Big)\Big) \cdot h(i) > 0\right),$$

$$f_2^o(i) := \mathbb{1}\left(A - \mathrm{E}\Big(V_0^o(i), \Delta_h^c \cdot \widehat{V}_0^o(i), h(i)\Big) + \Big(r + \delta - \mathrm{I}\Big(V_0^o(i), \Delta_h^c \cdot \widehat{V}_0^o(i), h(i)\Big)\Big) \cdot h(i) > 0\right),$$

$$\begin{aligned} f_3(i) := \mathbb{1}\Bigg(&A - \mathrm{E}\Big(V_0^o(i) - \theta\big(V_0(N_h) - H + h(i)\big), \Delta_h^c \cdot \widehat{V}_0(i) - \theta, h(i)\Big) \\ &+ \Big(r + \delta - \mathrm{I}\Big(V_0^o(i) - \theta\big(V_0(N_h) - H + h(i)\big), \Delta_h^c \cdot \widehat{V}_0(i) - \theta, h(i)\Big) \cdot h(i)\Big) > 0\Bigg), \end{aligned}$$

and $b_m(i) = 1 - f_m(i)$, $b_m^o(i) = 1 - f_m^o(i)$ for all m. In the above, I and E denote optimized values of gross investment i[39] and total investment expense $i + \Psi(i)$, respectively, given as

$$\mathrm{I}(v, v_h, h) = \frac{1}{\psi}\left(\frac{v}{v_h} - h - 1\right), \quad \mathrm{E}(v, v_h, h) = \frac{1}{2\psi}\left(\left(\frac{v - hv_h}{v_h}\right)^2 - 1\right).$$

As can be inferred from $f_1 = f_1^o = 1$, I use forward differences on $(\overline{h}, H]$ and $(\overline{h}^o, H]$. Define

$$\Delta_{hm}(i) := f_m(i)\Delta_h^f + b_m(i)\Delta_h^b, \quad \Delta_{hmo}(i) := f_m^o(i)\Delta_h^f + b_m^o(i)\Delta_h^b.$$

Let $m^*(i) \in \{1,2,3\}$ be the maximizing index in the set

$$\begin{aligned} \Big\{&\alpha V_0(i) - \beta(h(i))\Delta_{h1}(i) \cdot \widehat{V}_0(i), \xi\big(V_0(i), \Delta_{h2}(i) \cdot \widehat{V}_0(i), h(i)\big), \\ &\alpha\Big(\theta(V_0(N_h) - H + h(i)) + (1-\theta)V_0^o(i)\Big) - \beta(h(i))\Delta_{h3}(i) \cdot \widehat{V}_0(i) - \frac{1}{2}\sigma^2\Delta_h^2 \cdot \widehat{V}_0(i)\Big\}, \end{aligned} \tag{B.10}$$

and $m^{o*}(i) \in \{1,2\}$ in the set

$$\Big\{(\alpha+\gamma)V_0^o(i) - \beta(h(i))\Delta_{h1o}(i) \cdot \widehat{V}_0^o(i), \xi\big(V_0^o(i), \Delta_{h2o}(i) \cdot \widehat{V}_0^o(i), h(i)\big) + \gamma V_0(i)\Big\}. \tag{B.11}$$

[39] I and δ enter the drift in cash flow because h is internal funds *per capital*.

Lastly, let $f(i) := f_{m^*}(i)$, $f^o(i) := f^o_{m^{o*}}(i)$, $b(i) := 1 - f(i)$, $b^o(i) := 1 - f^o(i)$, and

$$\Delta_h(i) := f(i)\Delta_h^f + b(i)\Delta_h^b, \quad \Delta_h^o(i) := f^o(i)\Delta_h^f + b^o(i)\Delta_h^b.$$

These $\Delta_h(i)$, $\Delta_h^o(i) \in \mathbb{R}^3$ implement the upwinding for first differences in the construction of $M(\cdot)$, to which I now transition.

Constructing the matrix. As a reminder, $M(W_0)$ is a $2N_h \times 2N_h$ sparse matrix, because W_0 is a stacked vector of V_0, V_0^o. For each of $i = 2, 3, \ldots, N_h - 1$ rows,

$$M(i, i-1 : i+1 \mid W_0) := (0, \alpha, 0) - \beta(h(i))\Delta_h(i) - \frac{1}{2}\sigma^2\Delta_h^2,$$

where $M(i, i-1 : i+1)$ denotes the i^{th} row from the $(i-1)^{\text{th}}$ column to the $(i+1)^{\text{th}}$ column, in order to implement the left-hand side of Equation (B.6). Similarly, for each of $(N_h + i)^{\text{th}}$ rows with $i = 2, \ldots, N_h - 1$, implement the left-hand side of (B.7) by

$$M(N_h + i, N_h + i - 1 : N_h + i + 1 \mid W_0) := (0, \alpha + \gamma, 0) - \beta(h(i))\Delta_h^o(i) - \frac{1}{2}\sigma^2\Delta_h^2.$$

