The Zero-Beta Rate Revisited

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Abstract

The zero-beta rate is an important concept in asset pricing due to its implications for the security market line, beta anomaly, risk-free rate, etc. This paper revisits the estimation of the zero-beta rate and argues that existing methods produce high and volatile zero-beta rates arising from two channels: model misspecification and error-in-variables. Any model misspecification leads a non-uniqueness of the zero-beta rate. Measurement errors in betas increase noise in the estimation. Simulation analysis shows that both channels are quantitatively important for increasing the mean and volatility of the estimated zero-beta rate. In addition, I propose a new perspective on evaluating empirical factor models based on the theoretical result that a correctly specified model should feature a unique zero-beta rate. The new tests show that prominent factor models in the literature (e.g., Fama-French, q-factors, IPCA models) are misspecified.

1. Introduction

The zero-beta rate is defined as the expected return of an asset portfolio that is orthogonal (or zero-beta) with respect to the stochastic discount factor (SDF). This has always been an important topic in asset pricing studies. Going back to the Black (1972) version of CAPM without risk-free borrowing or lending, the expected return uncorrelated with the market portfolio (the zero-beta rate in CAPM) is much higher than the Treasury bill yield, indicating a much flatter security market line. The zero-beta rate is also related to the beta anomaly, which refers to the low (high) abnormal returns of stocks with high (low) beta (Frazzini and Pedersen, 2014; Hong and Sraer, 2016; Bali et al., 2017; etc.). Additionally, since the zero-beta rate is orthogonal to the SDF, it may naturally be interpreted as the risk-free rate, which can then be used to infer the US Treasury convenience yields[1] and explain the equity risk premium puzzle (Di Tella et al., 2023).

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[1]The US Treasury convenience yield is defined as the difference between the risk-free rate without convenience benefits and the US Treasury yield.

Since the zero-beta rate is unobserved, the literature has generally proposed two estimation methods. The first method is the two-stage Fama-MacBeth process, which runs time-series regressions to estimate the risk loadings (betas) followed by running cross-sectional regressions for each period. The zero-beta rate can be estimated using the intercept term in the cross-sectional regressions. The second method is proposed by Di Tella et al. (2023) and I refer to it as the zero-beta-portfolio approach. It analytically calculates the portfolio weights for the minimum-variance zero-beta portfolio, where betas are estimated from time-series regressions. The zero-beta rate is then estimated by predicting the expected return of the minimum-variance zero-beta portfolio with a set of macroeconomic variables. Using eight prominent factor models, I confirm that both methods produce high and volatile 1-month zero-beta rates in estimation compared to the 1-month Treasury bill yield, consistent with the evidence in the literature. Specifically, the Fama-MacBeth approach produces a zero-beta rate that is approximately 3 times higher and 20 times more volatile than the Treasury bill yield. The zero-beta-portfolio approach produces a zero-beta rate that is approximately 3 times higher and 2 times more volatile than the Treasury yield.

In this paper, I revisit the two approaches and assess the credibility of the estimation. I show that model specification and errors-in-variables are important factors contributing to the high level and volatility of the estimated zero-beta rate. I establish this argument in three steps.

First, I emphasize a theoretical result adapted from Roll (1980) regarding factor models: A correctly specified factor model should feature a unique zero-beta rate, whereas a misspecified model leads to multiplicity or indeterminacy of the zero-beta rate[2]. The multiplicity of the zero-beta rate can theoretically explain why it is high and volatile in estimation. Specifically, applying the Fama-MacBeth approach with misspecified factor models amounts to randomly selecting a value from an indeterminate set of zero-beta rates period by period. Since there are no restrictions on the time-series dimension due to the period-by-period cross-sectional regressions, the volatility of the estimated zero-beta rate can be very high. For comparison, the zero-beta-portfolio approach adds a time-series restriction by focusing on the minimum-variance zero-beta portfolio. This can be the reason why it produces much less volatile zero-beta rates than the Fama-MacBeth approach in estimation. However, the estimated zero-beta rate may still be different from the true risk-free rate owing to multiplicity caused by model misspecification.

Second, I develop new statistical tests of model misspecification that match the two

[2]The uniqueness of the zero-beta rate does not mean the zero-beta rate is constant over time. Instead, it means that the conditional mean of all zero-beta portfolios that are orthogonal to the SDF should be equalized. Hence, there is a unique time series of the zero-beta rate.

estimation methods (the Fama-MacBeth and zero-beta-portfolio approaches). These tests are based on the theoretical result that a correctly specified model implies a unique zero-beta rate. My testing results show that all eight prominent factor models studied in this paper are misspecified as they do not feature a unique zero-beta rate. Therefore, the theory that connects model misspecification with the multiplicity of the zero-beta rate is empirically relevant in explaining the high and volatile estimated zero-beta rate.

Third, I quantify the effect of model misspecification in simulation analysis. Because both estimation approaches are potentially subject to estimation errors in risk loadings (betas), I also account for the errors-in-variables (EIV) problem in the simulation. To start with, I apply both the Fama-MacBeth approach and the zero-beta-portfolio approach to the benchmark model that is correctly specified and without errors-in-variables. I find that both approaches are able to recover the true unobserved risk-free rate artificially created in simulation. In the meantime, the benchmark model passes my new tests for model misspecification. This evidence implies that the two approaches and the new tests are statistically valid. Next, I perform two exercises: (i) I increase the magnitude of model misspecification while keeping zero errors-in-variables and (ii) I increase the magnitude of errors-in-variables while keeping the correct model. These exercises separately investigate the quantitative effects of model misspecification and errors-in-variables. I show that both channels are nontrivial in increasing the level and volatility of the estimated zero-beta rate. In the second exercise, in particular, the correct model will not be rejected as I increase the magnitude of errors-in-variables, further validating the proposed tests. Finally, I examine these two channels together and find that model misspecification is a quantitatively dominant factor contributing to the high level and volatility of the estimated zero-beta rate. Comparing the two approaches, the zero-beta-portfolio approach proposed by Di Tella et al. (2023) is less prone to both model misspecification and errors-in-variables than the Fam-MacBeth approach. While the literature is indeed making progress on the estimation method, I argue that it is still too early to interpret the estimated zero-beta rate as the true unobserved risk-free rate.

During my inspection of the estimated zero-beta rate, I propose two new tests for factor models based on the idea that a correctly specified model should feature a unique zero-beta rate. This constitutes a new perspective on evaluating factor models. Conventionally, researchers use the Fama-MacBeth process to test factor models, where the cross-sectional R^2 is a common measure of goodness of fit. Researchers typically draw a scatter plot comparing the model-predicted expected (excess) return and the actual expected (excess) return. If a factor model has a high cross-sectional R^2 and the testing portfolios are located around the 45-degree line in the scatter plot, one would declare good performance for this model.

Based on this evidence, the literature usually proceeds to conduct inferences about factor risk premia, perform out-of-sample analyses, discuss the implications of risk pricing, etc., assuming the model is correctly specified. However, I believe that this literature protocol warrants further examination. Having a high cross-sectional R^2 or a lining-up scatter plot does not mean the model is correctly specified. My new perspective is broader and powerful: the SDF is correctly specified if and only if the ZBR is unique. This is a much stricter requirement than conventional ones. The conventional Fama-MacBeth test can only tell us the general power of the model in pricing the cross-section of stock returns. And it can test whether a particular factor represents a source of systematic risk. However, the news tests for model misspecification focus on whether the proposed factor model completely captures all the systematic risks.

Whether this new perspective is relevant depends on the purpose of the study. In conventional asset pricing studies, we aim to understand whether one particular factor is priced in the cross-section without needing to identify all risk factors. In this case, conventional tests are sufficient to provide an answer. In the zero-beta rate literature, however, constructing the zero-beta rate essentially requires us to rule out all risk sources, and therefore the new perspective is an important consideration before selecting factor models. Without this stricter requirement, we may fail to reject incorrect factor models and thus produce flawed estimates. In summary, I advocate that researchers and practitioners take a comprehensive and rigorous view of empirical factor models. I argue for the importance of evaluating models with risky assets alone and leveraging my tests based on the uniqueness of zero-beta rate.

I also apply this new perspective to an emerging literature that uses machine-learning techniques in asset pricing to deal with high dimensional asset characteristics (Fan et al., 2016; Kelly et al., 2019; Lettau and Pelger, 2020; Kozak et al., 2020; Chen et al., 2023; etc.). This new literature hopes to better approximate the SDF by entertaining the possibility of latent factors, nonlinear models, and large datasets of test assets. For example, the instrumented PCA (IPCA) approach developed by Kelly et al. (2019) performs dimension reduction in the characteristics space by modeling risk loadings as functions of firm characteristics. An SDF constructed using five principal component latent factors is shown to price the cross-section of returns reasonably well. Given its sucess and flexibility, this new literature may be ideally suited to tackle the question of the zero-beta rate extraction. I expore this possibility in the context of the IPCA approach. First, I run the IPCA approach using gross returns instead of excess returns as the true zero-beta rate is unobserved. In this setting, the zero-beta rate in IPCA models are statistically indifferent from zero because the latent factors and time-varying risk loadings are sufficient to explain the variations in the cross-section of gross returns. Next, I manually construct excess returns with random zero-beta rates. No matter what the zero-beta rate is, IPCA will eventually produce some latent factors and time-varying betas that nicely price the cross-section of the constructed excess returns. Therefore, IPCA models are essentially inconsistent with a theoretically unique zero-beta rate.

This paper contributes directly to the empirical literature on the estimation of the zero-beta rate (Black, 1972; Black et al., 1972; Frazzini and Pedersen, 2014; Hong and Sraer, 2016; Bali et al., 2017; Di Tella et al., 2023.). I investigate these two estimation methods and show that model misspecification and errors-in-variables are two crucial aspects that lead to the high level and volatility of the zero-beta rate. Hence, the implications of the zero-beta rate for the security market line, beta anomaly, risk-free rate, Treasury convenience yield, etc., may need to be reconsidered.

I also relate to the literature on testing empirical asset pricing models (Black et al., 1972; Fama and MacBeth, 1973; etc.). Conventional testing relies heavily on the cross-sectional R^2. Kan et al. (2013) develops R^2-based model misspecification tests recognizing the sampling uncertainty of the cross-sectional R^2. The model misspecification in Kan et al. (2013) is an empirical concept that refers to the nonzero aggregate pricing errors (sum of squares of residuals) in the cross-sectional regression. I theoretically link model misspecification with the uniqueness of the zero-beta rate. I contribute to this literature by proposing a new perspective on evaluating factor models.

The remainder of this paper is structured as follows. Section 2 describes existing methods to estimate the zero-beta rate. Section 3 demonstrates the role of model misspecification and errors-in-variables (EIV) in the level and volatility of the estimated zero-beta rate. Section 4 proposes a new perspective on evaluating factor models. Section 5 concludes.

2. Estimating the Zero-Beta Rate

There are two ways to estimate the zero-beta rate in the literature. The first one is the traditional two-stage Fama-MacBeth approach commonly used in empirical asset pricing. The second one is proposed by Di Tella et al. (2023), which I call the zero-beta-portfolio approach. This section reviews the above two approaches using eight prominent factor models that are popular in the literature. Table 1 lists the 8 factor models studied in this paper. In all of the models, the estimated zero-beta rate tends to be much higher and more volatile than the US Treasury yield, which is typically used as a proxy for the zero-beta rate.

Table 1: Factor Models Studied

Factors	**Labels**	**Papers**
MRP	CAPM	Sharpe (1964), Lintner (1965)
MRP, CG-NDG, CG-DG	D-CCAPM	Breeden (1979), Yogo (2006)
MRP, SMB, HML	FF3	Fama and French (1993)
MRP, SMB, HML, RMW, CMA, UMD, BAB	FF6+BAB	Fama and French (2018) Frazzini and Pedersen (2014)
MRP, ME, IA, ROE, EG	Q5	Hou, Mo, Xue, and Zhang (2021)
MRP, SMB, HML, LIQ	LIQ	Pástor and Stambaugh (2003)
MRP, SMB, DUR	DUR	Gormsen and Lazarus (2023)
MRP, SMB, HML,INM	INM	He, Kelly, and Manela (2017)

Notes: This table lists 8 prominent factor models studied in this paper.

2.1. The Fama-MacBeth Approach

The two-stage Fama-MacBeth process (Fama and MacBeth, 1973) is one of the most influential and practical tools in the empirical asset pricing literature. It is widely used to evaluate factor models. The first step runs time series regressions for each test asset to estimate the risk loadings (betas), and the second step runs cross-sectional regressions to estimate the factor risk premia. Using the Fama-MacBeth process, models are usually tested with excess asset returns, where the Treasury yield is used as a proxy for the zero-beta rate. In this paper, however, I will relax the constraint that the zero-beta rate equals the Treasury yield because the ultimate goal is to estimate the unobserved zero-beta rate. In other words, I proceed with the specification that there does not exist a traded risk-free asset (Black, 1972)[3].

In the first step, I run the following time series regressions for each test asset:

$$R_{i,t+1} = \alpha_i + \beta_{M,i} R_{M,t+1} + \sum_{k=2}^{K} \beta_{k,i} f_{k,t+1} + \varepsilon_{i,t+1} \tag{1}$$

where $R_{i,t+1}$ is the gross return for test asset i, $R_{M,t+1}$ is the gross market return, and $f_{k,t+1}$ are other excess return factors constructed using long-short portfolios. The market return is

[3] Precisely speaking, all traded risk-free assets have additional non-pecuniary "convenience" benefits so that they are not suited for testing factor models.

separately described in the equation because the market factor should have been the market excess return with respect to the zero-beta rate. Since I do not observe the zero-beta rate, I have to use the gross asset return and the gross market return on both sides of the equation. The other factors are zero-investment long-short portfolios that do not explicitly involve the zero-beta rate. Using the estimated risk loadings (betas), I run the following cross-sectional regressions at each date as the second step:

$$R_i = r_z + \beta_i \lambda + \epsilon_i \tag{2}$$

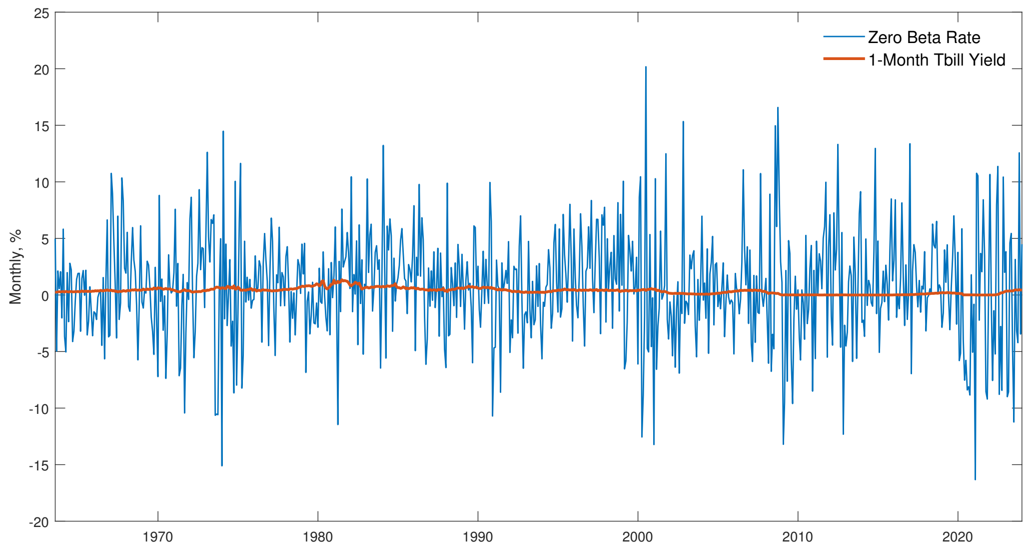
where R_i's are the gross returns for all test assets and β_i's are the estimated betas from the first step. This gives us the estimate of the zero-beta rate, $\hat{r_z}$, for each date.

In Figure 1, the blue line shows the estimated zero-beta rate using the Fama-MacBeth approach in the FF6+BAB model. The annualized mean and standard deviation of the zero-beta rate is 8.8% and 57.7%, respectively. As a comparison, the red line shows the 1-month Treasury bill yield from Kenneth French's website, which has an annualized mean of 4.4% and an annualized standard deviation of 3.2%. Surprisingly, the estimated zero-beta rate is 2.0 times higher in level and 18.1 times higher in volatility compared to the Treasury yield.

Appendix A.1 reviews all 8 factor models and describes the data, sample period, and test assets. Throughout the main text, I only report the empirical results using the FF6+BAB factor model. The estimated zero-beta rates in the other 7 factor models are very similar and are reported in Appendix A.2. Summarizing over all the 8 models, the zero-beta rate estimated by Fama-MacBeth is on average 3.0 times higher in level and 18.2 times more volatile than the Treasury yield. Figure A.1 shows the estimated zero-beta rate in all 8 factor models. Table A.1 reports the mean and standard deviations for each zero-beta rate series and shows that the correlation of estimated zero-beta rates across all 8 models is on average 0.57. In summary, the high level and high volatility of the estimated zero-beta rate is ubiquitous in all prominent factor models.

Notice that in the first step of the Fama-MacBeth approach, equation (1) uses the gross returns ($R_{i,t+1}$ and $R_{M,t+1}$) on both sides of the regression while the other factors ($f_{k,t+1}$) are excess returns. To alleviate the concern of inconsistent return units in the regression, Appendix A.3 provides an iterative Fama-MacBeth procedure to estimate the zero-beta rate. Briefly speaking, I initially guess a zero-beta rate and run regression (1) with excess asset returns and excess market returns. Next, I update the zero-beta rate using regression (2). Finally, I iterate the previous two steps until the zero-beta rate converges or until the beta estimates converge. Appendix A.3 shows that this iterative procedure does not essentially affect the results.

Figure. 1. Estimated Zero-Beta Rate (Fama-MacBeth)



Notes: This figure shows the monthly time series of the zero-beta rate estimated using the Fama-MacBeth approach in the FF6+BAB model from July 1963 to December 2023, in monthly percentages. The blue line is the estimated zero-beta rate. The red line is the 1-month US Treasury bill yield.