Next, construct the 'nonlinear' column vector $\mathbf{NL}(W_0) \in \mathbb{R}^{2N_h}$ as follows: for each of $i = 2, 3, \ldots, N_h - 1$ rows, $\mathbf{NL}(i \mid W_0)$ is the maximum in the set (B.10) and $\mathbf{NL}(N_h + i \mid W_0)$ in the set (B.11). The rows $i = 1, N_h, N_h + 1, 2N_h$ will be separately specified right below.

Lastly, the boundary conditions (B.8) and (B.9) are implemented as:

$$M(1, 1 \mid W_0) := \alpha,\ M(1, N_h \mid W_0) := -\theta\left(\frac{1}{\Delta t} + \alpha\right),\ \mathbf{NL}(1 \mid W_0) := -\theta\left(\frac{1}{\Delta t} + \alpha\right)H,$$
$$M(N_h, N_h - 1 \mid W_0) := -\left(\frac{1}{\Delta t} + \alpha\right),\ M(N_h, N_h \mid W_0) := \alpha,\ \mathbf{NL}(N_h \mid W_0) := \left(\frac{1}{\Delta t} + \alpha\right)\Delta H,$$
$$M(N_h + 1, N_h + 1 \mid W_0) := 0,\ \mathbf{NL}(N_h + 1 \mid W_0) := 0,$$
$$M(2N_h, 2N_h - 1 \mid W_0) := -\left(\frac{1}{\Delta t} + \alpha + \gamma\right),\ M(2N_h, 2N_h \mid W_0) := \alpha + \gamma,$$
$$\mathbf{NL}(2N_h \mid W_0) := \left(\frac{1}{\Delta t} + \alpha + \gamma\right)\Delta H.$$

Any unspecified element of $M(W_0)$ is set to zero, making it highly sparse.

Iteration to solution. Posit H. Start with some initial guess V_0, V_0^o, and stack them into W_0. Obtain W_1 that solves

$$\left(\frac{1}{\Delta t}\mathbb{I}_{2N_h} + M(W_0)\right) \cdot W_1 = \frac{1}{\Delta t}W_0 + \mathbf{NL}(W_0),$$

where $\mathbb{I}_{2N_h}$ is the $2N_h \times 2N_h$ identity matrix, also highly sparse. If W_1 is close enough to W_0, stop; $V := W_1(1 : N_h)$ and $V^o := W_1(N_h{+}1 : 2N_h)$. Otherwise, update $W_0 := aW_1 + (1-a)W_0$ for some weight $a \in (0, 1]$ and repeat.

If H is set too high, convergence might be achieved too slowly or even fail, and even upon success, the solution becomes unnecessarily coarse on $[0, \overline{h}]$. On the other hand, if H is too low, then possibly $H < \overline{h}$, in which case the algorithm fails. Therefore, I run the algorithm twice, an initializer and a verifier. During initializing, I use an adequate fraction (say 0.2) of the first-best value V^* as the initial H and use a high error tolerance. If convergence fails, I raise the initial H. If it succeeds, I choose a new H to be only slightly higher than $\max\{\overline{h}, \overline{h}^o\}$ and run the verifier with the main error tolerance.

B.1.2 Stochastic fluctuations

Stacked value. Let N_s denote the number of Markov states. For a continuous Markov chain, N_s is the number of grids in discretization. I use $s \in \{1, 2, \ldots, N_s\}$ to index the Markov states. Given $V_0(i_h, s)$ and $V_0^o(i_h, s)$, define

$$W_0 := \begin{pmatrix} V_0(:, 1) \\ \cdots \\ V_0(:, N_s) \\ V_0^o(:, 1) \\ \cdots \\ V_0^o(:, N_s) \end{pmatrix} \in \mathbb{R}^{2N_h N_s}$$

as their stacked column vector. $i \in \{1, \ldots, 2N_h N_s\}$ now jointly indexes (i_h, s) and inclusion/exclusion. The mapping $M : \mathbb{R}^{2N_h N_s} \to \mathbb{R}^{2N_h N_s \times 2N_h N_s}$ will be defined in a fashion overall identical to Appendix B.1.1 for *each* of the $N_h \times N_h$ blocks corresponding to $s \in \{1, 2, \ldots, N_s\}$ along the main diagonal $M\big((s-1)N_h + 1 : sN_h, (s-1)N_h + 1 : sN_h \mid W_0\big)$. There will be, however, an additional sparse matrix for the Markov chain and some changes to NL, NL^o.

Discrete Markov. Consider a Markov chain in Poisson arrival rates of transition given as

$$\begin{pmatrix} -\lambda^1 & \lambda_2^1 & \lambda_3^1 & \cdots & \lambda_{N_s}^1 \\ \lambda_1^2 & -\lambda^2 & \lambda_3^2 & \cdots & \lambda_{N_s}^2 \\ \lambda_1^3 & \lambda_2^3 & -\lambda^3 & \cdots & \lambda_{N_s}^3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_1^{N_s} & \lambda_2^{N_s} & \lambda_3^{N_s} & \cdots & -\lambda^{N_s} \end{pmatrix},$$

where $\lambda_{s'}^s \geq 0$ is the Poisson rate of transition at s to $s' \neq s$, and $\lambda^s := \sum_{s' \neq s} \lambda_{s'}^s$.

The combined HJB equations (B.6) and (B.7) are modified as follows: for $s \in \{1, \dots, N_s\}$,

$$
\begin{aligned}
&(\alpha(s) + \lambda^s)V(h,s) - \beta(h,s)V_h(h,s) - \frac{1}{2}\sigma^2 V_{hh}(h,s) - \sum_{s' \neq s} \lambda^s_{s'} V(h,s') = \mathrm{NL}(V, V^o, h, s) \\
&:= \max\Big\{(\alpha(s) + \lambda^s)V(h,s) - \beta(h,s)V_h(h,s) - \sum_{s' \neq s} \lambda^s_{s'} V(h,s'), \xi\big(V(h,s), V_h(h,s), h, s\big), \\
&\qquad (\alpha(s) + \lambda^s)\Big(\theta\big(V(H,s) - H + h\big) + (1-\theta)V^o(h,s)\Big) - \beta(h,s)V_h(h,s) - \frac{1}{2}\sigma^2 V_{hh}(h,s) \\
&\qquad\qquad - \sum_{s' \neq s} \lambda^s_{s'} V(h,s')\Big\}, \qquad \text{(B.12)}
\end{aligned}
$$

$$
\begin{aligned}
&(\alpha(s) + \lambda^s + \gamma(s))V^o(h,s) - \beta(h,s)V^o_h(h,s) - \frac{1}{2}\sigma^2 V^o_{hh}(h) - \sum_{s' \neq s} \lambda^s_{s'} V^o(h,s') = \mathrm{NL}^o(V, V^o, h, s) \\
&:= \max\Big\{(\alpha(s) + \lambda^s + \gamma(s))V^o(h,s) - \beta(h,s)V^o_h(h,s) - \sum_{s' \neq s} \lambda^s_{s'} V^o(h,s'), \\
&\qquad \xi\big(V^o(h,s), V^o_h(h,s), h, s\big) + \gamma(s)V(h)\Big\}. \qquad \text{(B.13)}
\end{aligned}
$$

The dependence of $\alpha, \beta, \gamma, \xi$ on s captures the fluctuating state variable. In case of fluctuating productivity in Section 6.6, only $\beta(h,s) := A(s) + \frac{1}{2\psi} + \left(r + \delta + \frac{1}{\psi}\right) h$ depends on s.

Define a Markov chain matrix for the entire (i,s) space by

$$
\Lambda := \begin{pmatrix}
-\lambda^1 \widetilde{\mathbb{I}}_{N_h} & \lambda^1_2 \widetilde{\mathbb{I}}_{N_h} & \lambda^1_3 \widetilde{\mathbb{I}}_{N_h} & \dots & \lambda^1_{N_s} \widetilde{\mathbb{I}}_{N_h} \\
\lambda^2_1 \widetilde{\mathbb{I}}_{N_h} & -\lambda^2 \widetilde{\mathbb{I}}_{N_h} & \lambda^2_3 \widetilde{\mathbb{I}}_{N_h} & \dots & \lambda^2_{N_s} \widetilde{\mathbb{I}}_{N_h} \\
\lambda^3_1 \widetilde{\mathbb{I}}_{N_h} & \lambda^3_2 \widetilde{\mathbb{I}}_{N_h} & -\lambda^3 \widetilde{\mathbb{I}}_{N_h} & \dots & \lambda^3_{N_s} \widetilde{\mathbb{I}}_{N_h} \\
\dots & \dots & \dots & \dots & \dots \\
\lambda^{N_s}_1 \widetilde{\mathbb{I}}_{N_h} & \lambda^{N_s}_2 \widetilde{\mathbb{I}}_{N_h} & \lambda^{N_s}_3 \widetilde{\mathbb{I}}_{N_h} & \dots & -\lambda^{N_s} \widetilde{\mathbb{I}}_{N_h}
\end{pmatrix} \in \mathbb{R}^{N_h N_s \times N_h N_s},
$$