2.2. The Zero-Beta-Portfolio Approach

Di Tella et al. (2023) (hereafter, DHKW) proposes an innovative method to estimate the zero-beta rate. The zero-beta rate is defined as the expected return of a zero-beta portfolio, which is zero-beta with respect to the risk factors or orthogonal to the stochastic discount factor. Directly motivated by this definition, the authors find the minimum-variance zero-beta portfolio and calculate its expected return by projecting the realized return onto a set of macroeconomic predictors. An assumption underlying this process is that the zero-beta rate is a linear function of a set of macroeconomic predictors: $r_{z,t} = \xi Y_t$, where $r_{z,t}$ is the zero-beta rate and Y_t is a set of macroeconomic variables. The estimation algorithm is as follows:

i. Guess the time series of the zero-beta rate, denoted as $r_{z,t}^{(0)}$ (I use the 1-month Treasury yield as the initial guess).

ii. Run the following time series regression for each test asset to estimate the betas:

$$R_{i,t+1} - r_{z,t}^{(0)} = \alpha_i + \beta_{M,i} \left(R_{M,t+1} - r_{z,t}^{(0)} \right) + \sum_{k=2}^{K} \beta_{k,i} f_{k,t+1} + \varepsilon_{i,t+1} \tag{3}$$

iii. Estimate the variance-covariance matrix of the test assets, Σ_R, using a non-linear shrinkage method developed by Ledoit and Wolf (2017) and Ledoit and Wolf (2020)[4].

iv. Construct the portfolio weights of the minimum-variance zero-beta portfolio using the following analytical formula:

$$\omega = \Sigma_R^{-1} \begin{bmatrix} \iota & \beta \end{bmatrix} \left(\begin{bmatrix} \iota' \\ \beta' \end{bmatrix} \Sigma_R^{-1} \begin{bmatrix} \iota & \beta \end{bmatrix} \right)^{-} \begin{bmatrix} 1 \\ \vec{0} \end{bmatrix} \tag{4}$$

where ω is an $N \times 1$ vector of portfolio weights, β is an $N \times K$ matrix of risk loading estimated in step ii, $\vec{0}$ is a $K \times 1$ vector of zeros, and ι is an $N \times 1$ vector of ones. N is the number of asset returns and K is the number of factors. The proof of equation (4) is in Appendix B.1.

v. The realized return of the minimum-variance zero-beta portfolio is $R_{z,t+1} = \omega' R_{t+1}$. To calculate the expected return, I project the realized return onto a set of macroeconomic predictors, Y_t, with an OLS regression:

$$R_{z,t+1} = \omega' R_{t+1} = \hat{\xi} Y_t + \varepsilon_{t+1} \tag{5}$$

where the fitted value $r_{z,t}^{(1)} \equiv \hat{\xi} Y_t$ estimate the zero-beta rate.

vi. Iterate steps i to v until the zero-beta rate $r_{z,t}$ converges.

In the original DHKW paper, their main specification uses seven factors: the five equity factors of Fama and French (2015) and the bond factor and the default factor of Fama and French (1993)[5]. The main macroeconomic predictors used in DHKW include the 1-month Treasury bill yield, the rolling average of the previous twelve-month inflation, the term spread (10-year minus 3-month Treasury yields), the excess bond premium (EBP) of Gilchrist and Zakrajšek (2012), the unemployment rate, and a constant term. The authors also show that their results are robust to including different factors and predictors.

In this paper, I follow DHKW's zero-portfolio-approach using the 8 prominent factor models for consistency. Due to data availability[6], I replace the excess bond premium with two additional predictors: the CAPE (Cyclical-adjusted price-to-earnings ratio) and the corporate bond spread[7]. As in the original paper, including more predictors does not significantly change the estimation results.

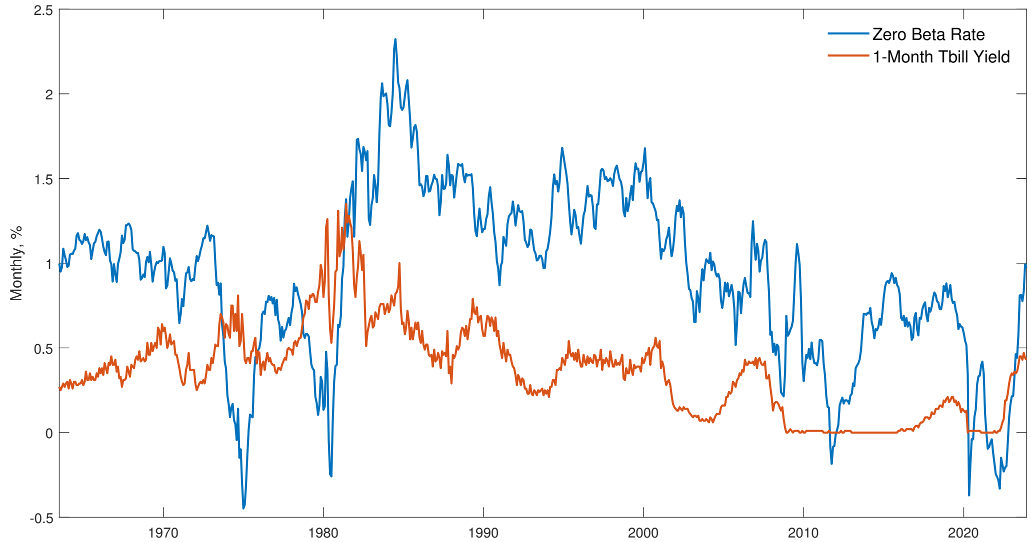
[4]The Matlab code for an analytical shrinkage estimator can be downloaded from Michael Wolf's website.

[5]Bond factor: the return of a 6 to 10-year Treasury bond portfolio over a 1-month Treasury bill. Default factor: the return of long-term corporate bonds over long-term Treasury bonds

[6]The excess bond premium (EBP) of Gilchrist and Zakrajšek (2012) is only available from January 1973, while my data samples for different factor models started between July 1963 and January 1970.

[7]Difference between the Moody's seasoned BAA corporate bond yields and Moody's seasoned AAA corporate bond yields.

Figure. 2. Estimated Zero-Beta Rate (DHKW)



Notes: This figure shows the monthly time series of the zero-beta rate estimated using the zero-beta-portfolio approach from DHKW in the FF6+BAB model from July 1963 to December 2023, in monthly percentages. The blue line is the estimated zero-beta rate. The red line is the 1-month US Treasury bill yield.

In Figure 2, the blue line shows the estimated zero-beta rate using the zero-beta-portfolio approach from DHKW in the FF6+BAB model. The annualized mean and standard deviation of the zero-beta rate is 11.2% and 6.0%, respectively. For comparison, the red line shows the 1-month Treasury bill yield from Kenneth French's website, which has an annualized mean of 4.4% and an annualized standard deviation of 3.2%. The estimated zero-beta rate is 2.6 times higher in level and 1.9 times higher in volatility compared to the Treasury yield. Table A.4 reports the point estimates and t-statistics for $\hat{\xi}$ in the predictive regression $R_{z,t+1} = \hat{\xi} Y_t + \varepsilon_{t+1}$.

The estimated zero-beta rates in the other 7 factor models are very similar and are reported in Appendix A.4. Summarizing over all the 8 models, the zero-beta rate estimated by DHKW is on average 2.6 times higher in level and 1.9 times more volatile than the Treasury yield. Figure A.3 shows the estimated zero-beta rate in all 8 factor models. Table A.3 reports the mean and standard deviations for each zero-beta rate series and shows that the correlation of estimated zero-beta rates across the 8 models is on average 0.96. Hence, I confirm the DHKW result that the estimated zero-beta rate is quite robust to different specifications of factor models. In summary, the high level and high volatility of the estimated zero-beta rate are ubiquitous in all prominent factor models.

The estimated zero-beta rate being higher and more volatile than the Treasury yield is consistent across two different approaches. However, the zero-beta-portfolio approach from DHKW produces much less volatile zero-beta rates than the Fama-MacBeth approach. In section 3, I will discuss the underlying reasons why there may be a big difference in the volatility between the two approaches and explain why the zero-beta-portfolio approach from DHKW makes good progress. In the meantime, I are going to argue that both methods are nontrivially subject to model misspecification and errors-in-variables, and thus researchers and practitioners should be cautious about both of them.

3. Model Misspecification and Errors-in-Variables

In the previous section, I followed two existing approaches in the literature to replicate the estimation of the zero-beta rate. The Fama-MacBeth approach produces a zero-beta rate that is approximately 3 times higher and 20 times more volatile than the Treasury yield. The zero-beta-portfolio approach from DHKW produces a zero-beta rate that is approximately 3 times higher and only around 2 times more volatile than the Treasury yield. In the second approach, DHKW claims that the estimated zero-beta rate is the correct intertemporal price of consumption, and it can be used as a proxy for the risk-free rate. Based on the evidence in DHKW, Di Tella et al. (2024) proceeds to provide a theory that can rationalize the high level and volatility of the zero-beta rate. This section takes a step back and asks a fundamental question about the existing evidence on the zero-beta rate: Is the zero-beta rate truly high and volatile or are the methods in trouble?

I hypothesize that both the Fama-MacBeth approach and the zero-beta-portfolio approach from DHKW tend to generate high and volatile zero-beta-rate estimates arising from two channels: model misspecification and errors-in-variables (EIV). To validate this hypothesis, I will take the following three logical steps. Section 3.1 theoretically shows that the zero-beta rate should be unique if the model is correctly specified. On the contrary, if the model is misspecified, the zero-beta rate is indeterminate. This multiplicity may lead to the high level and volatility of the estimated zero-beta rate. In section 3.2, I propose two tests based on the uniqueness of the zero-beta rate and show that the 8 prominent factor models are misspecified since they all fail to feature a unique zero-beta rate. Therefore, model misspecification could be a crucial candidate for explaining the level and volatility of the estimated zero-beta rate in section 2. Finally, section 3.3 performs simulation analyses, quantifying the effects of model misspecification on the estimation of the zero-beta rate. In the meantime, since both approaches potentially suffer from the issue of errors-in-variables (EIV), the simulation is also able to compare the quantitative effects of both channels.

3.1. Uniqueness of the Zero-Beta Rate

Roll (1980) discusses the zero-beta portfolios (orthogonal portfolios) in the CAPM environment. It proves that the zero-beta rate can take all levels if the market portfolio is not mean-variance efficient. This argument is generally correct for any multi-factor model. As an extension to Roll (1980), I emphasize the following Proposition about the uniqueness of the zero-beta rate for any factor model (The complete proof is in Appendix B.2).

Proposition 1. *The uniqueness of the zero-beta rate depends on whether the factor model is correctly specified.*

(i) If the factor model is correctly specified (no model misspecification), then the zero-beta rate is uniquely pinned down by the factors.

(ii) If the factor model is misspecified, then the zero-beta rate is indeterminate and it can take any values.

The intuition is as follows. A factor model defines what systematic risks are in the universe of traded assets by specifying risk factors. Then, it tells us what investors should be compensated for bearing these systematic risks. If a factor model correctly specifies the risk factors underlying all traded assets, it is able to uniquely recover the shadow return for a hypothetically risk-free asset even if it is not traded. This is the rate of return investors will earn if they are not exposed to any of the risk factors in the financial market. However, if a particular factor is misspecified meaning that it does not correctly specify the true risk factors, it recovers different levels of zero-beta rates because the corresponding zero-beta portfolios may still be exposed to different levels of omitted risks.

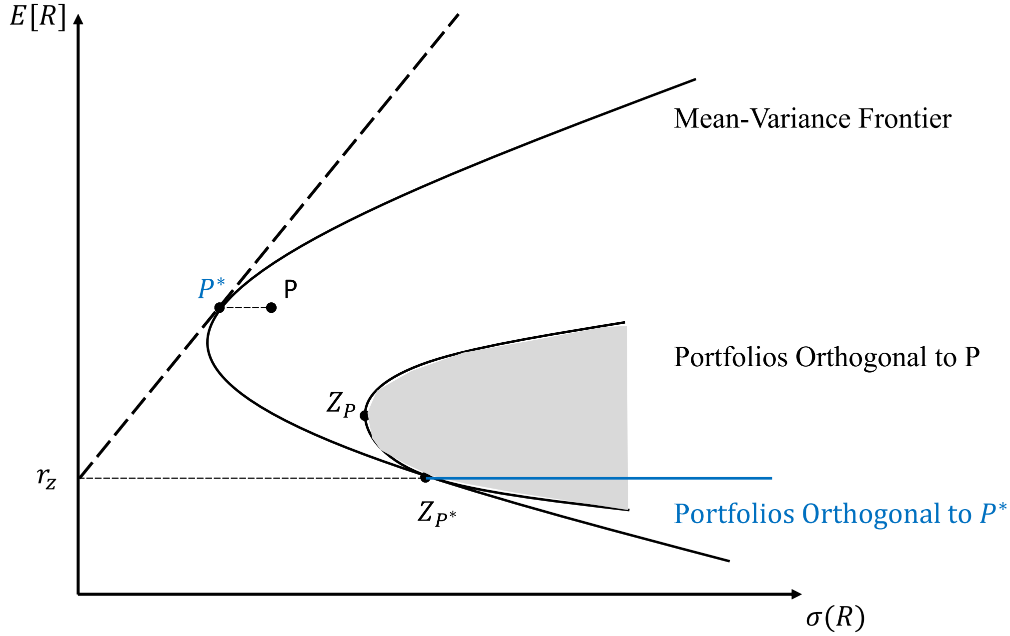
Let us further examine the nature of model misspecification and its effect on the zero-beta rate in the view of Proposition 1.*(ii)*. First of all, factor model theories suggest that any multi-factor model can be written as a single-factor model where the single factor portfolio is on the mean-variance frontier. Note that the general formula for a factor model is $E\left[R\right]-r_z = \beta\cdot\lambda = Cov\left(R,F\right)\Sigma_F^{-1}\cdot\lambda$ where $\beta \equiv Cov\left(R,F\right)\Sigma_F^{-1}$ is the risk loading. Then, the factor model can also be written as (the proof is in Appendix B.3):

$$E\left[R_i\right]-r_z = \tilde{\beta}\tilde{\lambda} \equiv \frac{Cov(R_i,\tilde{R})}{Var(\tilde{R})}\left(E[\tilde{R}]-r_z\right) \tag{6}$$

where $\tilde{R} \equiv \lambda'\Sigma_F^{-1}F$ is the single factor portfolio and $E[\tilde{R}]-r_z$ is the factor risk premium.

This link between multi-factor models and a single-factor model allows us to understand Proposition 1 in the traditional mean-standard deviation graph for any multi-factor models. A correctly specified factor model implies a constructed single factor $\tilde{R}$ on the mean-variance

Figure. 3. The Mean-Variance Frontier and Zero-Beta Frontiers



Notes: This figure shows the mean-variance frontier and the orthogonal portfolio (zero-beta portfolio) sets for two assets, P^* and P, in the mean-standard deviation space. P^* is on the mean-variance frontier. P is within the mean-variance frontier and it has the same mean as P^*. The blue horizontal line represents the orthogonal set with respect to P^*. The shaded area represents the orthogonal set with respect to P.

frontier. On the contrary, model misspecification indicates that the constructed single return factor $\tilde{R}$ is located within the mean-variance frontier. Interpreting the model misspecification from this mean-variance-frontier angle facilitates the discussion and allows us to borrow insights from Roll (1980). To avoid repetition, I will not prove the following statements because all the proofs can be found in Roll (1980).

- As is shown in Figure 3, portfolios orthogonal to a mean-variance efficient portfolio P^* are located on a horizontal ray (shown in blue in the figure) inside the mean-variance frontier. They have the same expected return—the zero-beta rate. One of them is on the mean-variance frontier, Z_{P^*}.
- Portfolios orthogonal to an inefficient portfolio P (see Figure 3) are located inside a half hyperbola in the mean-standard deviation space (shown in the shaded area in the figure). Hence, the zero-beta rate is indeterminate—there are infinite values of zero-beta rates. If $E[P] = E[P^*]$, then this hyperbola is tangent to the mean-variance frontier at the point Z_{P^*}. The minimum variance zero-beta portfolio Z_p falls inside the mean-variance frontier.

Now, I can develop intuition about the level and volatility of the zero-beta rate estimated using the two approaches described in section 2. In the Fama-MacBeth approach, the first-stage time series regressions estimate the betas, and the second stage essentially finds the expected return of the zero-beta portfolio through cross-sectional regressions. However, the cross-sectional regressions are run period-by-period, without imposing any restrictions from the time series perspective. Therefore, once the factor model is misspecified, I may end up randomly picking up the zero-beta portfolio from the shaded area shown in Figure 3 period by period. This may explain the high level and extremely high volatility of the estimated zero-beta rate if the factor model is misspecified. In the zero-beta-portfolio approach from DHKW, I look for the zero-beta portfolio of minimum variance (Z_P in Figure 3). This variance minimization problem introduces time series constraints and thus significantly reduces the volatility of the estimated zero-beta rate. However, this approach may still produce a high and volatile zero-beta rate if the factor model is misspecified.

In summary, model misspecification generates the multiplicity (or indeterminacy) of the zero-beta rate, which could potentially lead to the high level and volatility of the estimation due to the nature of the two approaches. In addition, since both approaches rely on the estimation of risk loadings (betas) as a building rock, it is reasonable to be concerned about the issue of errors-in-variables. In the simulation analysis of section 3.3, I will take into consideration both channels: model misspecification and errors-in-variables.

3.2. Testing for the Uniqueness of the Zero-Beta Rate

Section 3.1 theoretically connects model misspecification with the multiplicity of the zero-beta rate, which may potentially provide an explanation of the high level and volatility of the estimated zero-beta rate reported in section 2. To establish this logic, I would like to assess where there exists model misspecification in the 8 prominent factor models I studied. In this section, I propose two factor model tests that focus on the uniqueness of the zero-beta rate. I show that all the 8 prominent models are misspecified in the sense that they do not feature a unique zero-beta rate. It may sound astonishing to claim that the well-acknowledged, widely used factor models are misspecified. After all, they are popular in the literature because they price the cross-section of stock returns pretty well according to some conventional tests. I will further discuss this new perspective of testing factor models based on the uniqueness of the zero-beta rate in section 4.

Test 1: A Time Series Regression Test

The first test works with the Fama-MacBeth time series regressions. Although the time series regressions are run asset by asset, the uniqueness of the zero-beta rate should impose restrictions that connect all the separate regressions. Recall that theories suggest the traditional time series regression with excess returns has the following form:

$$R_{i,t+1} - r_{z,t} = \beta_{M,i}\left(R_{M,t+1} - r_{z,t}\right) + \sum_{j=2}^{K} \beta_{j,i} f_{j,t+1} + \varepsilon_{i,t+1} \tag{7}$$

where $r_{z,t}$ is the unobserved zero-beta rate that appears on both sides of the regression. Since I do not observe the zero-beta rate, I have to run regression (1) with gross asset returns and gross market returns mentioned in section 2.1. Rearranging equation (7) I get:

$$R_{i,t+1} = (1 - \beta_{M,i}) r_{z,t} + \beta_{M,i} R_{M,t+1} + \sum_{j=2}^{K} \beta_{j,i} f_{j,t+1} + \varepsilon_{i,t+1} \tag{8}$$

If the factor model is correctly specified, then the zero-beta rate should be unique. Comparing equation (1) and equation (8), I know that a unique series of the zero-beta rate $r_{z,t}$ imposes restrictions connecting the intercept terms of the time series regressions for all test assets. Specifically, the null hypothesis I are testing is:

$$\mathcal{H}_0: \quad \text{the time series regression intercept } \alpha_i = (1 - \beta_{M,i})\bar{r}_z \quad \forall i \tag{9}$$

where $\bar{r}_z$ is the sample mean of the zero-beta rate and $\beta_{M,i}$ is the market beta. Here is the testing procedure:

(i) For each test asset i, run regression (1) and use the intercept coefficient to compute the implied mean of the zero-beta rate: $\bar{r}_z^i = \alpha_i/(1 - \beta_{M,i})$.

(ii) The implied mean of the zero-beta rate should be equal across all test assets. Therefore, for another test asset $j \neq i$, conduct the F-test[8]: $\alpha_j = (1 - \beta_{M,j})\bar{r}_z^i$.