where $\widetilde{\mathbb{I}}_{N_h}$ is the $N_h \times N_h$ identity matrix but with the first and the last main diagonal elements replaced with zero; the first and the last rows in each block are preserved for the boundary conditions. Proceed to extend Λ to both inclusion and exclusion by defining

$$
\boldsymbol{\Lambda} := \begin{pmatrix} \Lambda & \mathbf{0} \\ \mathbf{0} & \Lambda \end{pmatrix} \in \mathbb{R}^{2N_h N_s \times 2N_h N_s}.
$$

The zero off-diagonal blocks indicate that Λ is orthogonal to inclusion and exclusion.

The construction of each main diagonal block of $M(W_0)$—i.e. $M\big((s-1)N_h+1 : sN_h, (s-1)N_h+1 : sN_h \mid W_0\big)$—is unchanged, both for the interior rows $i \in \{(s-1)N_h+2, \dots, sN_h-1\}$ that implement the combined HJB (B.12), (B.13)[40] and for the boundaries $i \in \{(s-1)N_h + 1, sN_h\}$ that implement the same boundary conditions. The construction of $\mathbf{NL}(W_0)$ is adjusted slightly for the interior, in accordance with the modified definition of $\mathrm{NL}, \mathrm{NL}^o$ in

[40] Note that the maximizing indices m^*, m^{o*} are based on the modified NL and NL^o in (B.12), (B.13).

(B.12), (B.13). Once the mappings $M(\cdot)$, $\mathbf{NL}(\cdot)$ are constructed, iteratively solve

$$\left(\frac{1}{\Delta t}\mathbb{I}_{2N_hN_s} + M(W_0) - \mathbf{\Lambda}\right) \cdot W_1 = \frac{1}{\Delta t}W_0 + \mathbf{NL}(W_0).$$

Continuous Markov chain. Let s_t follow

$$ds_t = \mu_s(s_t)\ dt + \sigma_s(s_t)\ dZ_t.$$

Discretize the state space into N_s grids with size ΔS. Let $s \in \{1, 2, \ldots, N_s\}$ index the state space increasingly. The above law of motion is 'discretized' into a Markov chain such that

$$\mu_s(s) \geq 0 \implies \lambda^s_{s-1} = \frac{\sigma_s(s)^2}{2\Delta S^2}, \quad \lambda^s_{s+1} = \frac{\mu_s(s)}{\Delta S} + \frac{\sigma_s(s)^2}{2\Delta S^2},$$
$$\mu_s(s) < 0 \implies \lambda^s_{s-1} = -\frac{\mu_s(s)}{\Delta S} + \frac{\sigma_s(s)^2}{2\Delta S^2}, \quad \lambda^s_{s+1} = \frac{\sigma_s(s)^2}{2\Delta S^2}.$$

As for the endpoints $s \in \{1, N_s\}$, mean reversion will generally allow upwinding of the first-order terms $\pm\frac{\mu_s(s)}{\Delta S}$. The second-order terms, however, cannot be correctly computed as they go outside the grid. I therefore use W_0 to compute second as well as *third* finite differences at $s \in \{2, N_s - 1\}$, and use them to linearly approximate the endpoint second derivatives.[41]

Once the discretized Markov chain has been set up, follow the same procedure as above.

B.1.3 Investment irreversibility

The algorithm remains mostly the same and is modified only as follows. First, whenever $\frac{1}{\psi}$ shows up in the above algorithm, multiply it by $1 + (\phi - 1) \cdot \mathbb{1}(\frac{V_0}{\Delta_h \cdot V_0} - h < 1)$; use V_0^o, Δ_h^o instead of V_0, Δ_h appropriately. The indicator function tracks whether the firm divests. Second, move all terms involving this modified expression to the inside of **NL**, as investment versus divestment makes the system highly nonlinear.

B.2 Closed-form solution procedure for exogenous cash flow

Sections 3.2 and 3.3 inform a general procedure for analytically solving the equilibrium for exogenous cash flow models in Section 2 when the HJB equation (3) admits an explicit solution (which requires $r = 0$). By Lemma 2, the equilibrium is fully characterized by a pair $(\underline{h}, \overline{h})$, $0 \leq \underline{h} < \overline{h}$. The procedure is as below:

1. In all cases, $V'(\overline{h}) = 1$, and if $\sigma > 0$, then $V''(\overline{h}) = 0$.
2. Solve the model with $\gamma = 0$. By Corollary 1, $\underline{h} = 0$, and $\overline{h}$ is implicitly defined by $V(0) = \theta\big(V(\overline{h}) - \overline{h}\big)$.

[41] For these endpoints, the 'discrete jump' interpretation might not hold, but the algorithm still works.

3. For $\gamma > 0$, first determine whether $\underline{h} = 0$ or $\underline{h} > 0$. This can be done as follows: (i) posit the value of $\overline{h}$ obtained in Step 2, and (ii) evaluate Inequality (13). If $\underline{h} = 0$, assign to $\overline{h}$ the value obtained in Step 2.

4. If $\underline{h} > 0$, then use the following conditions to determine $(\underline{h}, \overline{h})$:
   (a) Stationary Recursion: $V(\overline{h}) - V(\underline{h}) = \left(1 + \frac{\rho - r}{\gamma}\right) \cdot \Delta h$,
   (b) Threshold Indifference: for $h \in [0, \underline{h}]$,

$$\rho V_o(h) - rhV_o'(h) = \gamma\theta\big(V(\overline{h}) - \overline{h} + \underline{h} - V_o(\overline{h})\big) + \Lambda(V_o)(h) + \mathcal{H}(V_o)(h),$$

   with boundary conditions $V_o(0) = 0$, $G(\underline{h}) = 0$,[42] and
   (c) Smooth Pasting: $V'(\underline{h}) = \theta + (1 - \theta)V_o'(\underline{h})$.

[42] See the full proof of Lemma 2 in Appendix A.

Appendix C Analytic Derivations

Here, I go through analytic solutions to the startup and operating firm examples. The main purpose is to *prove* comparative statics in business parameters, which will motivate extending the framework with investment choice. An analytic solution to the HJB equation requires that the internal yield be zero $r = 0$. Note, as an aside, that regardless of the existence of an analytic solution for the value function, all the formal results in the main documents hold.

C.1 Solving startup equilibrium

The business incurs a fixed flow expense $\kappa\ dt$, $\kappa > 0$, until success arrives at Poisson rate $\lambda > 0$ upon which the business terminates with one-time payoff $\Pi > 0$. As discussed, assume $r = 0$. Let us reiterate the first part of Assumption 1 as a reference.

Assumption C.1.1 (Positive net present value)**.** $\lambda\Pi > \kappa$.

Then, V on $(\underline{h}, \overline{h})$ satisfies the following ODE:

$$\rho V(h) = \lambda\big(\Pi + h - V(h)\big) - \kappa V'(h)$$
$$\implies V(h) = -c \cdot e^{-\frac{\rho+\lambda}{\kappa}h} + \frac{\lambda}{\rho+\lambda}\left(\Pi + h - \frac{\kappa}{\rho+\lambda}\right),$$

for some $c \in \mathbb{R}$. In addition, since $V'(\overline{h}) = 1$, we have

$$V(h) = \frac{1}{\rho+\lambda}\left(\lambda \cdot \Big(\Pi + h\Big) - \frac{\kappa}{\rho+\lambda} \cdot \Big(\lambda + \rho \cdot e^{\frac{\rho+\lambda}{\kappa}(\overline{h}-h)}\Big)\right), \tag{C.1}$$

$$V(\overline{h}) = \frac{1}{\rho+\lambda}\left(\lambda \cdot \Big(\Pi + \overline{h}\Big) - \kappa\right). \tag{C.2}$$

C.1.1 Baseline: no re-inclusion

First consider $\gamma = 0$. Since $\underline{h} = 0$ by Corollary 1, the equilibrium—just $\overline{h}$ in this case—is implicitly defined by the stationary recursion as follows:

$$V(0) = x(0)V(\overline{h}) = \theta(V(\overline{h}) - \overline{h})$$
$$\iff \theta\rho\big((\rho+\lambda)\overline{h} + \kappa\big) + (1-\theta)\lambda\big((\rho+\lambda)\Pi - \kappa\big) = \rho\kappa \cdot \exp\left(\frac{\rho+\lambda}{\kappa} \cdot \overline{h}\right). \tag{C.3}$$

In the first line, $V_o(0) = 0$ is used. Note that the solution to Equation C.3 is positive if and only if Assumption C.1.1 holds.

With Equation (C.3), comparative statics is straightforward.