Suppose I have N test assets and K risk factors, then this pair-wise testing procedure ends up generating $N \times (N-1)$ separate tests and F-statistics. Figure 4 plots the histogram of $N \times (N-1)$ F-statistics in my first test for the FF6+BAB seven-factor model. Specifically,

[8] This is the textbook test for a linear restriction on the OLS coefficients, α_j and $\beta_{M,j}$:

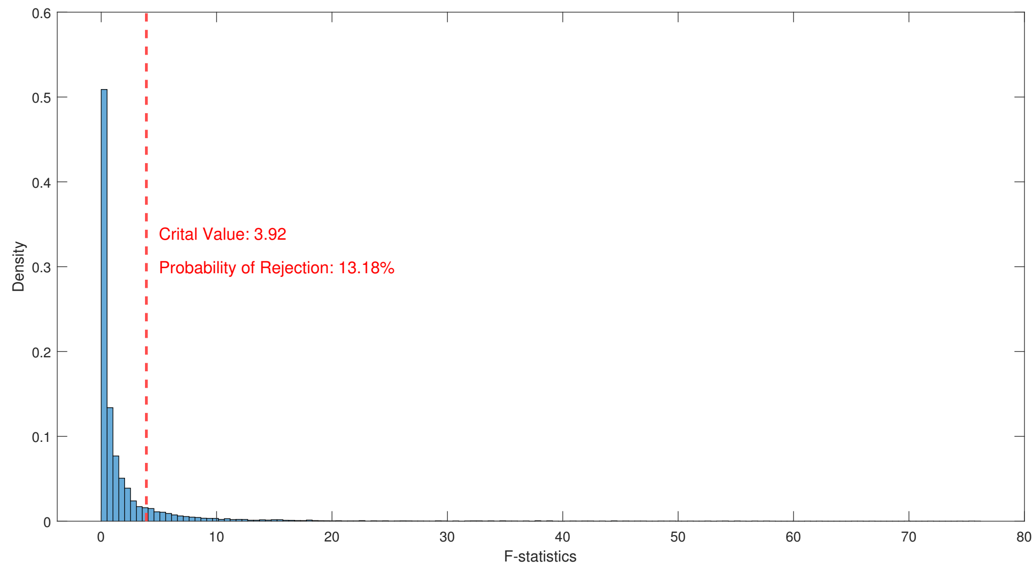
$$\alpha_j = (1 - \beta_{M,j})\bar{r}_z^i \quad \Leftrightarrow \quad \alpha_j + \bar{r}_z^i \beta_{M,j} = \bar{r}_z^i$$

rewrite this linear restriction in the matrix form: $H \cdot \hat{B} = \bar{r}_z^i$ where $H = \begin{bmatrix} 1 & \bar{r}_z^i & 0 & \cdots & 0 \end{bmatrix}$ and $\hat{B}$ is the OLS coefficient vector: $\begin{bmatrix} \alpha_j & \beta_{M,j} & \beta_{2,j} & \cdots & \beta_{K,j} \end{bmatrix}$. Then, I have the F-statistics and its distribution:

$$F = \frac{(H\hat{B} - \bar{r}_z^i)' \left[H(X'X)^{-1}H' \right] (H\hat{B} - \bar{r}_z^i)}{\hat{e}'\hat{e}/(N-K-1)} \quad \sim \quad F(1, N-K-1)$$

where N is the number of assets, K is the number of factors, X is the factor matrix including the constant term, and $\hat{e}$ is the sample residual vector.

Figure. 4. Histogram of F-Statistics



Notes: This figure plots the histogram of $N \times (N-1)$ F-statistics in the Fama-MacBeth time series regression test for the FF6+BAB model. The red vertical dashed line represents the critical value (3.92) of the F distribution with degrees of freedom 1 and $N-K-1$ with a 5% significance level.

I have $135 \times 134 = 18,090$ pair-wise tests and F-statistics out of 135 testing portfolios. The red vertical dashed line represents the critical value (3.92) of the F distribution with degrees of freedom 1 and 1 $(N-K-1)$ with a 5% significance level. Among the 18,090 separate tests, 13.18% of them reject the individual null hypothesis: $\alpha_j = (1-\beta_{M,j})\bar{r}_z^i$. Since the probability of rejection (13.18%) is higher than 5%, I conclude that the overall null hypothesis is rejected. That is, the zero-beta rate is not unique in the FF6+BAB model.

The first row of Table 2 reports the probabilities of rejection for all 8 factor models. All of them are higher than the 5% significance threshold, indicating that all the 8 factor models are rejected based on this test. They do not feature a unique zero-beta rate and thus are not consistent with the factor model theories. Figure A.4 plots the histogram of $N \times (N-1)$ F-statistics for all 8 factor models.

Test 2: A Zero-Beta-Portfolio Test

The second test works with the zero-beta-portfolio approach from DHKW. Recall that DHKW focuses on the minimum-variance zero-beta portfolio and estimates the zero-beta rate by predicting the expected return of that portfolio. However, I should acknowledge that there exists an infinite number of zero-beta portfolios for any factor model. If the factor model is correctly specified, then the zero-beta rate is unique. This means that the expected return of all zero-beta portfolios should be equalized. Motivated by this idea, I look at a

Table 2: Model Testing based on the Uniqueness of the Zero-Beta Rate

Models	CAPM	D-CCAPM	FF3	FF6+BAB	Q5	LIQ	DUR	INM
Test 1 Pr(Rejection)	21.25%	11.54%	18.01%	13.18%	11.54%	17.52%	18.67%	11.61%
Test 2 p-value	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Notes: This table reports the model testing results based on the uniqueness of the zero-beta rate for all 8 prominent factor models. Section 3.2 describes the two tests. The first row reports the probabilities of rejection for test 1. The second row reports the p-values for test 2.

large number of zero-beta portfolios and test whether they have the same expected return in a statistical sense. Specifically, the null hypothesis I are testing is:

$$\mathcal{H}_0 : \quad \text{the means of all zero-beta portfolios are equalized} \tag{10}$$

Here is the testing procedure:

(i) Completely follow the zero-beta-portfolio approach from DHKW described in section 2.2. This procedure produces the beta estimates and the minmum-variance zero-beta portfolio weights.

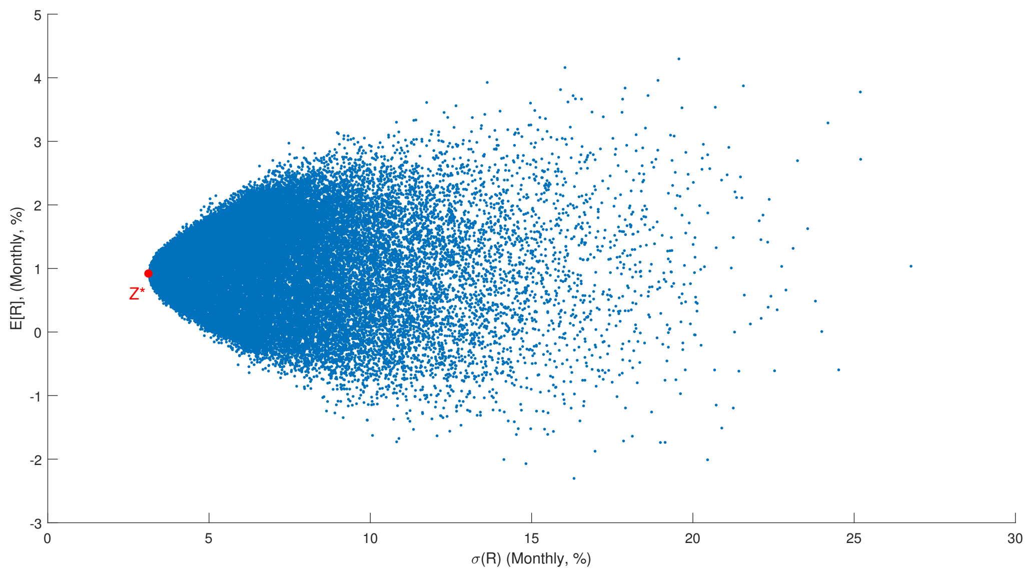
(ii) Construct multiple zero-beta portfolios that are close to the minimum-variance zero-beta portfolio. I want to make sure that the constructed zero-beta portfolios are close to the minimum-variance one for better visualization in the mean-standard deviation space. I do it in the following steps. First. I construct the null space of the betas[9]. Second, I project the minimum-variance zero-beta portfolio weight onto the beta null space and obtain the coefficients. Next, I perturb the coefficients by adding tiny normal shocks to construct 10,000 new coefficients. Finally, I multiply the new coefficients by the basis of the beta null space and get the new portfolio weights after normalization. In this way, I can make sure that the constructed portfolio weights are close to the minimum-variance zero-beta portfolio weight.

(iii) With the 10,000 zero-beta portfolio weights, I construct 10,000 zero-beta portfolios that are close to the minimum-variance zero-beta portfolio. Then, I perform the one-way ANOVA to test whether these 10,000 portfolios have the same expected return.

Figure 5 plots the zero-beta portfolios constructed around the zero-beta portfolio of minimum variance in the mean standard deviation space for the FF6+BAB model. The

[9]The null space of betas is characterized by a set of basis vectors. Any linear combination of the basis vectors will be zero-beta by definition.

Figure. 5. Constructed Zero-Beta Portfolios



Notes: This figure plots the constructed zero-beta portfolios around the minimum-variance zero-beta portfolio in the mean-standard deviation space for the FF6+BAB model. The red dot Z^* represents the minimum-variance zero-beta portfolio. The blue dots denote the random zero-beta portfolios.

red dot Z^* represents the minimum-variance zero-beta portfolio. The blue dots denote the random zero-beta portfolios. Interestingly, the shape of the zero-beta portfolios is highly consistent with being bounded by a half hyperbola, as is shown in section 3.1 (see Figure 3). Obviously, it is hard to believe that all zero-beta portfolios have the same mean. Indeed, the F-statistic for the ANOVA is 4.678 with a p-value of 0.00. Hence, I conclude that the overall null hypothesis is rejected. That is, the zero-beta rate is not unique in the FF6+BAB model.

The second row of Table 2 reports the ANOVA p-values for all 8 factor models. All of them are 0.00, indicating that all the 8 factor models are rejected based on this test. They do not feature a unique zero-beta rate and thus are not consistent with the factor model theories. Figure A.5 plots the constructed zero-beta portfolios for all 8 factor models.

In summary, I have proposed two tests (a time series regression test and a zero-beta portfolio test) to evaluate the factor models. I show that all the 8 prominent factor models fail my tests indicating that they do not feature a unique zero-beta rate and thus are not consistent with factor model theories. Combined with Proposition 1, it is reasonable to believe that the high level and volatility of the estimated zero-beta rate in section 2 could potentially come from model misspecification in theory. Now the question is: How quantitatively important is model misspecification? A simulation analysis can give us the answer.

3.3. Simulation Analysis

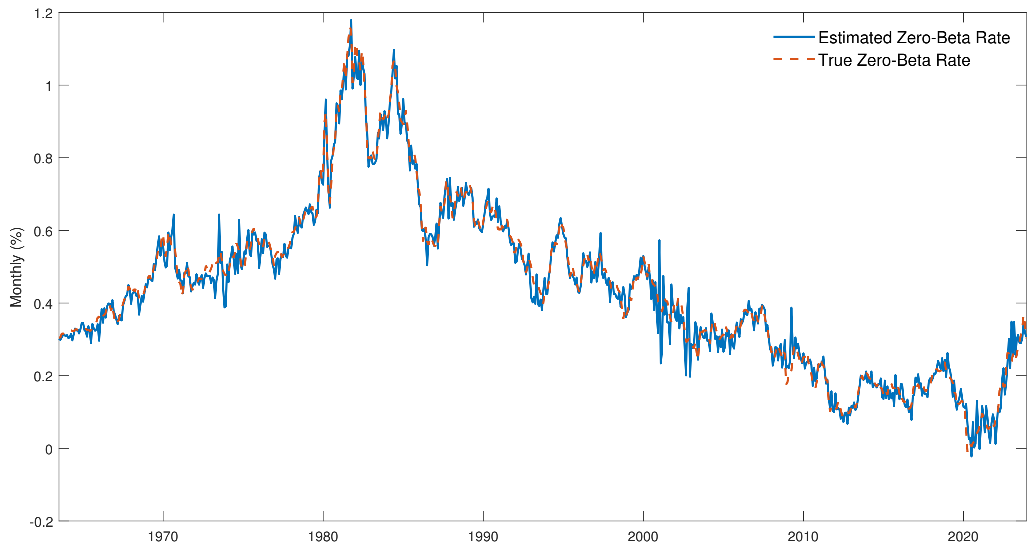
Up to now, I know that model misspecification could theoretically generate a high level and volatility of the estimated zero-beta rate, and our 8 prominent factor models are indeed misspecified in the view of a unique zero-beta rate. I proceed to perform simulation analyses to quantify the importance of model misspecification in the estimation of the zero-beta rate using the previous two approaches (the Fama-MacBeth approach and the zero-beta-portfolio approach). Simulation can also achieve other desired purposes. First, I show that if a particular model is correctly specified, it is able to recover the true unobserved zero-beta rate with the two approaches and it is able to pass my two tests. This means that the two estimation approaches and the two tests are statistically valid. There is nothing wrong with the methods. Second, since both approaches potentially suffer from the issue of errors-in-variables (EIV), the simulation is also able to quantify and compare the effects of both model misspecification and errors-in-variables. It turns out that both channels contribute nontrivially to the high level and volatility of the estimated zero-beta rate.

3.3.1. Simulating the asset returns

Let me start from describing my data simulation. First, I create an artificial zero-beta rate which is assumed to be a linear function of a set of macroeconomic predictors following DHKW: $r_{z,t} = a'Y_t$. In fact, I are using the same set of macroeconomic predictors, Y_t, as in the zero-beta-portfolio approach to estimating the zero-beta rate. Recall that Y_t includes: the 1-month Treasury bill yield, the rolling average of the previous twelve-month inflation, the term spread, the unemployment rate, the CAPE, the corporate bond spread, and a constant term. In the simulation, as a result, I make sure that the zero-beta-portfolio approach does not make mistakes in specifying the set of macroeconomic predictors, isolating the effects of model misspecification and errors-in-variables. $a = [0.007, 1, -0.1, 1, -0.005, 0.0002, 0.001]$, which is manually designed such that the true 1-month zero-beta rate is generally above the 1-month Treasury yield. My simulation results are robust to the artificial design of the zero-beta rate.

To make the simulated data as close to the real data as possible, I use the seven real-life factors, that is, $f_t =$ MRP, SMB, HML, RMW, CMA, UMD, BAB} downloaded from Kenneth French's website. I also use the real-life betas, that is, β's are estimated using regression (1) for the N test assets with the above seven factors. The simulated return is computed as: $R_{t+1} = r_{z,t} + \beta f_{t+1} + \sigma\varepsilon_{t+1}$, where σ adds noises to the estimation. Intuitively, the magnitude of σ captures the severity of errors-in-variables. Thus, I will vary the level of σ and see how errors-in-variables affect the estimated zero-beta rate. Because the factors

Figure. 6. Estimated Zero-Beta Rate vs True Zero-Beta Rate (Fama-MacBeth)



Notes: This figure compares the estimated zero-beta rate using the Fama-MacBeth approach versus the true zero-beta rate. The blue line is the estimated zero-beta rate in the simulation with the correctly specified model and minimal estimation errors in betas ($\sigma = 0$). The red dashed line is the artificially constructed true zero-beta rate.

and betas come directly from the real life, the simulated sample has the same length of 726 months as the factors, which range from July 1963 to December 2023.

3.3.2. The Fama-MacBeth approach in simulation

I apply the Fama-MacBeth approach in the simulated sample to estimate the zero-beta rate. To start with, I use the full seven-factor model and assume $\sigma = 0$. When $\sigma = 0$, the maximum standard error among estimating the seven betas for all test assets in regression (1) is as low as 0.0062[10]. In this way, I start from evaluating the Fama-MacBeth approach when the model is correctly specified and the errors-in-variables concern is minimal. Figure 6 compares the estimated zero-beta rate versus the true zero-beta rate. The blue line is the estimated zero-beta rate in the simulation with the correctly specified model and minimal estimation errors in betas ($\sigma = 0$). The red dashed line is the artificially constructed true zero-beta rate. The estimation is pretty good as the estimated series closely tracks the true value, although it is not perfect because of the non-zero beta estimation errors.

Let us introduce model misspecification and errors-in-variables and see what happens. I will look at the two channels separately and then combine them together.

[10]This number is not exactly zero because regression (1) uses the gross returns and its intercept term α_i does not consider the time variation in the zero-beta rate.

Table 3: Estimated Zero-Beta Rate (Fama-MacBeth), Model Misspecification

	Mean			Volatility			Test #1 Pr(Rejection)		
	Avg	Max	Min	Avg	Max	Min	Avg	Max	Min
True ZB	0.443			0.233					
Correct Model	0.439			0.231			0.000		
Omit 1 Factor	0.485	1.219	0.073	1.955	4.547	0.411	0.135	0.321	0.000
Omit 2 Factors	0.601	1.276	0.032	3.099	8.301	0.886	0.178	0.514	0.000
Omit 3 Factors	0.799	1.390	0.023	3.984	8.469	1.964	0.157	0.569	0.000
Omit 4 Factors	1.015	1.596	0.102	4.419	8.043	2.590	0.115	0.566	0.000
Omit 5 Factors	1.178	1.706	0.163	4.458	7.963	3.228	0.075	0.463	0.000
Omit 6 Factors	1.238	1.384	1.055	4.512	4.912	4.072	0.041	0.276	0.000

Notes: This table shows the estimation results using the Fama-MacBeth approach under model misspecification with minimal errors-in-variables in the simulated sample. The first three columns report the mean of the true zero-beta rate and the estimated zero-beta rates using the correct model versus misspecified models. The middle three columns report the volatility and the last three columns report the probability of rejection from the time series regression test described in section 3.2. The first row reports the mean and volatility of the true zero-beta rate as a benchmark. Moving from the second row to the last row, the magnitude of model specification rises from none to the maximum. Since there are multiple ways to omit a certain number of factors, I report the average, max, and min of the statistics.