Proposition C.1.1 (Comparative statics — Startups without re-inclusion)**.**

$$\frac{\partial \overline{h}}{\partial \theta} = -\frac{1}{\rho} \cdot \frac{\lambda\Pi - \kappa - \rho\overline{h}}{\exp\left(\frac{\rho+\lambda}{\kappa} \cdot \overline{h}\right) - \theta} < 0, \ \lim_{\theta\to 0} \overline{h} < \frac{\lambda\Pi - \kappa}{\rho}, \ \lim_{\theta\to 1} \overline{h} = 0, \quad \textit{and} \ \lim_{\theta\to 1} \frac{\partial \overline{h}}{\partial \theta} = -\infty, \tag{C.4}$$

$$\frac{\partial \overline{h}}{\partial \Pi} = \frac{\lambda}{\rho} \cdot \frac{1-\theta}{\exp\left(\frac{\rho+\lambda}{\kappa} \cdot \overline{h}\right) - \theta} > 0, \tag{C.5}$$

$$\lim_{\lambda\to\kappa/\Pi} \overline{h} = \lim_{\lambda\to\infty} \overline{h} = 0. \tag{C.6}$$

Proof. Most results are straightforward. For the first inequality in (C.4), see Appendix A.3. □

C.1.2 General comparative statics

Consider the general case of $\gamma \geq 0$. Inequality (11) translates into: $\underline{h} > 0$ if and only if.

$$\rho\overline{h} < \frac{(1-\theta)\gamma}{\rho + \lambda + (1-\theta)\gamma} \cdot (\lambda\Pi - \kappa). \tag{C.7}$$

By Section B.2, the following result is obtained.

Proposition C.1.2 (Startup financing)**.** *Denote*

$$\eta := \frac{(1-\theta)\gamma}{\rho + \lambda + (1-\theta)\gamma}, \quad \xi := \frac{\lambda\Pi - \kappa}{\rho}.$$

The equilibrium is characterized by $\underline{h} = 0$ and $\overline{h}$ implicitly defined by Equation (C.3) if

$$\theta\rho\big((\rho+\lambda)\eta\xi + \kappa\big) + (1-\theta)\lambda\big((\rho+\lambda)\Pi - \kappa\big) \geq \rho\kappa \cdot \exp\left(\frac{\rho+\lambda}{\kappa} \cdot \eta\xi\right). \tag{C.8}$$

If the inequality is strictly reversed, then $\underline{h} = \overline{h} - \Delta h > 0$ and $\overline{h} > \Delta h$ is implicitly defined by

$$\begin{aligned} &\frac{1-\theta}{\rho+\lambda}\left((\rho+\lambda+\theta\gamma)\lambda\Pi - \theta\rho\gamma\left(\overline{h} + \frac{\kappa}{\rho+\lambda+\theta\gamma}\right) - (\lambda+\theta\gamma)\kappa\right) \\ &\qquad = \rho\left(\frac{\rho+\lambda+\gamma}{\gamma}\Delta h + (1-\theta)\frac{\kappa}{\rho+\lambda+\theta\gamma}\right) \cdot \exp\left(\frac{\rho+\lambda+\theta\gamma}{\kappa} \cdot (\overline{h} - \Delta h)\right), \end{aligned} \tag{C.9}$$

where $\Delta h = \overline{h} - \underline{h} > 0$ is given by

$$1 + \frac{\rho+\lambda}{\kappa} \cdot \left(1 + \frac{\rho+\lambda}{\gamma}\right) \cdot \Delta h = \exp\left(\frac{\rho+\lambda}{\kappa} \cdot \Delta h\right). \tag{C.10}$$

Proof sketch. Inequality (C.8) is simply a combination of Equation (C.3) and Inequality (C.7). When $\underline{h} = 0$, $\overline{h}$ satisfies the same Stationary Recursion as when $\gamma = 0$. Suppose Inequality (C.8) fails. Equation (C.9) derives from Stationary Recursion, while Equation (C.10) from

Smooth Pasting. Threshold Indifference is redundant in the startup model. For details, see Appendix A.3. □

Proposition C.1.3 (Comparative statics — startups)**.** *$\overline{h}$ strictly increases in Π. When $\underline{h} > 0$, $\underline{h}$ strictly increases in Π and Δh is constant in Π and strictly decreasing in λ. Lastly, $\overline{h}$, $\underline{h}$ and Δh converge to zero as either (i) Π goes to κ/λ, or (ii) λ goes to either κ/Π or ∞.*

Proof. Immediate from Proposition C.1.2. □

Proposition C.1.4 (Breakeven re-inclusion — Startups)**.** *$\underline{\gamma}$ strictly decreases in Π, and converges to zero as Π goes to ∞. It goes to ∞ as Π goes down to κ/λ, the lower bound in Assumption C.1.1.*

Proof. See Appendix A.3. □

C.2 Solving operating firm equilibrium

The second example involves a fixed average profit but with volatility. That is, the business's underlying cash flow is captured by

$$\pi \ dt + \sigma \ dB_t,$$

with $\pi, \sigma^2 > 0$, where B_t is a standard Brownian motion. Again, assume $r = 0$ for simplicity.