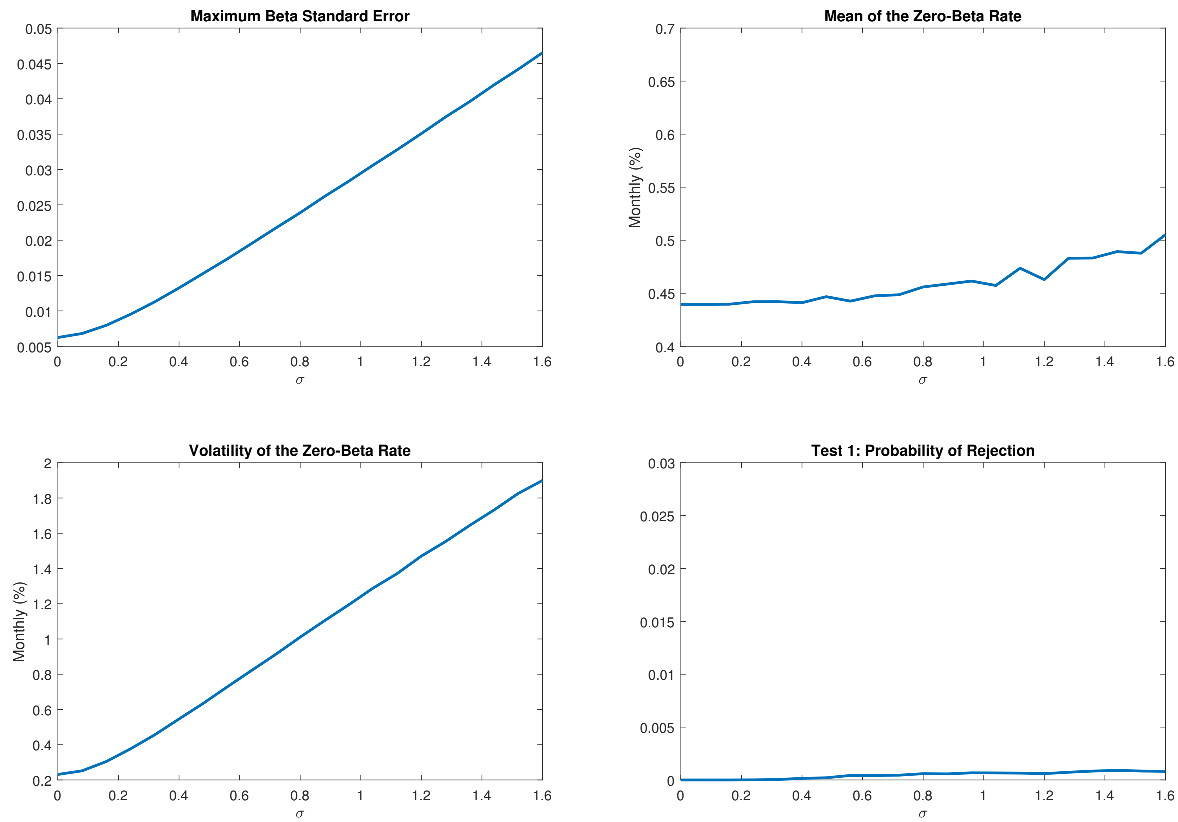
First, I randomly omit factors in my estimation while keeping $\sigma = 0$. This aims to separately examine model misspecification with minimal errors-in-variables. Since the full correct model has seven factors, I are able to randomly omit one up to six factors. For each misspecified model that misses one or several factors, I apply the Fama-MacBeth approach to estimate the zero-beta rate. Table 3 shows the estimation results under model misspecification with minimal errors-in-variables in the simulated sample. The first three columns report the mean of the true zero-beta rate and the estimated zero-beta rates using the correct model versus misspecified models. The middle three columns report the volatility and the last three columns report the probability of rejection from the time series regression test described in section 3.2.

The first row reports the mean and volatility of the true zero-beta rate as a benchmark. Moving from the second row to the last row, the magnitude of model specification rises from none to the maximum. Since there are multiple ways to omit a certain number of factors, I report the average, max, and min of the statistics. For example, there are 7 possible six-factor models in the third row since there are 7 ways to randomly omit one factor from the seven factors. For all these 7 misspecified models, I estimate the 7 zero-beta rates, report the average, max, and min of them, and perform the test. The second row shows that the mean and volatility of the zero-beta rate are very close to the true value (also see Figure 6). In addition, the time series regression test produces a probability of rejection of zero, confirming that the model is correctly specified in the sense that it features a unique zero-beta rate. However, when I start to have model misspecification by omitting factors, the mean and volatility of the estimated zero-beta rate are monotonically and dramatically increasing. The average volatility of the zero-beta rate already goes up more than 8 times even if there is only one factor missing. The misspecified models on average fail the time series test according to the last three columns in Table 3 with the exception of omitting 6 factors. It may seem weird that the average probability of rejection is not monotonically increasing with the magnitude of model misspecification and that there always exist models that are misspecified but pass the test as suggested by the last "min" column. This could be because the estimation uncertainty also accumulates with larger model misspecification, reducing the power of the time series regression test. Therefore, the interpretation of the time series regression test result should be treated with caution. If a model passes the test, I cannot say that it is correctly specified. However, if a model fails the test, which is the case for all 8 prominent factor models as is shown in section 3.2, I can claim for misspecification (at some significance level).

Second, I gradually increase σ while keeping the correct factor model. This aims to separately examine the errors-in-variables without model misspecification. I increase σ from 0 to 1.6 (monthly percentage), which is the calibrated average monthly standard deviation of real-life idiosyncratic returns. Figure 7 shows the estimation results with increasing errors-in-variables and the correct model in the simulated sample. The top-left panel plots the maximum standard error of beta estimates as a function of σ. This confirms our intuition that larger σ leads to larger beta estimation errors. The top-right panel shows that the mean of the estimated zero-beta rate is increasing in σ, but the slope is pretty flat. The bottom-left panel shows that the volatility of the estimated zero-beta rate is monotonically increasing in σ with a slope that is much steeper than the previous panel. The bottom-right panel plots the probability of rejection from the time series test. The almost flat line tells us that the correct model always passes the time series regression test despite the increasing errors-in-variables.

Table 3 and Figure 7 show that both model misspecification and errors-in-variables separately contribute to the high level and volatility of the zero-beta rate using the Fama-MacBeth approach in a nontrivial way. I also compare the quantitative effects of both chan-

Figure. 7. Estimated Zero-Beta Rate (Fama-MacBeth), Errors-in-Variables



Notes: This figure shows the estimation results using the Fama-MacBeth approach with increasing errors-in-variables and the correct model in the simulated sample. As a function of σ, the top-left panel plots the maximum standard error of beta estimates, the top-right panel plots the mean of the estimated zero-beta rate, the bottom-left panel plots the volatility of the estimated zero-beta rate, and the bottom-right panel plots the probability of rejection from the time series test.

nels by combining them together. Table 4 shows the quantitative effects of both channels (model misspecification and errors-in-variables) on the mean and volatility of the estimated zero-beta rate. Along the rows, the magnitude of model specification rises from none to the maximum when I move from the correct model to omitting 6 random factors. Along the columns, the magnitude of errors-in-variables rises from the minimum to the maximum when I increase σ from 0 to 1.6 (monthly percentage). For each pair of model misspecification and σ, I estimate the zero-beta rates and report the average multiples of the mean or volatility relative to those of the true zero-beta rate. With the correct model, the mean and volatility of the estimated zero-beta rate rise to 1.1 times and 8.2, times of those of the true zero-beta rate, respectively. However, the mean and volatility jump to 1.1 times and 8.4 times when I only drop one random factor. This implies that model misspecification is quantitatively contributing more to the high level and volatility of the estimated zero-beta rate.

Table 4: Quantification of Both Channels (Fama-MacBeth)

	Mean					Volatility				
σ	0	0.4	0.8	1.2	1.6	0	0.4	0.8	1.2	1.6
Correct Model	1.0	1.0	1.0	1.1	1.1	1.0	2.3	4.3	6.3	8.2
Omit 1 Factors	1.1	1.1	1.1	1.2	1.2	8.4	8.7	9.5	10.4	11.5
Omit 2 Factors	1.4	1.4	1.4	1.4	1.4	13.3	13.4	13.7	14.1	14.7
Omit 3 Factors	1.8	1.8	1.8	1.8	1.9	17.1	17.1	17.2	17.4	17.6
Omit 4 Factors	2.3	2.3	2.3	2.3	2.3	18.9	18.9	19.0	19.0	19.0
Omit 5 Factors	2.7	2.7	2.7	2.7	2.7	19.1	19.1	19.1	19.1	19.2
Omit 6 Factors	2.8	2.8	2.8	2.8	2.8	19.3	19.3	19.3	19.4	19.4

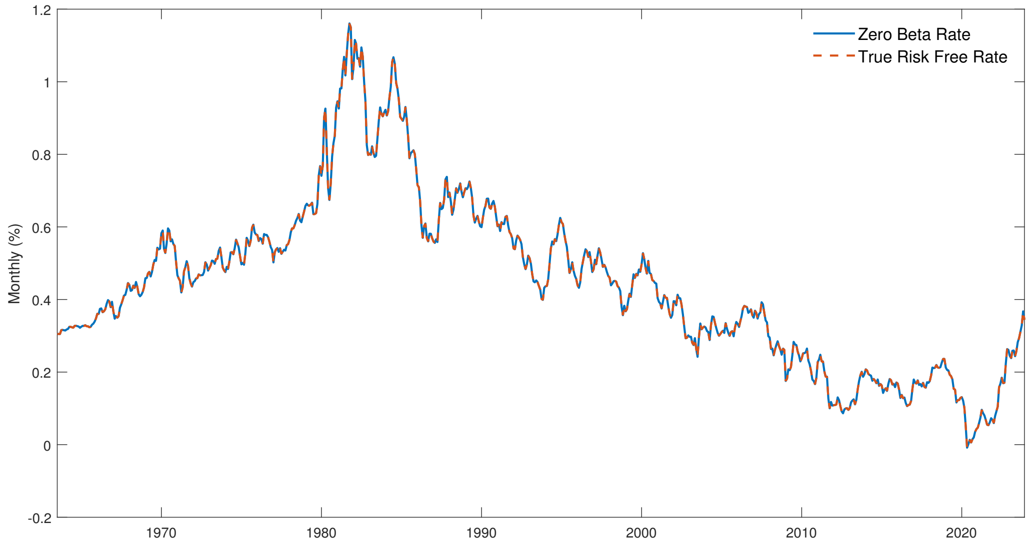
Notes: This table shows the quantitative effects of both channels (model misspecification and errors-in-variables) on the mean and volatility of the estimated zero-beta rate using the Fama-MacBeth approach. Along the rows, the magnitude of model specification rises from none to the maximum when I move from the correct model to omitting 6 random factors. Along the columns, the magnitude of errors-in-variables rises from the minimum to the maximum when I increase σ from 0 to 1.6 (monthly percentage). For each pair of model misspecification and σ, I estimate the zero-beta rates and report the average multiples of the mean or volatility relative to those of the true zero-beta rate.

3.3.3. The zero-beta-portfolio approach in simulation

I apply the zero-beta-portfolio approach from DHKW in the simulated sample to estimate the zero-beta rate. Since this approach requires an estimation of the variance-covariance matrix of returns (Σ_R in equation 4), I assume in the simulation that this matrix is the same as the real-life variance-covariance matrix obtained by a non-linear shrinkage estimator. Similarly, I start with using the full seven-factor model and assume $\sigma = 0$. When $\sigma = 0$, the maximum standard error among estimating the seven betas for all test assets in the zero-beta-portfolio approach from DHKW is zero[11]. In this way, I start from evaluating the zero-beta-portfolio approach when the model is correctly specified and the errors-in-variables concern is zero. Figure 8 compares the estimated zero-beta rate versus the true zero-beta rate. The blue line is the estimated zero-beta rate in the simulation with the correctly specified model and zero estimation errors in betas ($\sigma = 0$). The red dashed line is the

[11] Recall in the Fama-MacBeth approach, the beta estimation errors are minimal but not exactly zero even when $\sigma = 0$. Here, the zero-beta-portfolio approach is an iterative procedure that guesses and verifies the zero-beta rate (see section 2.2). Excess returns with respect to the zero-beta rate are used in each iteration to estimate betas. Hence, the beta estimation errors converge to zero as the zero-beta rate converges.

Figure. 8. Estimated Zero-Beta Rate vs True Zero-Beta Rate (DHKW)



Notes: This figure compares the estimated zero-beta rate using the zero-beta-portfolio approach versus the true zero-beta rate. The blue line is the estimated zero-beta rate in the simulation with the correctly specified model and zero estimation errors in betas ($\sigma = 0$). The red dashed line is the artificially constructed true zero-beta rate.

artificially constructed true zero-beta rate. The estimation is perfect as the estimated series perfectly overlaps with the true value.

Similar to the previous exercise, I will introduce model misspecification and errors-in-variables separately followed by combining the two channels together.

First, I randomly omit factors in my estimation while keeping $\sigma = 0$. This aims to separately examine model misspecification without errors-in-variables. For each misspecified model that misses one or several factors, I apply the zero-beta-portfolio approach to estimate the zero-beta rate. Table 5 shows the estimation results under model misspecification without errors-in-variables in the simulated sample. The first three columns report the mean of the true zero-beta rate and the estimated zero-beta rates using the correct model versus misspecified models. The middle three columns report the volatility and the last three columns report the ANOVA p-value from the zero-beta-portfolio test described in section 3.2. The first row reports the mean and volatility of the true zero-beta rate as a benchmark. Moving from the second row to the last row, the magnitude of model specification rises from none to the maximum. The second row shows that the mean and volatility of the zero-beta rate are exactly the same as the true value (also see Figure 8). In addition, the zero-beta-portfolio test produces a p-value of one, confirming that the model is correctly specified in the sense that it features a unique zero-beta rate. However, when I start to have model misspecification by omitting factors, the mean and volatility of the estimated zero-

Table 5: Estimated Zero-Beta Rate (DHKW), Model Misspecification

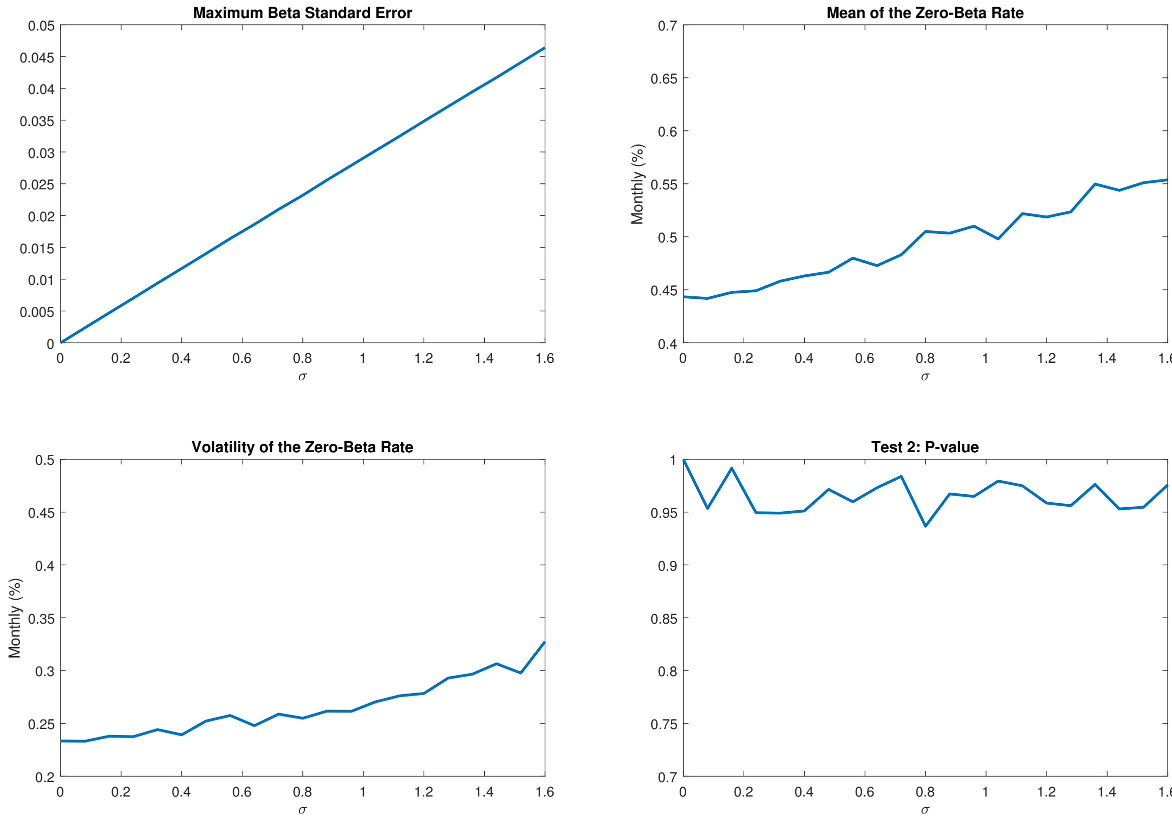
	Mean			Volatility			Test #1 Pr(Rejection)		
	Avg	Max	Min	Avg	Max	Min	Avg	Max	Min
True ZB	0.443			0.233					
Keep All Factors	0.443			0.233			1.000		
Omit 1 Factor	0.498	0.785	0.381	0.262	0.354	0.217	0.000	0.000	0.000
Keep 5 Factors	0.559	0.831	0.374	0.291	0.417	0.218	0.000	0.000	0.000
Keep 4 Factors	0.625	0.840	0.369	0.322	0.420	0.221	0.000	0.000	0.000
Keep 3 Factors	0.692	0.841	0.386	0.354	0.420	0.234	0.000	0.000	0.000
Keep 2 Factors	0.755	0.842	0.413	0.384	0.420	0.242	0.000	0.000	0.000
Keep 1 Factors	0.806	0.842	0.678	0.407	0.419	0.353	0.000	0.000	0.000

Notes: This table shows the estimation results using the zero-beta-portfolio approach under model misspecification without errors-in-variables in the simulated sample. The first three columns report the mean of the true zero-beta rate and the estimated zero-beta rates using the correct model versus misspecified models. The middle three columns report the volatility and the last three columns report the probability of rejection from the time series regression test described in section 3.2. The first row reports the mean and volatility of the true zero-beta rate as a benchmark. Moving from the second row to the last row, the magnitude of model specification rises from none to the maximum. Since there are multiple ways to omit a certain number of factors, I report the average, max, and min of the statistics.

beta rate are monotonically increasing. Compared to the Fama-MacBeth approach in Table 3, the speed of increase is much lower with the zero-beta-portfolio approach, suggesting its superiority over the former method. According to the last three columns, all of the misspecified models fail the zero-beta-portfolio test. If a model fails the test, which is the case for all 8 prominent factor models as is shown in section 3.2, I can claim for misspecification (at some significance level).

Second, I gradually increase σ while keeping the correct factor model. This aims to separately examine the errors-in-variables without model misspecification. I increase σ from 0 to 1.6 (monthly percentage), which is the calibrated average monthly standard deviation of real-life idiosyncratic returns. Figure 9 shows the estimation results with increasing errors-in-variables and the correct model in the simulated sample. The top-left panel plots the maximum standard error of beta estimates as a function of σ. This confirms our intuition that larger σ leads to larger beta estimation errors. The top-right panel shows that the mean of the estimated zero-beta rate is increasing in σ. The bottom-left panel shows that

Figure. 9. Estimated Zero-Beta Rate (DHKW), Errors-in-Variables



Notes: This figure shows the estimation results using the zero-beta-portfolio approach with increasing errors-in-variables and the correct model in the simulated sample. As a function of σ, the top-left panel plots the maximum standard error of beta estimates, the top-right panel plots the mean of the estimated zero-beta rate, the bottom-left panel plots the volatility of the estimated zero-beta rate, and the bottom-right panel plots the ANOVA p-value from the zero-beta-portfolio.

the volatility of the estimated zero-beta rate is also increasing in σ with a slope that is similar to the previous panel. The bottom-right panel plots the ANOVA p-value from the zero-beta-portfolio test. That the p-values are close to 1 implies that the correct model always passes the zero-beta-portfolio test despite the increasing errors-in-variables.