Note that V on $(\underline{h}, \overline{h})$ satisfies the following ODE:

$$\rho V(h) = \pi V'(h) + \frac{1}{2}\sigma^2 V''(h).$$

In addition, since shareholders will receive dividends at $h_t = \overline{h}$ such that $\overline{h}$ becomes a reflection boundary, both smooth pasting and super contact conditions must hold at $\overline{h}$, i.e. $V'(\overline{h}) = 1$, $V''(\overline{h}) = 0$. Therefore,

$$V(h) = \frac{1}{\Phi + \phi}\left(\frac{\Phi}{\phi} \cdot e^{-\phi(\overline{h}-h)} - \frac{\phi}{\Phi} \cdot e^{\Phi(\overline{h}-h)}\right), \tag{C.11}$$

$$V(\overline{h}) = \frac{\pi}{\rho}, \tag{C.12}$$

where $\Phi := (\sqrt{\pi^2 + 2\rho\sigma^2} + \pi)/\sigma^2$ and $\phi := (\sqrt{\pi^2 + 2\rho\sigma^2} - \pi)/\sigma^2$.

C.2.1 Baseline: no re-inclusion

First suppose that $\gamma = 0$. Then, again by Corollary 1, $\underline{h} = 0$ and $\overline{h}$ is implicitly defined by $V(0) = \theta\left(V(\overline{h}) - \overline{h}\right)$, which is simply

$$\theta\left(\frac{\pi}{\rho} - \overline{h}\right) = \frac{1}{\Phi + \phi}\left(\frac{\Phi}{\phi} \cdot e^{-\phi \cdot \overline{h}} - \frac{\phi}{\Phi} \cdot e^{\Phi \cdot \overline{h}}\right). \tag{C.13}$$

From above, we can derive comparative statics.

Proposition C.2.1 (Comparative statics — Operating firms without re-inclusion)**.**

$$\frac{\partial \overline{h}}{\partial \theta} = -\frac{1}{\rho} \cdot \frac{\pi - \rho \overline{h}}{\frac{\Phi}{\Phi+\phi} \cdot e^{-\phi \cdot \overline{h}} + \frac{\phi}{\Phi+\phi} \cdot e^{\Phi \cdot \overline{h}} - \theta} < 0,$$

$$\lim_{\theta \to 0} \overline{h} < \frac{\pi}{\rho}, \ \lim_{\theta \to 1} \overline{h} = 0, \ \textit{and} \ \lim_{\theta \to 1} \frac{\partial \overline{h}}{\partial \theta} = -\infty, \tag{C.14}$$

$$\frac{\partial \overline{h}}{\partial \sigma^2} = \frac{\rho}{\pi^2 + 2\rho\sigma^2} \cdot \frac{\overline{h}\sqrt{\pi^2 + 2\rho\sigma^2}\left(e^{\Phi \cdot \overline{h}} + e^{-\phi \cdot \overline{h}}\right) - \sigma^2 \left(e^{\Phi \cdot \overline{h}} - e^{-\phi \cdot \overline{h}}\right)}{\sqrt{\pi^2 + 2\rho\sigma^2}\left(e^{\Phi \cdot \overline{h}} + e^{-\phi \cdot \overline{h}} - 2\theta\right) - \pi \left(e^{\Phi \cdot \overline{h}} - e^{-\phi \cdot \overline{h}}\right)} > 0,$$

$$\lim_{\sigma^2 \to 0} \overline{h} = 0, \ \lim_{\sigma^2 \to \infty} \overline{h} = \frac{\pi}{\rho}, \ \lim_{\sigma^2 \to 0} \frac{\partial \overline{h}}{\partial \sigma^2} = \infty, \ \textit{and} \ \lim_{\sigma^2 \to \infty} \frac{\partial \overline{h}}{\partial \sigma^2} = 0, \tag{C.15}$$

$$\lim_{\pi \to 0} \overline{h} = \lim_{\pi \to \infty} \overline{h} = 0. \tag{C.16}$$

Proof. See Appendix A.4. □

C.2.2 General comparative statics

Now consider $\gamma \geq 0$. Let us first evaluate Inequality (11): $\underline{h} > 0$ if and only if

$$\rho \overline{h} < \frac{(1-\theta)\gamma}{\rho + (1-\theta)\gamma} \cdot \pi. \tag{C.17}$$

Proposition C.2.2 (Operating firm financing)**.** *Denote*

$$\eta := \frac{(1-\theta)\gamma}{\rho + (1-\theta)\gamma}, \quad \xi := \frac{\pi}{\rho}.$$