Table 5 and Figure 9 show that both model misspecification and errors-in-variables separately contribute to the high level and volatility of the zero-beta rate using the zero-beta-portfolio approach in a nontrivial way. I also compare the quantitative effects of both channels by combining them together. Table 6 shows the quantitative effects of both channels (model misspecification and errors-in-variables) on the mean and volatility of the estimated zero-beta rate. Along the rows, the magnitude of model specification rises from none to the maximum when I move from the correct model to omitting 6 random factors. Along

Table 6: Quantification of Both Channels (DHKW)

	Mean					Volatility				
σ	0	0.4	0.8	1.2	1.6	0	0.4	0.8	1.2	1.6
Correct Model	1.0	1.0	1.1	1.2	1.2	1.0	1.0	1.2	1.2	1.3
Omit 1 Factors	1.1	1.2	1.2	1.3	1.4	1.1	1.2	1.2	1.4	1.4
Omit 2 Factors	1.3	1.3	1.4	1.4	1.5	1.2	1.3	1.4	1.4	1.5
Omit 3 Factors	1.4	1.4	1.5	1.5	1.6	1.4	1.4	1.5	1.6	1.6
Omit 4 Factors	1.6	1.6	1.6	1.7	1.8	1.5	1.6	1.6	1.7	1.8
Omit 5 Factors	1.7	1.7	1.8	1.8	1.9	1.6	1.7	1.8	1.8	1.9
Omit 6 Factors	1.8	1.8	1.9	2.0	2.0	1.7	1.8	1.9	1.9	2.0

Notes: This table shows the quantitative effects of both channels (model misspecification and errors-in-variables) on the mean and volatility of the estimated zero-beta rate using the Fama-MacBeth approach. Along the rows, the magnitude of model specification rises from none to the maximum when I move from the correct model to omitting 6 random factors. Along the columns, the magnitude of errors-in-variables rises from the minimum to the maximum when I increase σ from 0 to 1.6 (monthly percentage). For each pair of model misspecification and σ, I estimate the zero-beta rates and report the average multiples of the mean or volatility relative to those of the true zero-beta rate.

the columns, the magnitude of errors-in-variables rises from zero to the maximum when I increase σ from 0 to 1.6 (monthly percentage). For each pair of model misspecification and σ, I estimate the zero-beta rates and report the average multiples of the mean or volatility relative to those of the true zero-beta rate. With the correct model, the mean and volatility of the estimated zero-beta rate rise to 1.2 times and 1.3 times of those of the true zero-beta rate, respectively. However, the mean and volatility jump to 1.3 times and 1.2 times when I only drop two random factors. This implies that model misspecification is quantitatively contributing more to the high level and volatility of the estimated zero-beta rate using the zero-beta-portfolio approach.

Now I are able to answer the question at the beginning of section 3: Is the zero-beta rate truly high and volatile or are the estimation methods in trouble? I show that the estimated zero-beta rate tends to be too high and too volatile because of both model misspecification and errors-in-variables. Section 3.1 establishes the theoretical result that model misspecification leads to the multiplicity of the zero-beta rate, which in turn contributes to the high level and volatility of the estimated zero-beta rate given the nature of the two approaches in section 2. Section 3.2 confirms that the 8 prominent factor models studied in this paper all fail the two new tests that I designed to test for the uniqueness of the zero-beta rate. The simulation analyses in section 3.3 show that model misspecification is quantitatively nontrivial in explaining the high level and volatility of the estimated zero-beta rate. the errors-in-variables problem is also another contributing factor, but it has less power than model misspecification. Comparing the two approaches, the zero-beta-portfolio approach from DHKW is less prone to both model misspecification and errors-in-variables than the Fam-MacBeth approach. The good news is that the literature is indeed making progress on the estimation side. The bad news, however, is that I may still not be able to interpret the estimated zero-beta rate as the true unobserved risk-free rate.

4. A New Perspective on Evaluating Factor Models

In section 3, I proposed two new tests for factor models based on the idea that a correctly specified factor model should feature a unique zero-beta rate. Unfortunately, all the 8 prominent factor models studied in this paper fail these tests in the sense that they all imply an indeterminate zero-beta rate. This constitutes a new perspective on evaluating factor models. To understand how this new perspective connects to the model testing literature, section 4.1 reviews the conventional asset pricing tests, and 4.2 discusses the relationship between the existing tests and our new perspective of evaluating factor models. In section 4.3, I show that the new perspective also applies to a more recent advanced factor model technique: characteristic-dense factor models[12]. Taking Kelly et al. (2019) as an example, I evaluate the instrumented PCA (IPCA) factor models from the new perspective and find that they are not consistent with the uniqueness of the zero-beta rate, feither.

4.1. Conventional Factor Model Testing

I test the 8 prominent factor models using the conventional two-stage Fama-MacBeth process, which probably remains to the be the most popular testing method in the literature. It estimates the risk loadings (betas) from the time series regressions, followed by the estimation of the factor risk premia via cross-sectional regressions. I proceed under two specifications for the Fama-MacBeth process. The first specification uses the 1-month Treasury yield as the risk-free/zero-beta rate and tests factor models with excess returns. That is, I

[12] This term comes from the machine learning literature, which refers to the traditional Fama-French style factor models as characteristic-sparse models since they pre-specify factors based on a small number of firm characteristics. On the contrary, models that extract pricing information from a large number of firm characteristics are referred to as characteristic-dense models.

Table 7: Model Testing Statistics in Fama-MacBeth

Statistics	**Papers**	**Descriptions**
	Panel A: Model Specification Tests	
R^2		Cross-sectional R^2
$se(R^2)$	Kan et al. (2013)	Standard Error of R^2
$p(R^2 = 1)$	Kan et al. (2013)	p-value, $H_0 : R^2 = 1$
$p(R^2 = 0)$	Kan et al. (2013)	p-value, $H_0 : R^2 = 0$
$p(\lambda_K = 0_K)$		p-value, H_0 : all factor premia are zero
	Panel B: Factor Risk Premium Tests	
$t\text{-}stat_{fmb}$	Fama and MacBeth (1973)	t-stat in Fama-MacBeth
$t\text{-}stat_{s}$	Shanken (1992)	t-stat adj. for EIV under homoskedasticity
$t\text{-}stat_{jw}$	Jagannathan and Wang (1998)	t-stat adj. for EIV under HAC
$t\text{-}stat_{rks}$	Kan et al. (2013)	t-stat adj. for EIV, HAC, and misspecification

Notes: This table lists the testing statistics I will report in the Fama-MacBeth process. Panel A and Panel B list the model specification test statistics and the factor risk premium test statistics, respectively, corresponding to the two testing objectives.

run regression (7) with $r_{z,t}$ restricted to be equal to the 1-month Treasury yield in the first step. In the second step, I run a single cross-sectional regression where the left-hand side is a vector of sample mean excess returns and the right-hand side includes the beta estimates without a constant term[13]. The second specification assumes that the zero-beta rate is unavailable (risk-free assets are not traded) and tests factor models with gross returns. That is, I run regression (1) using gross returns in the first step. In the second step, I run a single cross-sectional regression where the left-hand side is a vector of sample mean gross returns and the right-hand side includes the beta estimates with a constant term. A constant term is included here to capture the unobserved zero-beta rate.

The Fama-MacBeth process achieves two testing objectives. First, it can test for model specification, investigating whether a factor model explains the cross-sectional differences in expected asset (excess) returns in terms of asset exposures (betas) to model specified systematic risk factors. Second, it can test whether a risk factor is priced in the cross-section of the asset (excess) returns, and compute the factor risk premium. I list the testing statistics I will consider in the Fama-MacBeth process in Table 7. The choice of these criteria

[13]It is a standard practice in the literature to exclude a constant term in the cross-sectional regression since I have excess returns on the left-hand side. There are also papers including a constant term.

mainly follows Kan et al. (2013) (KRS, hereafter). Panel A lists the model specification test statistics. R^2 is the most common measure of goodness of fit, indicating the extent to which the factor model accounts for the cross-sectional variations in mean (excess) returns. The next three statistics come from KRS, which recognizes the sampling uncertainty of the R^2 and derives its asymptotic distribution. $se(R^2)$ is the asymptotic sampling standard error of R^2. $p(R^2 = 1)$ is the p-value of $\mathcal{H}_0 : R^2 = 1$, testing whether I have a correctly specified factor model ($R^2 = 1$). At the other extreme, $p(R^2 = 0)$ is the p-value of: $\mathcal{H}_0 : R^2 = 0$, testing whether the model has any explanatory power for (excess) returns. The last one in Panel A, $p(\lambda_K = 0_K)$, is the p-value of a standard Wald test: $\mathcal{H}_0 : \lambda_K = 0_K$, checking whether all the factor risk premia are zero. Panel B lists the factor risk premium test statistics. $t - stat_{fmb}$ is the standard t-statistics of the factor risk premium presuming a correctly specified model (Fama and MacBeth, 1973). $t - stat_s$ is the t-statistics adjusted for errors-in-variables under homoskedasticity (Shanken, 1992). $t - stat_{jw}$ is the t-statistics adjusted for errors-in-variables allowing for conditional heteroskedasticity and auto-correlated errors (Jagannathan and Wang, 1998). $t - stat_{rks}$ is the t-statistics adjusted for errors-in-variables with HAC errors and model specification (KRS).

Table 8 reports the model specification testing results. For all the 8 prominent factor models, I use the same set of test assets[14] since I would like to run a horse race to compare all the models. Panel A and C assume that the zero-beta rate is equal to the 1-month Treasury yield using OLS and GLS, respectively, for the cross-sectional regression. Note the 135 test assets mainly include long-short portfolios constructed using multiple firm characteristics: size, value, operating profitability, investment, momentum, and market beta. It is well expected that the "FF6+BAB" and "Q5" models would perform better since they are designed to capture more anomalies. Indeed, only "FF6+BAB" or "Q5" have a cross-sectional R^2 above 73% with OLS and 7% with GLS, while other smaller models have lower or even negative R^2's[15] shown in the first row of each panel. In the fifth row, the Wald tests show that I can reject that all factor risk premia are zero for the 8 models.

Most papers stop right here claiming good performances for "FF6+BAB" and "Q5" after observing decent R^2's. However, a main contribution in KRS is the sampling uncertainty of the cross-sectional R^2. The second row of each panel in Table 8 reports the sampling asymptotic standard errors of R^2 and the third and fourth rows test whether $R^2 = 1$ and $R^2 = 0$, respectively. Our results show that all models should reject $R^2 = 1$[16], indicating model misspecification. All models should not reject $R^2 = 0$ with OLS estimation and only

[14]See Appendix A.1 for the identities of the 135 test assets.

[15]Some models such as CAPM have negative cross-sectional R^2's. This is because I do not include a constant term for the second-stage cross-sectional regression in the Fama-MacBeth process.

[16]I omit the significance levels when I discuss the hypothesis tests for conciseness.

Table 8: Model Specification Testing Results

Panel A: OLS, Zero-Beta Rate = 1M TBill Yield								
Model	CAPM	D-CCAPM	FF3	FF6+BAB	Q5	LIQ	DUR	INM
R^2	-0.428	-0.048	-0.105	0.732	0.738	-0.165	-0.306	0.264
$se(R^2)$	0.245	0.333	0.253	0.083	0.083	0.271	0.279	0.208
$p(R^2 = 1)$	0.000	0.021	0.000	0.000	0.038	0.000	0.000	0.000
$p(R^2 = 0)$	1.000	1.000	1.000	0.426	0.451	1.000	1.000	0.746
$p(\lambda_K = 0_K)$	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.000
Panel B: OLS, Zero-Beta Rate Unavailable								
Model	CAPM	D-CCAPM	FF3	FF6+BAB	Q5	LIQ	DUR	INM
R^2	0.004	0.177	0.294	0.749	0.741	0.243	0.284	0.400
$se(R^2)$	0.030	0.169	0.171	0.079	0.082	0.160	0.169	0.146
$p(R^2 = 1)$	0.000	0.002	0.000	0.000	0.025	0.000	0.000	0.000
$p(R^2 = 0)$	0.806	0.419	0.094	0.005	0.008	0.289	0.104	0.087
$p(\lambda_K = 0_K)$	0.806	0.207	0.054	0.000	0.000	0.205	0.052	0.001
Panel C: GLS, Zero-Beta Rate = 1M TBill Yield								
Model	CAPM	D-CCAPM	FF3	FF6+BAB	Q5	LIQ	DUR	INM
R^2	-0.070	-0.065	-0.046	0.070	0.124	-0.042	-0.065	-0.010
$se(R^2)$	0.025	0.027	0.027	0.033	0.043	0.026	0.026	0.030
$p(R^2 = 1)$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
$p(R^2 = 0)$	1.000	1.000	1.000	0.000	0.000	1.000	1.000	1.000
$p(\lambda_K = 0_K)$	0.000	0.004	0.000	0.000	0.000	0.000	0.003	0.000
Panel D: GLS, Zero-Beta Rate Unavailable								
Model	CAPM	D-CCAPM	FF3	FF6+BAB	Q5	LIQ	DUR	INM
R^2	0.000	0.013	0.020	0.119	0.151	0.015	0.008	0.037
$se(R^2)$	0.000	0.013	0.010	0.027	0.036	0.009	0.006	0.018
$p(R^2 = 1)$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
$p(R^2 = 0)$	0.947	0.378	0.028	0.000	0.000	0.208	0.334	0.006
$p(\lambda_K = 0_K)$	0.947	0.349	0.019	0.000	0.000	0.112	0.288	0.005
#Obs.	726	726	726	726	684	660	726	648

Notes: This table reports the model specification testing results. R^2 is the cross-sectional R^2. $se(R^2)$ is the asymptotic sampling standard error of R^2. $p(R^2 = 1)$ is the p-value of: $\mathcal{H}_0 : R^2 = 1$. At the other extreme, $p(R^2 = 0)$ is the p-value of: $\mathcal{H}_0 : R^2 = 0$. $p(\lambda_K = 0_K)$ is the p-value of a standard Wald test: $\mathcal{H}_0 : \lambda_K = 0_K$. Panel A and C assume that the zero-beta rate is equal to the 1-month Treasury yield, while Panel B and D assume that the zero-beta rate is unavailable. Panel A and B use OLS, while Panel C and D use GLS for the cross-sectional regression.

Table 9: Factor Risk Premium Testing Results ("FF6+BAB")

Panel A: OLS, Zero-Beta Rate = 1M TBill Yield

Factors	ZBR	MRP	SMB	HML	RMW	CMA	UMD	BAB
Estimate		0.591	0.221	0.271	0.220	0.210	0.671	0.412
$t\text{-}stat_{fm}$		3.520	1.930	2.363	2.425	2.497	4.215	2.334
$t\text{-}stat_{s}$		3.517	1.927	2.355	2.402	2.476	4.206	2.270
$t\text{-}stat_{jw}$		3.515	1.907	2.332	2.358	2.479	4.121	2.244
$t\text{-}stat_{krs}$		3.513	1.904	2.355	2.295	2.394	4.131	2.110

Panel B: OLS, Zero-Beta Rate Unavailable

Factors	ZBR	MRP	SMB	HML	RMW	CMA	UMD	BAB
Estimate	0.731	0.236	0.215	0.270	0.231	0.199	0.645	0.077
$t\text{-}stat_{fm}$	4.101	0.951	1.883	2.350	2.556	2.387	4.050	0.367
$t\text{-}stat_{s}$	3.933	0.928	1.881	2.344	2.537	2.372	4.043	0.357
$t\text{-}stat_{jw}$	3.884	0.926	1.866	2.336	2.506	2.381	3.992	0.367
$t\text{-}stat_{krs}$	3.578	0.890	1.864	2.359	2.430	2.280	3.998	0.306
TS Mean of the CSR Estimates	0.731	0.236	0.215	0.270	0.231	0.199	0.645	0.077
TS Mean of Data	0.363	0.568	0.215	0.292	0.283	0.273	0.600	0.770
TS Std.Dev of the CSR Estimates	4.808	6.686	3.081	3.097	2.440	2.249	4.292	5.629
TS Std.Dev of Data	0.266	4.497	3.033	2.995	2.225	2.077	4.213	3.262

Panel C: GLS, Zero-Beta Rate = 1M TBill Yield

Factors	ZBR	MRP	SMB	HML	RMW	CMA	UMD	BAB
Estimate		0.582	0.209	0.294	0.234	0.273	0.647	0.354
$t\text{-}stat_{fm}$		3.485	1.854	2.627	2.789	3.478	4.115	2.424
$t\text{-}stat_{s}$		3.485	1.853	2.625	2.784	3.471	4.112	2.380
$t\text{-}stat_{jw}$		3.486	1.851	2.622	2.772	3.464	4.108	2.388
$t\text{-}stat_{krs}$		3.487	1.847	2.600	2.778	3.459	4.072	2.105

Panel D: GLS, Zero-Beta Rate Unavailable

Factors	ZBR	MRP	SMB	HML	RMW	CMA	UMD	BAB
Estimate	0.915	0.027	0.207	0.289	0.234	0.267	0.641	-0.008
$t\text{-}stat_{fm}$	7.933	0.133	1.833	2.578	2.797	3.403	4.076	-0.050
$t\text{-}stat_{s}$	7.582	0.131	1.833	2.576	2.793	3.397	4.074	-0.049
$t\text{-}stat_{jw}$	7.670	0.132	1.832	2.572	2.788	3.398	4.073	-0.051
$t\text{-}stat_{krs}$	5.882	0.114	1.827	2.556	2.790	3.395	4.041	-0.041
TS Mean of the CSR Estimates	0.915	0.027	0.207	0.289	0.234	0.267	0.641	-0.008
TS Mean of Data	0.363	0.568	0.215	0.292	0.283	0.273	0.600	0.770
TS Std.Dev of the CSR Estimates	3.110	5.466	3.040	3.018	2.260	2.116	4.239	4.406
TS Std.Dev of Data	0.266	4.497	3.033	2.995	2.225	2.077	4.213	3.262

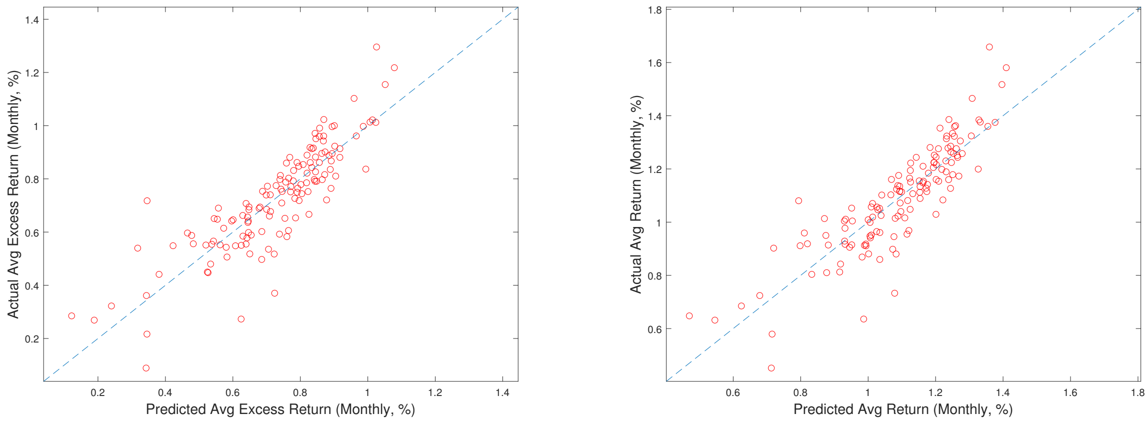
"FF6+BAB" and "Q5" reject $R^2 = 0$ with GLS estimation. To sum up Panel A and C, "FF6+BAB" and "Q5" stand out among the 8 models for our 135 test assets. However, testing for whether $R^2 = 1$ implies that they are also misspecified.

Let us move to Panel B and D where I assume the zero-beta rate is unavailable. The testing results are remarkably similar to what Panel A and C imply. "FF6+BAB" and "Q5" stand out among the 8 models according to the cross-sectional R^2. Looking at the Wald test in the fifth row of each panel, only "FF6+BAB", "Q5", and "INM" can reject that all factor risk premia are zero. For the R^2-based tests from KRS, only "FF6+BAB", "Q5", and "INM" reject $R^2 = 0$ and all models reject $R^2 = 1$. Hence, all models are misspecified in the sense that the cross-sectional pricing errors are non-zero. Comparing Panel A and C with Panel B and D, I find that our assumption about the zero-beta rate does not make a significant difference in the Fama-MacBeth testing. Whether I restrict the zero-beta rate to be equal to the Treasury yield or assume the zero-beta rate to be unavailable does not change our conclusions on model performance and comparison.

Since the "FF6+BAB" model performs well according to Table 8, I take it as an example to report the factor risk premium test results in Table 9. Panel A and C assume that the zero-beta rate is equal to the 1-month Treasury yield using OLS and GLS, respectively, for the cross-sectional regression. I find that all factors except "SMB" carry significant risk premia according to all 4 t-statistics. Generally speaking, $t\text{-}stat_{fmb}$ is the largest, and $t\text{-}stat_{krs}$ is the smallest among the 4 since the latter accounts for EIV, HAC errors and model misspecification. $t\text{-}stat_{s}$ and $t\text{-}stat_{jw}$ are generally in between as they account for EIV with either homoskedasticity or HAC errors. Let us move to Panel B and D where I assume the zero-beta rate is unavailable. Since there is an extra constant term in the cross-sectional regression, I also report the constant estimate and its t-statistics under column "ZBR". First, the constant term is an estimate of the average zero-beta rate and it is significantly different from zero according to the 4 t-statistics. Second, I observe that the market factor and the BAB factor lose their pricing power in this case. In the last 4 rows of Panel B and D, I run cross-sectional regressions period-by-period and report the time series mean and standard deviation of regression coefficients[17]. I also report the time series mean and standard deviation of the realized factor returns for comparison ("ZBR" is compared with the 1-month Treasury). First, the estimated zero-beta rate is much higher and more volatile than the Treasury yield, as discussed in section 2. Second, the estimated market factor risk premium and the BAB factor risk premium are much smaller than their counterparties in the data. This evidence suggests that relaxing the restriction on the zero-beta rate may violate the estimation of the risk premia of some factors such as MRP and BAB.

[17]This is the Fama-MacBeth approach to estimating the zero-beta rate described in 2.1

Figure. 10. Model-Predicted vs Actual Returns ("FF6+BAB")



(a) Zero-Beta Rate = 1M TBill Yield

(b) Zero-Beta Rate Unavailable

Notes: This figure plots the model predicted versus actual returns in Figure 10. Panel (a) shows the pairs of model-predicted expected excess returns and actual expected excess returns in red circles, assuming the zero-beta rate is equal to the Treasury yield. Panel (b) shows the pairs of model-predicted expected gross returns and actual expected gross returns in red circles, assuming the zero-beta rate is unobservable. The blue dashed lines are the 45-degree lines.

4.2. Conventional Testing vs Zero-Beta-Rate Testing

In the factor model testing literature, the cross-sectional R^2 remains to be the most common measure of goodness of fit. Although relaxing the restriction on the zero-beta rate may violate some risk premia estimates in the "FF6+BAB" model (see Table 9), Table 8 indicates that it generates good R^2 in pricing the cross-section of stock returns. Hence, from the conventional view, I tend to conclude that the "FF6+BAB" model works well. To reinforce this conclusion, I plot the model predicted versus actual returns in Figure 10. Panel (a) shows the pairs of model-predicted expected excess returns and actual expected excess returns in red circles, assuming the zero-beta rate is equal to the Treasury yield. Panel (b) shows the pairs of model-predicted expected gross returns and actual expected gross returns in red circles, assuming the zero-beta rate is unobservable. The blue dashed lines are the 45-degree lines. No matter what I assume about the zero-beta rate, the "FF6+BAB" model produces expected (excess) returns that are almost perfectly in line with the data. In fact, many papers present scatter plots similar to Figure 10 and declare good performances of their models.

I think this literature protocol may be a bit hasty. Having a high cross-sectional R^2 (a good-looking Figure 10) does not mean that the model is correctly specified. Unfortunately, the literature typically takes a high-R^2 model as the correct model and proceeds to conduct inferences about the risk premia, perform out-of-sample analyses, discuss the implications of risk pricing, etc. KRS is one of the first papers to propose additional model specification tests. It recognizes the sampling uncertainty of R^2 and advocates tests of whether $R^2 = 1$ or $R^2 = 0$. If a model rejects $R^2 = 1$, it is misspecified according to KRS. Consequently, KRS proposes to test the factor risk premia using t-statistics that adjust for errors-in-variables (EIV), heterogeneous errors with autocorrelation (HAC), and model misspecification.

Note that model misspecification in KRS is an empirical concept, meaning the non-zero aggregate pricing errors (sum of squares of residuals) in the cross-sectional regression. In other words, there exists model misspecification as long as the cross-sectional R^2 is statistically different from 1. Our view of model misspecification comes from the theory side, emphasizing the theoretical uniqueness of the zero-beta rate. In section 3.2, I proposed two model specification tests investigating whether the factor model is consistent with the underlying theory that permits a unique zero-beta rate. Through the lens of our zero-beta rate test, all 8 prominent factor models are rejected, meaning that they are inherently misspecified. Although it may sound astonishing to claim that these popular models are misspecified, I are not trying to deny their contributions to the empirical asset pricing literature. In fact, most of the factor models indeed do a good job of pricing the cross-section of stock returns and identifying the sources of systematic risks. Moreover, they inspire and guide the development of the finance industry especially in asset management businesses. Instead, I argue for a comprehensive perspective on evaluating factor models.

The conventional perspective of model testing focuses on whether a particular risk factor candidate earns a risk premium. Hence, conventional tests are able to tell us whether a particular factor represents a source of systematic risks. Nevertheless, when I evaluate the factor models from a different angle based on the uniqueness of the zero-beta rate, these factor models are shown to be inconsistent with the underlying factor model theories. This is a new perspective of evaluating factor models, focusing on whether the proposed factor model completely captures all the systematic risks. Therefore, the new perspective is much stricter than the conventional perspective. Whether this new perspective is relevant depends on different purposes. Typically, I aim to understand whether one particular factor is priced in the cross-section without needing to identify all risk factors. In this case, conventional tests suffice to provide an answer. In the zero-beta rate literature, however, constructing the zero-beta rate essentially requires us to rule out all the risk sources, where our new perspective is an important consideration before selecting factor models. Ignoring this new perspective may lead us to falsely accept wrong factor models, and thus producing flawed estimates. In addition, this new perspective of evaluating factor models may also explain the poor out-of-sample performances of existing models. If the selected factor model is essentially incorrectly specified, taking to model to the out-of-sample is highly likely to be in vain. In summary, I advocate researchers and practitioners to take a critical and rigorous view on empirical factor models and I argue for the importance of the new perspective of evaluating factor models based on the uniqueness of the zero-beta rate.

4.3. Instrumented PCA (IPCA) Factor Models

The 8 models studied in this paper are all characteristic-sparse where they pre-specify risk factors based on a small number of stock characteristics (e.g., size, book-to-market ratio, momentum, etc) and assume constant risk loadings in the Fama-French style. Recently, an emerging literature proposes characteristic-dense factor models (Fan et al., 2016; Kelly et al., 2019; Lettau and Pelger, 2020; Kozak et al., 2020; Chen et al., 2023; etc.) where the time-varying risk loadings are modeled as functions of a large number of stock characteristics. Their PCA-based methods allow us to estimate latent risk factors that are relevant in pricing the stock returns without manually constructing the long-short portfolios. In this section, I apply our new model testing perspective to this literature using the instrumented PCA (IPCA) factor models from Kelly et al. (2019) (KPS, hereafter) as examples. Specifically, I ask whether the IPCA models feature a unique zero-beta rate.

The IPCA model specification for the excess return vector is:

$$\begin{aligned} R_{t+1} - r_{z,t} &= \alpha_t + \beta_t F_{t+1} + \varepsilon_{i,t+1} \\ \alpha_t = Z_t \Gamma_\alpha, &\quad \beta_t = Z_t \Gamma_\beta \end{aligned} \tag{11}$$

where $R_{i,t}$ is an N-dimension vector gross returns, F_{t+1} is a K-dimension vector of latent factors, $\alpha_{t(N\times 1)}$ and $\beta_{t(N\times K)}$ are assumed to be time-varying and are linear functions of L firm characteristics, Z_t (an $N \times L$ matrix). The original KPS paper uses excess returns for estimation, and thus $r_{z,t}$ is restricted to be equal to the 1-month Treasury yield. Now, I adapt this system to our interested case where the zero-beta rate is unobservable:

$$\begin{aligned} R_{t+1} &= r_{z,t} + \beta_t F_{t+1} + \varepsilon_{i,t+1} = \delta_t Y_t + \beta_t F_{t+1} + \varepsilon_{i,t+1} \\ \delta_t &= Z_t \Gamma_\delta, \quad \beta_t = Z_t \Gamma_\beta \end{aligned} \tag{12}$$

where β_t is assumed to be time-varying and is a linear function of firm characteristics, Z_t, as in KPS. I also assume the zero-beta rate $r_{z,t} = \delta_t Y_t$ is a linear function of M macroeconomic predictors, Y_t (an M-dimension vector), following Di Tella et al. (2023). In the spirit of KPS, I assume $\delta_{t(N\times M)}$ is a linear function of firm characteristics, Z_t. Since both macro predictors and firm characteristics include constant terms, α_t will be captured in $\delta_t Y_t$. The zero-beta rate can be decomposed as:

$$
\begin{aligned}
r_{z,t} = \delta_t Y_t = Z_t \Gamma_\delta Y_t = \begin{bmatrix} \tilde{Z}_t & \mathbb{1}_N \end{bmatrix} \begin{bmatrix} \tilde{\Gamma}_\delta \\ \Gamma_0 \end{bmatrix} Y_t \\
= \underbrace{\tilde{Z}_t \tilde{\Gamma}_\delta Y_t}_{\text{asset-specific component}} + \underbrace{\mathbb{1}_N \Gamma_0 Y_t}_{\text{common component}}
\end{aligned} \tag{13}
$$

This system is general in the sense that I allow for an asset-specific component in the zero-beta rate. Factor model theories suggest that $\tilde{\Gamma}_\delta$ should be zero, which constitutes our test for the uniqueness of the zero-beta rate. If $\tilde{\Gamma}_\delta = 0$, then $r_{z,t} = \Gamma_0 Y_t$ for all assets, implying a unique zero-beta rate. If $\tilde{\Gamma}_\delta \neq 0$, then there exists an asset-specific component of the zero-beta rate, inconsistent with the theories.

Rearrange equation (12) I have:

$$
\begin{aligned}
&R_{t+1} = Z_t \Gamma_\delta Y_t + Z_t \Gamma_\beta F_{t+1} + \varepsilon_{i,t+1} \equiv Z_t \Gamma_G G_{t+1} + \varepsilon_{i,t+1} \\
&\Gamma_G = \begin{bmatrix} \Gamma_\delta & \Gamma_\beta \end{bmatrix}, \quad G_{t+1} = \begin{bmatrix} Y_t \\ F_{t+1} \end{bmatrix}
\end{aligned} \tag{14}
$$

The estimation objective is to minimize the sum of squared pricing errors:

$$
\begin{aligned}
\min_{\{F_{t+1}, \Gamma_G\}} \sum_{t=1}^{T} \left(R_{t+1} - Z_t \Gamma_\delta Y_t - Z_t \Gamma_\beta F_{t+1}\right)' \left(R_{t+1} - Z_t \Gamma_\delta Y_t - Z_t \Gamma_\beta F_{t+1}\right) \\
= \sum_{t=1}^{T} \left(R_{t+1} - Z_t \Gamma_G G_{t+1}\right)' \left(R_{t+1} - Z_t \Gamma_G G_{t+1}\right)
\end{aligned} \tag{15}
$$

The first-order conditions for this minimization problem are:

$$
F_{t+1} = \left(\Gamma_\beta' Z_t' Z_t \Gamma_\beta\right)^{-1} \Gamma_\beta' Z_t' \left(R_{t+1} - Z_t \Gamma_\delta Y_t\right) \tag{16}
$$

$$
vec(\Gamma_G) = \left(\sum_{t=1}^{T} (Z_t' Z_t) \otimes (G_{t+1} G_{t+1}')\right)^{-1} \left(\sum_{t=1}^{T} \left(Z_t' \otimes G_{t+1}\right) R_{t+1}\right) \tag{17}
$$

Following KPS, I use an alternating least squares (ALS) algorithm to solve for F_{t+1} and Γ_G from equation (16) and (17). Three identification assumptions are needed: (i) $\Gamma_\beta' \Gamma_\beta = \mathbb{1}_K$, (ii) $\Gamma_\delta' \Gamma_\beta = 0_{(M \times K)}$, and (iii) factors F_{t+1} are orthogonal to each other and their means are non-negative. These assumptions help to uniquely identify solutions to the first-order conditions without placing economic restrictions. I also follow the original paper to report two model performance measures:

$$\text{Total } R^2 = 1 - \frac{\sum_{t=1}^{T} \left(R_{t+1} - Z_t\Gamma_\delta Y_t - Z_t\Gamma_\beta F_{t+1}\right)' \left(R_{t+1} - Z_t\Gamma_\delta Y_t - Z_t\Gamma_\beta F_{t+1}\right)}{\sum_{t=1}^{T} R'_{t+1}R_{t+1}} \tag{18}$$

$$\text{Predictive } R^2 = 1 - \frac{\sum_{t=1}^{T} \left(R_{t+1} - Z_t\Gamma_\delta Y_t - Z_t\Gamma_\beta \Lambda_{t+1}\right)' \left(R_{t+1} - Z_t\Gamma_\delta Y_t - Z_t\Gamma_\beta \Lambda_{t+1}\right)}{\sum_{t=1}^{T} R'_{t+1}R_{t+1}} \tag{19}$$

The total R^2 represents the fraction of individual gross return variations that can be explained by the model. The predictive R^2 differs from the total R^2 only by replacing F_{t+1} by Λ_{t+1}, which is the mean of F_{t+1}. Hence, the predictive R^2 represents the faction of individual gross return variations that can be explained by the model's description of conditional expected returns. That is, it is measuring the model performance when I use the expected value of factors to predict the gross returns. Recall from KPS that although IPCA models have the benefit of operating on individual stocks, they can be interpreted as pricing characteristics-managed portfolios in consistency with the traditional Fama-French style models. The model performance can also be computed with respect to the L characteristics-managed portfolios: $X_{t+1} = Z'_t R_{t+1}$. Denote $W_t = Z'_t Z_t$, then for X_{t+1}:

$$\text{Total } R^2 = 1 - \frac{\sum_{t=1}^{T} \left(X_{t+1} - W_t\Gamma_\delta Y_t - W_t\Gamma_\beta F_{t+1}\right)' \left(X_{t+1} - W_t\Gamma_\delta Y_t - W_t\Gamma_\beta F_{t+1}\right)}{\sum_{t=1}^{T} X'_{t+1}X_{t+1}} \tag{20}$$

$$\text{Predictive } R^2 = 1 - \frac{\sum_{t=1}^{T} \left(X_{t+1} - W_t\Gamma_\delta Y_t - W_t\Gamma_\beta \Lambda_{t+1}\right)' \left(X_{t+1} - W_t\Gamma_\delta Y_t - W_t\Gamma_\beta \Lambda_{t+1}\right)}{\sum_{t=1}^{T} X'_{t+1}X_{t+1}} \tag{21}$$

To be comparable with the original KPS paper, I use the same data sample of stock returns and firm characteristics[18]. I use the same 6 macroeconomic predictors, Y_t, as in 2.2[19]. Note the key difference between our exercises with KPS. I use IPCA models to

[18] The data sample ranges from July 1962 to May 2014, including 1,403,544 stock-month observations for 12,813 unique stocks and 36 firm characteristics for each stock. See Kelly et al. (2019) for details.

[19] The 6 macroeconomic predictors are: the 1-month Treasury bill yield, the rolling average of the previous twelve-month inflation, the term spread, the unemployment rate, the CAPE, the corporate bond spread, and a constant term.

Table 10: IPCA Model Performance and Testing

		6	7	8	9	10	11	12
Panel A: KPS Data Sample								
Total R^2	Individual Stocks	19.21	19.42	19.57	19.69	19.80	19.90	20.00
	Managed Portfolios	98.93	99.20	99.35	99.43	99.49	99.53	99.59
Predictive R^2	Individual Stocks	1.17	1.16	1.13	1.13	1.10	1.07	1.07
	Managed Portfolios	4.76	4.71	4.48	4.51	4.33	4.20	4.42
W_δ p-value		6.95	7.72	6.56	5.79	5.41	5.79	6.95
Panel B: Simulated Data Sample								
		6	7	8	9	10	11	12
Total R^2	Individual Stocks	99.73	98.78	99.51	99.98	99.98	99.98	99.98
	Managed Portfolios	99.55	99.80	99.93	1.00	1.00	1.00	1.00
Predictive R^2	Individual Stocks	6.23	5.94	5.83	5.85	5.85	5.85	5.85
	Managed Portfolios	5.35	5.24	5.16	5.44	5.44	5.44	5.44
W_δ p-value		10.42	8.88	5.41	8.49	8.48	9.27	10.04

Notes: This table reports the total and predictive R^2 in percentages for IPCA models with 6 to 12 factors both at the individual stock level and at the characteristics-managed portfolios. It also reports the bootstrapping p-values of $W_\delta = 0$. Panel A uses the same data sample of stock returns and firm characteristics as Kelly et al. (2019). Panel B uses a simulated data sample. I only run IPCA models with 6 to 12 factors in the simulated data because I have 6 macroeconomic predictors that may also contain pricing information about the stock returns.

explain the variations of gross returns assuming a functional form for the zero-beta rate, while KPS runs IPCA on excess returns where the zero-beta rate is assumed to be the 1-month Treasury yield. Panel A of Table 10 reports the total and predictive R^2 in percentages for IPCA models with 6 to 12 factors at the level of both individual stocks and characteristics-managed portfolios. I only run IPCA models with 6 to 12 factors in the simulated data because I have 6 macroeconomic predictors that may also contain pricing information about the stock returns. KPS finds that the model performance does not significantly increase with more than 5 factors, which is confirmed in our results. The IPCA models with 7 or more factors can explain more than 19.0% and 98% of the total variations in gross returns at the level of individual stocks and characteristics-managed portfolios, respectively. The predictive R^2's are above 1% and 4%.