The equilibrium is characterized by $\underline{h} = 0$ and $\overline{h}$ implicitly defined by Equation (C.13) if

$$\theta(1-\eta)\xi \leq \frac{1}{\Phi + \phi}\left(\frac{\Phi}{\phi} \cdot \exp\left(-\phi \cdot \eta\xi\right) - \frac{\phi}{\Phi} \cdot \exp\left(\Phi \cdot \eta\xi\right)\right). \tag{C.18}$$

If the inequality is strictly reversed, then $\underline{h} = \overline{h} - \Delta h > 0$ and $\overline{h} > \Delta h$ is implicitly defined by

$$\begin{aligned}
&\left(\frac{\pi}{\rho + \theta\gamma} - \frac{1}{1-\theta}\left(1 + \frac{\rho}{\gamma}\right) \cdot \Delta h\right) \cdot \frac{\phi_o \exp\big(\phi_o(\overline{h} - \Delta h)\big) + \Phi_o \exp\big(-\Phi_o(\overline{h} - \Delta h)\big)}{\exp\big(\phi_o(\overline{h} - \Delta h)\big) - \exp\big(-\Phi_o(\overline{h} - \Delta h)\big)} \\
&\quad + \theta\left(\Phi_o + \phi_o\right)\frac{\gamma}{\rho}\left(\left(1 + \frac{\rho}{\rho + \theta\gamma}\right)\frac{\pi}{\rho} - \overline{h}\right) \cdot \frac{\exp\left(-2\pi(\overline{h} - \Delta h)/\sigma^2\right)}{\exp\big(\phi_o(\overline{h} - \Delta h)\big) - \exp\big(-\Phi_o(\overline{h} - \Delta h)\big)} \\
&\quad = \frac{\rho + \theta\gamma}{(1-\theta)\rho}\left(\phi\left(\frac{\pi}{\rho} - \left(1 + \frac{\rho}{\gamma}\right) \cdot \Delta h\right) + \frac{\phi}{\Phi} \cdot \exp\big(\Phi \cdot \Delta h\big)\right) - \frac{\theta}{1-\theta}\left(1 + \frac{\gamma}{\rho}\right),
\end{aligned} \tag{C.19}$$

where

$$\Phi_o := (\sqrt{\pi^2 + 2(\rho + \theta\gamma)\sigma^2} + \pi)/\sigma^2, \quad \phi_o := (\sqrt{\pi^2 + 2(\rho + \theta\gamma)\sigma^2} - \pi)/\sigma^2$$

and $\Delta h = \overline{h} - \underline{h} > 0$ *is given by*

$$\frac{\pi}{\rho} - \left(1 + \frac{\rho}{\gamma}\right) \cdot \Delta h = \frac{1}{\Phi + \phi}\left(\frac{\Phi}{\phi} \cdot e^{-\phi \cdot \Delta h} - \frac{\phi}{\Phi} \cdot e^{\Phi \cdot \Delta h}\right). \tag{C.20}$$

Proof sketch. The equilibrium claim for $\underline{h} = 0$ is straightforward. If Inequality (C.18) fails, first obtain V_o on $[0, \underline{h}]$ by the appropriate ODE along with two boundary conditions $V_o(0) = 0$ and $G(\underline{h}) = 0$ from Equation (9) (i.e. Threshold Indifference). Stationary Recursion, with $V(\underline{h})$ evaluated by Equation (C.11) and with $V_o(\underline{h})$ that solves $G(\underline{h}) = 0$, gives (C.20). Smooth Pasting at $\underline{h}$, with some algebra, gives (C.19). For details, see Appendix A.4. □

Proposition C.2.3 (Comparative statics — Operating firms)**.** *Both $\overline{h}$ and Δh strictly increase in σ^2. There exists $\overline{\sigma}^2 > 0$ such that $\sigma^2 \geq \overline{\sigma}^2$ if and only if $\underline{h} = 0$. Above it, $\overline{h} = \Delta h$ converge to $\frac{\pi}{\rho}$ as $\sigma^2 \to \infty$. $\overline{h}$, $\underline{h}$, Δh converge to zero as either σ^2 goes to zero or π goes to zero. Δh converges to zero as π goes to ∞. Lastly, there exists $\underline{\pi} > 0$ such that $\pi \leq \underline{\pi}$ if and only if $\underline{h} = 0$.*

Proof. See Appendix A.4. □

Proposition C.2.4 (Breakeven re-inclusion — Operating firms)**.** *$\underline{\gamma}$ strictly increases in σ^2, strictly decreases in π, and diverges to ∞ as either σ^2 goes to ∞ or π goes to zero. It converges to zero as either σ^2 goes to zero or π goes to ∞.*

Proof. See Appendix A.4. □