Most importantly, I would like to test whether the zero-beta rate is unique in the IPCA framework. Previously, I mentioned that testing for a unique zero-beta rate is equivalent to testing $\mathcal{H}_0 : \tilde{\Gamma}_\delta = 0$. Let us push it further and test $\mathcal{H}_0 : \Gamma_\delta = 0$. If $\Gamma_\delta = 0$, then I are in an extreme case where the whole zero-beta rate $r_{z,t} = \delta_t Y_t = Z_t \Gamma_\delta Y_t = 0$ for all assets. Following KPS, I construct a Wald-like test statistics for $\Gamma_{\delta(N\times M)}$: $W_\delta = vec(\Gamma_\delta)' vec(\Gamma_\delta)$, capturing the distance between Γ_δ and zero. Inferences are conducted via bootstrapping (see KPS for details). First, I compute the residuals of characteristics-managed portfolios, $\hat{d}_{t+1} = Z_t' \varepsilon_{t+1}$, from the estimated models. Next, I randomly resample with replacement from the realized residuals and reconstruct the gross returns imposing $\Gamma_\delta = 0$. Then, I estimate the IPCA models and compute the Wald-like statistics, $\hat{W}_\delta$. Finally, I repeat this process for 1000 bootstrapping samples. The p-value is calculated as the fraction of sample $\hat{W}_\delta$ that is larger than the true W_δ. The bootstrapping p-values are reported in percentages in the last row of Panel A in Table 10. Since they are all above 5%, I cannot reject the null: $\Gamma_\delta = 0$ at the 5% significance level. Surprisingly, the testing results show that the zero-beta rates in IPCA models are statistically zero with 6 or more factors. Our intuition is that the latent factors together with the dynamics of time-varying risk loadings are already sufficient to explain the variations in the cross-section of gross returns, completely subsuming the zero-beta rate.

In KPS, the zero-beta rate is restricted to be the 1-month Treasury yield and IPCA is operating on excess returns. I manually construct excess returns with random zero-beta rates. No matter what the zero-beta rate is, IPCA will eventually produce some latent factors and time-varying betas that nicely price the cross-section of the constructed excess returns. Above all, I conclude that the IPCA models are inconsistent with the factor model theories in the sense that they are not able to feature a unique zero-beta rate.

To further validate this argument, I run IPCA models on a simulated sample of gross returns. First, I create an artificial zero-beta rate which is assumed to be a linear function of the 6 macroeconomic predictors: $r_{z,t} = a' Y_t$, where $a = [0.007, 1, -0.1, 1, -0.005, 0.0002, 0.001]$. Next, I simulate the individual gross stock returns by $R_{t+1} = r_{z,t} + \beta_t F_{t+1} = r_{z,t} + Z_t \Gamma_\beta F_{t+1}$. To make the simulated sample as close to the real data as possible, I use the estimated 9 latent factors (F_{t+1}) and their estimated risk loadings (Γ_β) from running IPCA on the KPS data sample. I also use the real-life firm characteristics (Z_t) in simulation. The idiosyncratic noise term ε_{t+1} is omitted for simplicity. Panel B of Table 10 reports the same R^2's for IPCA models at different aggregation levels in the simulated sample. As is expected, IPCA models perform very well at the level of both individual stocks and managed portfolios. With 9 or more factors, the model performances do not change because the simulated data only features 9 latent factors. Again, I test the null hypothesis $\mathcal{H}_0 : W_\delta = 0$ and find that all the p-values are above 5%, failing to reject the null at 5% significance level. Even if the simulated data is well structured and the IPCA models perform well according to the total and predictive R^2, the IPCA approach cannot recover the unobserved zero-beta rate.

The IPCA approach does its best to explain the variations in the cross-section of gross returns, leaving no space for the zero-beta rate if I leave the zero-beta rate unrestricted. That said, I are not denying the contributions of IPCA in empirical asset pricing models. If I are willing to assume a unique zero-beta rate (e.g., the 1-month Treasury yield), I are able to improve the cross-sectional pricing of excess returns with IPCA models. However, our results show that (i) IPCA factor models are inherently inconsistent with factor model theories regarding the zero-beta rate, and (ii) the IPCA approach cannot help to estimate the zero-beta rate.

5. Conclusion

In this paper, I revisit the estimation of the zero-beta rate and find that existing methods (the Fama-MacBeth approach and the zero-beta-portfolios approach from Di Tella et al. (2023)) tend to produce high and volatile zero-beta rates. I argue that the high level and volatility of the estimated zero-beta rate arise from two channels: model misspecification and errors-in-variables. First, I show in theory that model misspecification leads to multiplicity of the zero-beta rate. This indeterminacy may be a major factor in increasing the mean and volatility of the estimated zero-beta rate. In the simulation analysis, I confirm that both channels are quantitatively nontrivial. The literature has been making progress because the newer zero-beta-portfolio approach is less prone to model misspecification and errors-in-variables than the traditional Fama-MacBeth approach. Our results, however, call for caution when I estimate and interpret the zero-beta rate using either approach. For example, it may not be appropriate to interpret the estimated zero-beta rate as the true risk-free rate if I do not systematically rule out the two channels above. In the meantime, the theoretical link between model misspecification and multiplicity of the zero-beta rate provides a new perspective on evaluating factor models. I develop statistical tests that examine whether a factor model implies a unique zero-beta rate. Based on this new perspective, prominent factor models (either Fama-French style models or PCA-based models) may be inherently misspecified. Conventional tests such as the Fama-MacBeth process are valid if I want to understand whether a particular risk factor is priced in the cross-section of asset returns. However, in other exercises that require factor models to be correctly specified such as estimating the zero-beta rate, I argue that it may be important to consider the new perspective regarding the uniqueness of the zero-beta rate.

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Appendix A. Additional Analysis

A.1. 8 Factor Models

The fundamental equation for risk premium—$E[R_i] - r_z = -r_z Cov(m, R_i)$—tells us that the key object in studying risk premium is the covariance term $Cov(m, R_i)$. Note that $r_z \equiv 1/E[m]$ is the zero-beta rate. However, the covariance term is generally impossible to compute because of the extremely high dimension of the calculation. This is where factor models kick in because they essentially reduce the dimension and make the problem tractable. A general factor model has the following beta pricing formula:

$$E[R] - r_z = \underbrace{-r_z Cov(R, m) = \beta \cdot \lambda = Cov(R, F)\, \Sigma_F^{-1} \cdot \lambda}_{\text{dimension reduction}} \tag{A.1}$$

where $R \in \mathcal{R}^N$ is a vector of returns for all N risky assets, β is a $N \times K$ matrix capturing the exposure of N assets to a set of K risk factors, λ is a $K \times 1$ matrix capturing the prices of K risk factors or factor risk premiums. By the projection theory, $\beta = Cov(R, F)\, \Sigma_F^{-1}$, where $F = \{f_1, f_2, ..., f_K\}$ are the factors and Σ_F denotes the covariance matrix for the factors.

This section reviews the 8 prominent factor models studied in this paper. Specifically, I will briefly describe the model function, key intuition, data sources, sample period, etc. for each factor model. I use the same set of testing portfolios in this paper since I would like to compare the model performances with the same set of anomalies. Specifically, I use 135 testing portfolios consisting of 25 portfolios sorted by size and book-to-market sorted portfolios, 25 portfolios sorted by size and operating profitability, 25 portfolios sorted by size and investment, 25 portfolios sorted by size and momentum, 25 portfolios sorted by size and market beta, and 10 industry portfolios. These testing portfolios can be downloaded from Kenneth French's website.

A.1.1. CAPM

The capital asset pricing model (Sharpe, 1964; Lintner, 1965) remains to be one of the most popular tools in academia and the business industry due to its simplicity and beautiful intuition. CAPM is a single-factor model where the market risk is assumed to be the only source of systematic risk. The beta-pricing formula for CAPM is:

$$E[R_{t+1}] - r_{z,t} = \beta_{mkt} \lambda_{mkt,t+1} \tag{A.2}$$

where $r_{z,t}$ is the zero-beta rate and $\lambda_{mkt,t+1}$ is the market factor risk premium. The market returns come from Kenneth French's website.. The data sample ranges from July 1963 to

December 2023.

A.1.2. D-CCAPM

The consumption capital asset pricing model (CCAPM) is an extension of CAPM that emphasizes the consumption risk instead of the market risk to explain the risk premium. It replaces the market beta with the consumption beta in the model specification. Introduced since Breeden (1979), there have been multiple versions of CCAPM proposed in the literature, among which I choose the Yogo (2006) model. It highlights the cyclical role of durable consumption in addition to non-durable consumption in asset pricing, and thus I label it as D-CCAPM. The beta-pricing formula for D-CCAPM is:

$$E\left[R_{t+1}\right] - r_{z,t} = \beta_{mkt}\lambda_{mkt,t+1} + \beta_{cg-ndg}\lambda_{cg-ndg,t+1} + \beta_{cg-dg}\lambda_{cg-dg,t+1} \tag{A.3}$$

where $\lambda_{cg-ndg,t+1}$ the risk premium of the non-durable consumption risk factor (growth rate of real per capita non-durable consumption) and $\lambda_{cg-dg,t+1}$ is the risk premium of the durable consumption risk factor (growth rate of real per capita durable consumption). Both consumption series are seasonally adjusted at annual rates, downloaded from the Bureau of Economic Analysis (BEA). Following Vissing-Jørgensen and Attanasio (2003) and Kan et al. (2013), I linearly interpolate the quarterly consumption series into monthly series so that the model estimation is at the monthly frequency consistent with other factor models. The data sample ranges from July 1963 to December 2023.

A.1.3. FF3

The Fama-French three-factor model (Fama and French, 1993) extends the CAPM by adding two factors: size and book-to-market ratio. These two factors are motivated by empirically observing that (1) small stocks tend to outperform large stocks and (2) value stocks tend to outperform growth stocks. The beta-pricing formula for FF3 is:

$$E\left[R_{t+1}\right] - r_{z,t} = \beta_{mkt}\lambda_{mkt,t+1} + \beta_{smb}\lambda_{smb,t+1} + \beta_{hml}\lambda_{hml,t+1} \tag{A.4}$$

where $\lambda_{smb,t+1}$ is the risk premium of the size factor (the excess return of small-minus-big long-short portfolio) and $\lambda_{hml,t+1}$ is the risk premium of the value factor (the excess return of high-minus-low long-short portfolio). These factor returns can be downloaded from Kenneth French's website. The data sample ranges from July 1963 to December 2023.

A.1.4. FF6+BAB

Fama and French (2015) and Fama and French (2018) extend the classic Fama-French three-factor model to five-factor and six-factor models. The three additional factors are related to operating profitability, investment, and momentum (Efficiency, 1993). They are empirically motivated by the facts that (1) firms with robust profitability tend to outperform firms with weak profitability, (2) firms with conservative investment tend to outperform firms with aggressive investment, and (3) past winners tend to outperform past losers. I also include a betting-against-beta factor (BAB) proposed by Frazzini and Pedersen (2014) as high market beta stocks tend to earn lower risk-adjusted returns than low market beta stocks. The beta-pricing formula for this seven-factor model (FF6+BAB) is:

$$\begin{aligned} E\left[R_{t+1}\right] - r_{z,t} = \beta_{mkt}\lambda_{mkt,t+1} + \beta_{smb}\lambda_{smb,t+1} + \beta_{hml}\lambda_{hml,t+1} + \beta_{rmw}\lambda_{rmw,t+1} \\ + \beta_{cma}\lambda_{cma,t+1} + \beta_{umd}\lambda_{umd,t+1} + \beta_{bab}\lambda_{bab,t+1} \end{aligned} \tag{A.5}$$

where $\lambda_{rmw,t+1}$ is the risk premium of the profitability factor (the excess return of robust-minus-weak profitability long-short portfolio), $\lambda_{cma,t+1}$ is the risk premium of the investment factor (the excess return of conservative-minus-aggressive investment long-short portfolio), $\lambda_{umd,t+1}$ is the risk premium of the momentum factor (the excess return of up-minus-down long-short portfolio), and $\lambda_{bab,t+1}$ is the risk premium of the BAB factor (the excess return of low-beta-minus-high-beta long-short portfolio). These factor returns can be downloaded from Kenneth French's website. The data sample ranges from July 1963 to December 2023.

A.1.5. Q5

Production-based asset pricing models claim that productivity shocks are related to the changes in the firm investment opportunity set, and thus should be priced in stock returns. Hou et al. (2015) shows that an investment-to-asset factor and a return-on-equity factor are associated with productivity shocks. In Hou et al. (2021), the authors augment the previous q-factor model with an expected growth factor. The beta-pricing formula for the new five-factor, which I label as Q5, is:

$$E\left[R_{t+1}\right] - r_{z,t} = \beta_{mkt}\lambda_{mkt,t+1} + \beta_{me}\lambda_{me,t+1} + \beta_{ia}\lambda_{ia,t+1} + \beta_{roe}\lambda_{roe,t+1} + \beta_{eg}\lambda_{eg,t+1} \tag{A.6}$$

where $\lambda_{me,t+1}$ is the same size factor risk premium as $\lambda_{smb,t+1}$, $\lambda_{ia,t+1}$ is the risk premium of the investment factor (the excess return of firms with low versus high levels of new investments), $\lambda_{roe,t+1}$ is the risk premium of the profitability factor (the excess return of firms with high versus low ROE), and $\lambda_{eg,t+1}$ is the risk premium of the expected growth factor (the excess return of firms with high versus low expected growth). These factor returns can be downloaded from Lu Zhang's website. The data sample ranges from January 1967 to December 2023.

A.1.6. LIQ

Pástor and Stambaugh (2003) augments the Fama-French three-factor model with a liquidity factor to capture the idea that asset returns are closely related to market-wide liquidity. The aggregate market liquidity is measured as the cross-sectional average of individual stocks' liquidity measures. The beta-pricing formula for the four-factor model, which I label as LIQ, is:

$$E\left[R_{t+1}\right] - r_{z,t} = \beta_{mkt}\lambda_{mkt,t+1} + \beta_{smb}\lambda_{smb,t+1} + \beta_{hml}\lambda_{hml,t+1} + \beta_{liq}\lambda_{liq,t+1} \tag{A.7}$$

where $\lambda_{liq,t+1}$ is the risk premium of the liquidity factor (the excess return of a long-short portfolio that long stocks that are more sensitive to liquidity shocks and short stocks that are less sensitive to liquidity shocks). The liquidity factor returns can be downloaded from Robert Stambaugh's website. The data sample ranges from January 1968 to December 2023.

A.1.7. DUR

Gormsen and Lazarus (2023) shows that the premia on a number of equity risk factors (value, profitability, investment, etc.) can be explained by a single duration factor. The intuition is that these risk factors are essentially constructed by investing in stocks with lower duration and selling stocks with higher duration. Hence, the near-future premium can summarize all these factors. They proceed to propose a three-factor model which I label as DUR. The beta-pricing formula for DUR is:

$$E\left[R_{t+1}\right] - r_{z,t} = \beta_{mkt}\lambda_{mkt,t+1} + \beta_{smb}\lambda_{smb,t+1} + \beta_{dur}\lambda_{dur,t+1} \tag{A.8}$$

where $\lambda_{dur,t+1}$ is the risk premium of the duration factor (the excess return of a low-duration-minus-high-duration long-short portfolio). The duration factor returns can be replicated following the instructions in the original paper as well as Weber (2018). Or they can be downloaded from the Global Factor Data website built by Jensen et al. (2023). The data sample ranges from July 1963 to December 2023.

A.1.8. INM

The last type of models I consider is the intermediary-based asset pricing models (Adrian et al., 2014; He et al., 2017). The key idea is that households may not be the marginal investors of financial assets, and thus household consumption may not be the only source of systematic risk in asset pricing. On the contrary, financial intermediaries are sophisticated and actively participate in the financial markets. It is reasonable to believe that intermediary balance sheets contain additional information about the pricing of the financial assets. The intermediary-based model used in this paper is from He et al. (2017), which I label as INM. The beta-pricing formula for INM is:

$$E\left[R_{t+1}\right] - r_{z,t} = \beta_{mkt}\lambda_{mkt,t+1} + \beta_{smb}\lambda_{smb,t+1} + \beta_{hml}\lambda_{hml,t+1} + \beta_{inm}\lambda_{inm,t+1} \tag{A.9}$$

where $\lambda_{inm,t+1}$ is the risk premium of the intermediary capital risk factor (AR(1) innovations to the market-based capital ratio of primary dealers). The intermediary capital risk factor returns can be downloaded from Zhiguo He's website. The data sample ranges from January 1970 to December 2023.

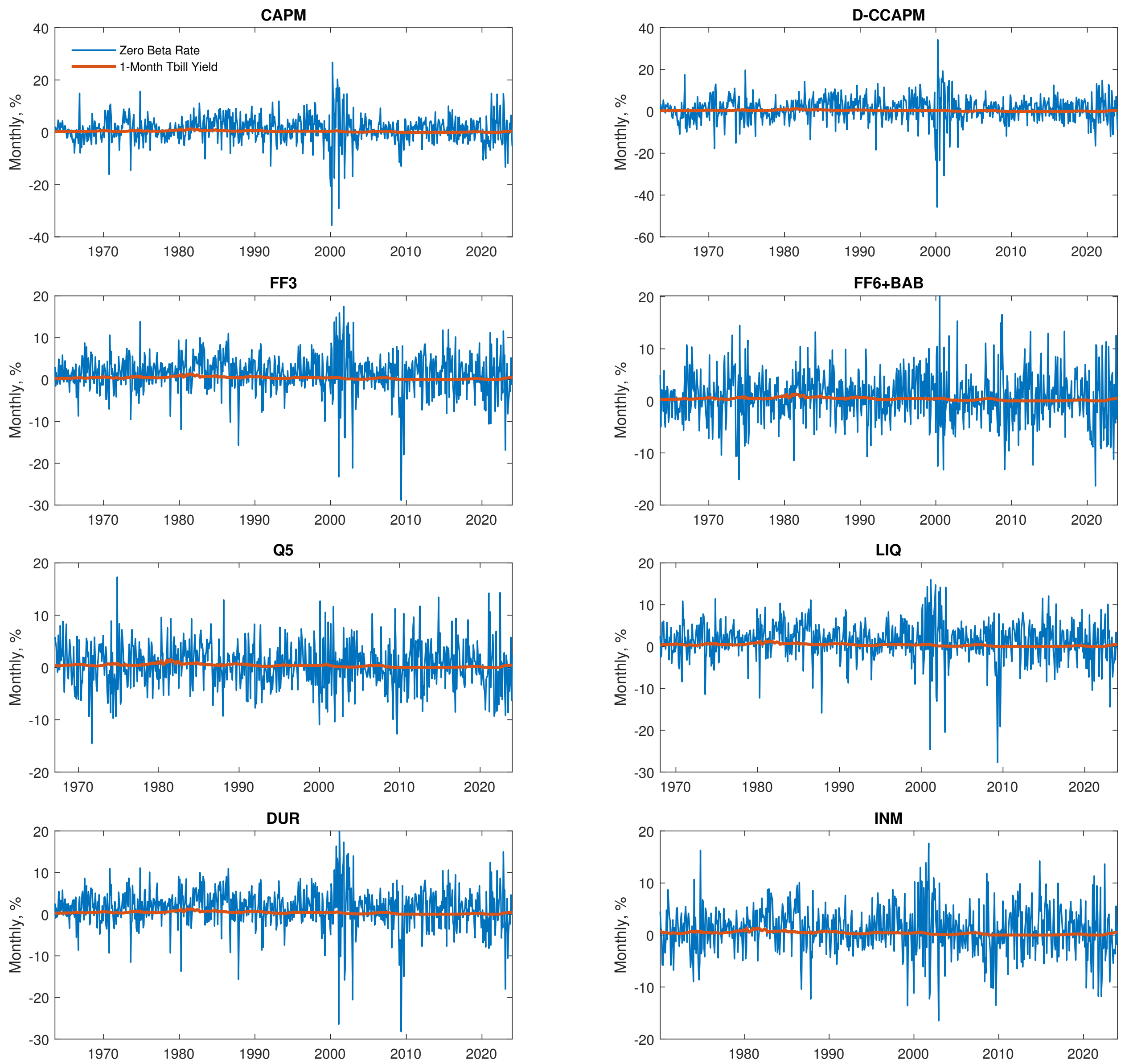
A.2. Zero-Beta Rates Estimated using the Fama-MacBeth Approach

Figure A.1 shows the monthly time series of the zero-beta rate estimated using the Fama-MacBeth approach for all 8 prominent factor models. Table A.1 reports the summary statistics of these estimated zero-beta rates. In Panel A, μ_z is the mean of the estimated zero-beta rate, and μ_z/μ_y is the ratio between the mean of zero-beta rate and the mean of 1-month Treasury yield. In Panel B, σ_z is the volatility of the estimated zero-beta rate and σ_z/σ_y is the ratio between the volatility of the zero-beta rate and the volatility of the 1-month Treasury yield. Panel C reports the correlation matrix of the 8 zero-beta rates. Summarizing over all the 8 models, the zero-beta rate estimated by Fama-MacBeth is on average 3.0 times higher in level and 18.2 times more volatile than the Treasury yield. The correlation of estimated zero-beta rates across all 8 models is on average 0.57. In summary, the high level and high volatility of the estimated zero-beta rate is ubiquitous in all prominent factor models.

A.3. An Iterative Fama-MacBeth Procedure

To alleviate the concern of inconsistent return units in regression (1) for the Fama-MacBeth approach, this section provides an iterative Fama-MacBeth procedure:

Figure. A.1. Estimated Zero-Beta Rates (Fama-MacBeth)



Notes: This figure shows the monthly time series of the zero-beta rate estimated using the Fama-MacBeth approach for all 8 prominent factor models, in monthly percentages. The blue lines are the estimated zero-beta rates. The red lines are the 1-month US Treasury bill yields.

Table A.1: Estimated Zero-Beta Rates (Fama-MacBeth)

	CAPM	D-CCAPM	FF3	FF6+BAB	Q5	LIQ	DUR	INM
				Panel A: Mean				
μ_z	13.96	11.92	16.50	8.78	5.34	16.43	18.11	11.88
μ_z/μ_y	3.21	2.74	3.79	2.02	1.22	3.75	4.16	2.74
				Panel B: Volatility				
σ_z	63.14	72.63	56.46	57.71	52.33	56.50	57.02	53.87
σ_z/σ_y	19.82	22.80	17.72	18.11	15.79	17.09	17.90	16.07
				Panel C: Correlation Matrix				
CAPM	1.00							
D-CCAPM	0.97	1.00						
FF3	0.55	0.43	1.00					
FF6+BAB	0.19	0.12	0.46	1.00				
Q5	0.31	0.24	0.57	0.73	1.00			
LIQ	0.55	0.43	0.99	0.41	0.53	1.00		
DUR	0.63	0.48	0.98	0.41	0.50	0.97	1.00	
INM	0.45	0.34	0.83	0.61	0.76	0.77	0.77	1.00

Notes: This table reports the summary statistics of the estimated zero-beta rates for 8 prominent factor models using the Fama-macBeth approach. In Panel A, μ_z is the annualized mean of the estimated zero-beta rate and, μ_z/μ_y is the ratio between the mean of zero-beta rate and the mean of 1-month Treasury yield. In Panel B, σ_z is the annualized volatility of the estimated zero-beta rate and σ_z/σ_y is the ratio between the volatility of the zero-beta rate and the volatility of the 1-month Treasury yield. Panel C reports the correlation matrix of the 8 zero-beta rates.

i. Guess the time series of the zero-beta rate, denoted as $r_{z,t}^{(0)}$ (I use the 1-month Treasury yield as the initial guess).

ii. Run the following time series regression for each test asset to estimate the betas:

$$R_{i,t+1} - r_{z,t}^{(0)} = \alpha_i + \beta_{M,i}\left(R_{M,t+1} - r_{z,t}^{(0)}\right) + \sum_{k=2}^{K} \beta_{k,i} f_{k,t+1} + \varepsilon_{i,t+1} \tag{A.10}$$

iii. Run the cross-sectional regressions (2) at each date to update the zero-beta rate, denoted as $r_{z,t}^{(1)}$.

iv. Iterate steps i to iii until $r_{z,t}$ converges or until beta estimates converge.

Summarizing over all the 8 models, the zero-beta rate estimated by Fama-MacBeth is

Table A.2: Estimated Zero-Beta Rates (Iterative Fama-MacBeth)

	CAPM	D-CCAPM	FF3	FF6+BAB	Q5	LIQ	DUR	INM
				Panel A: Mean				
μ_z	10.37	10.03	21.15	10.88	11.56	21.56	21.54	22.04
μ_z/μ_y	2.38	2.30	4.86	2.50	2.64	4.92	4.95	5.08
				Panel B: Volatility				
σ_z	79.66	84.40	66.39	80.19	72.66	69.52	64.07	71.20
σ_z/σ_y	25.00	26.49	20.84	25.17	21.92	21.03	20.11	21.24
				Panel C: Correlation Matrix				
CAPM	1.00							
D-CCAPM	1.00	1.00						
FF3	0.28	0.26	1.00					
FF6+BAB	-0.04	-0.06	0.12	1.00				
Q5	-0.06	-0.08	0.51	0.69	1.00			
LIQ	0.30	0.29	0.99	-0.00	0.43	1.00		
DUR	0.34	0.32	0.97	0.11	0.43	0.96	1.00	
INM	0.26	0.25	1.00	0.09	0.49	0.99	0.97	1.00

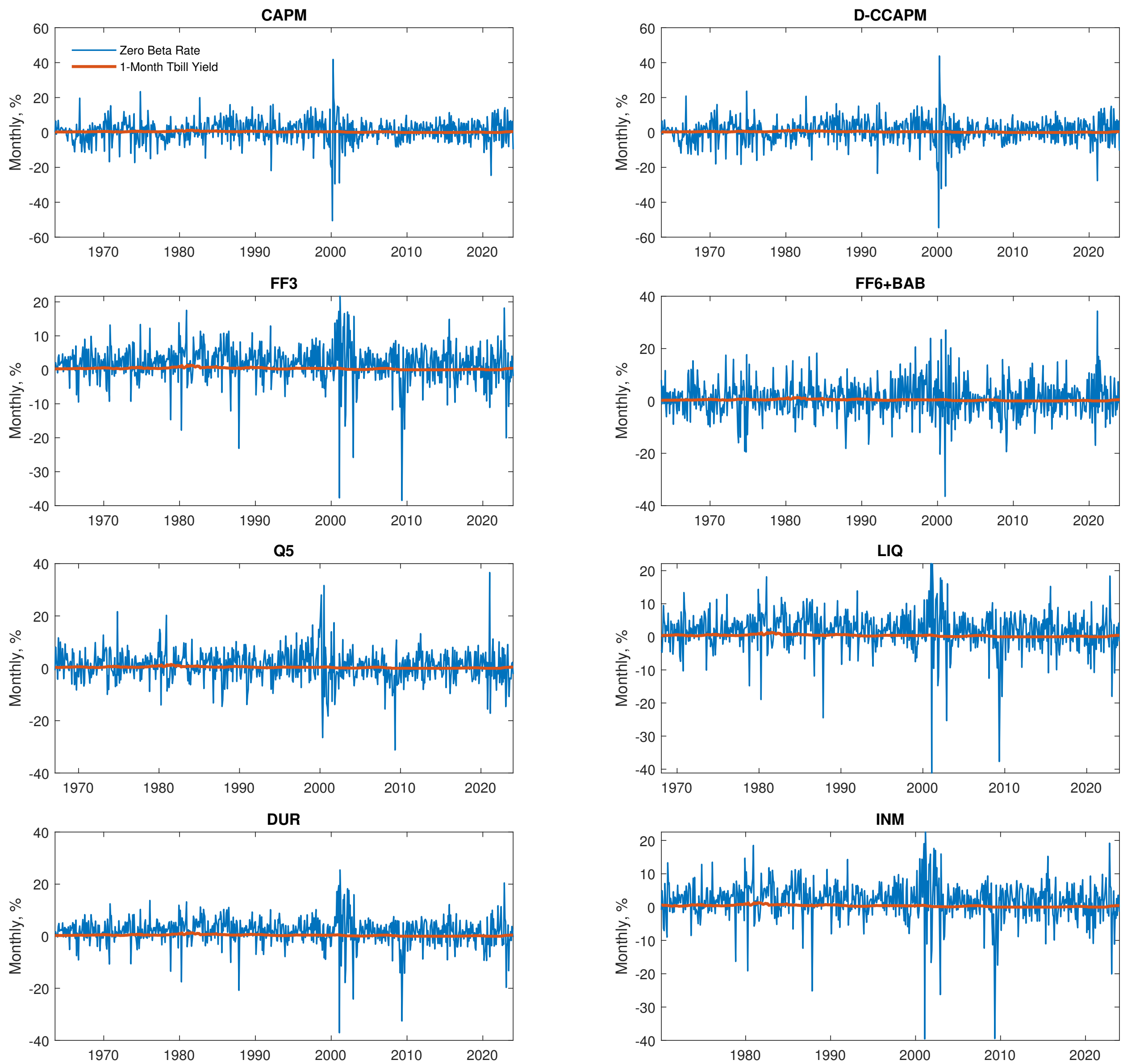
Notes: This table reports the summary statistics of the estimated zero-beta rates for 8 prominent factor models using the iterative Fama-macBeth approach. In Panel A, μ_z is the annualized mean of the estimated zero-beta rate, and μ_z/μ_y is the ratio between the mean of zero-beta rate and the mean of 1-month Treasury yield. In Panel B, σ_z is the annualized volatility of the estimated zero-beta rate and σ_z/σ_y is the ratio between the volatility of the zero-beta rate and the volatility of the 1-month Treasury yield. Panel C reports the correlation matrix of the 8 zero-beta rates.

on average 3.7 times higher in level and 22.7 times more volatile than the Treasury yield. Figure A.2 shows the estimated zero-beta rate in all 8 factor models. Table A.2 reports the mean and standard deviations for each zero-beta rate series and shows that the correlation of estimated zero-beta rates across all 8 models is on average 0.42. Obviously, the iterative procedure does not essentially affect the results in the main text.

A.4. Zero-Beta Rates Estimated using the Zero-Beta-Portfolio Approach

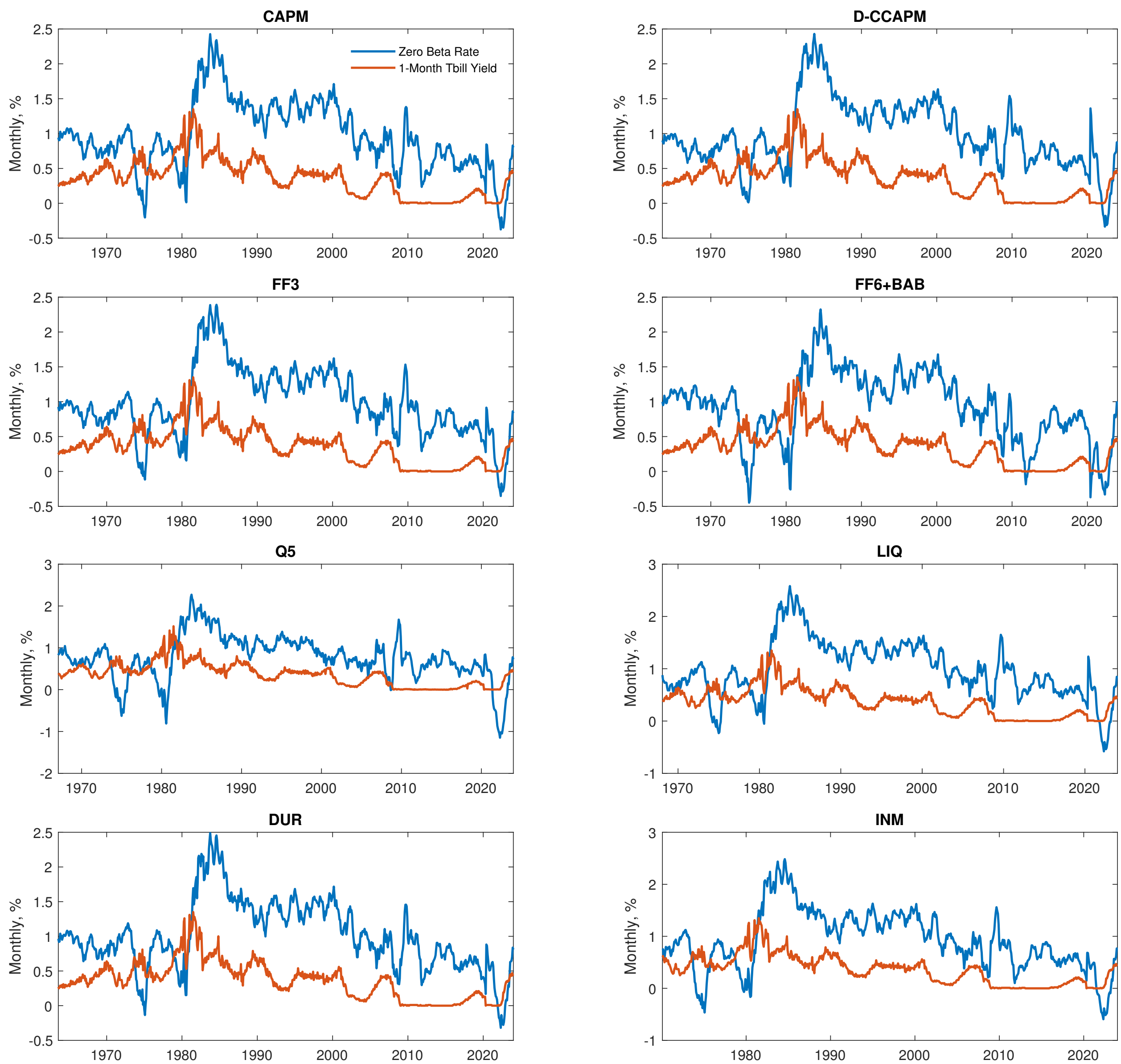
Figure A.3 shows the monthly time series of the zero-beta rate estimated using the zero-beta-portfolio approach from DHKW for all 8 prominent factor models. Table A.3 reports the summary statistics of these estimated zero-beta rates. In Panel A, μ_z is the mean of the

Figure. A.2. Estimated Zero-Beta Rates (Iterative Fama-MacBeth)



Notes: This figure shows the monthly time series of the zero-beta rate estimated using the iterative Fama-MacBeth approach for all 8 prominent factor models, in monthly percentages. The blue lines are the estimated zero-beta rates. The red lines are the 1-month US Treasury bill yields.

Figure. A.3. Estimated Zero-Beta Rates (DHKW)



Notes: This figure shows the monthly time series of the zero-beta rate estimated using the zero-beta-portfolio approach from DHKW for all 8 prominent factor models, in monthly percentages. The blue lines are the estimated zero-beta rates. The red lines are the 1-month US Treasury bill yields.

Table A.3: Estimated Zero-Beta Rates (DHKW)

	CAPM	D-CCAPM	FF3	FF6+BAB	Q5	LIQ	DUR	INM
				Panel A: Mean				
μ_z	11.52	11.82	11.78	11.17	9.18	11.76	12.10	11.02
μ_z/μ_y	2.65	2.72	2.71	2.57	2.10	2.68	2.78	2.54
				Panel B: Volatility				
σ_z	5.90	5.71	5.84	5.98	6.38	6.60	5.97	6.89
σ_z/σ_y	1.85	1.79	1.83	1.88	1.92	2.00	1.87	2.05
				Panel C: Correlation Matrix				
CAPM	1.00							
D-CCAPM	0.98	1.00						
FF3	0.99	0.99	1.00					
FF6+BAB	0.94	0.89	0.93	1.00				
Q5	0.91	0.91	0.93	0.89	1.00			
LIQ	0.99	0.99	1.00	0.91	0.94	1.00		
DUR	1.00	0.99	0.99	0.93	0.91	0.99	1.00	
INM	0.99	0.98	1.00	0.94	0.94	0.99	0.99	1.00

Notes: This table reports the summary statistics of the estimated zero-beta rates for 8 prominent factor models using the zero-beta-portfolio approach from DHKW. In Panel A, μ_z is the annualized mean of the estimated zero-beta rate and μ_z/μ_y is the ratio between the mean of zero-beta rate and the mean of 1-month Treasury yield. In Panel B, σ_z is the annualized volatility of the estimated zero-beta rate and σ_z/σ_y is the ratio between the volatility of zero-beta rate and the volatility of 1-month Treasury yield. Panel C reports the correlation matrix of the 8 zero-beta rates.

estimated zero-beta rate, and μ_z/μ_y is the ratio between the mean of zero-beta rate and the mean of 1-month Treasury yield. In Panel B, σ_z is the volatility of the estimated zero-beta rate and σ_z/σ_y is the ratio between the volatility of the zero-beta rate and the volatility of the 1-month Treasury yield. Panel C reports the correlation matrix of the 8 zero-beta rates. Summarizing over all the 8 models, the zero-beta rate estimated by Fama-MacBeth is on average 2.6 times higher in level and 1.9 times more volatile than the Treasury yield. The correlation of estimated zero-beta rates across all 8 models is on average 0.96. In summary, the high level and high volatility of the estimated zero-beta rate is ubiquitous in all prominent factor models.

Table A.4: Predictive Regression Results (DHKW)

	$Const$	Y_{1M}	$INFL$	TMS	UMP	$CAPE$	CSP	R^2
Coefficient	0.68	2.58	-2.15	2.88	-0.09	0.01	-0.69	2.50
t-statistics	0.63	3.67	-2.95	2.03	-0.86	0.39	-0.17	

Notes: This table reports the point estimates and t-statistics for coefficients of the predictive regression in the zero-beta-portfolio approach from DHKW. This regression predicts the realized return of the minimum-variance zero-beta portfolio using a constant term and 6 macroeconomic variables: Y_{1M} (1-month Treasury yield), $INFL$ (rolling average of the previous twelve-month inflation), TMS (term spread), UMP (unemployment rate), $CAPE$, and CSP (corporate bond spread). Predictive R^2 is also reported in percentages.

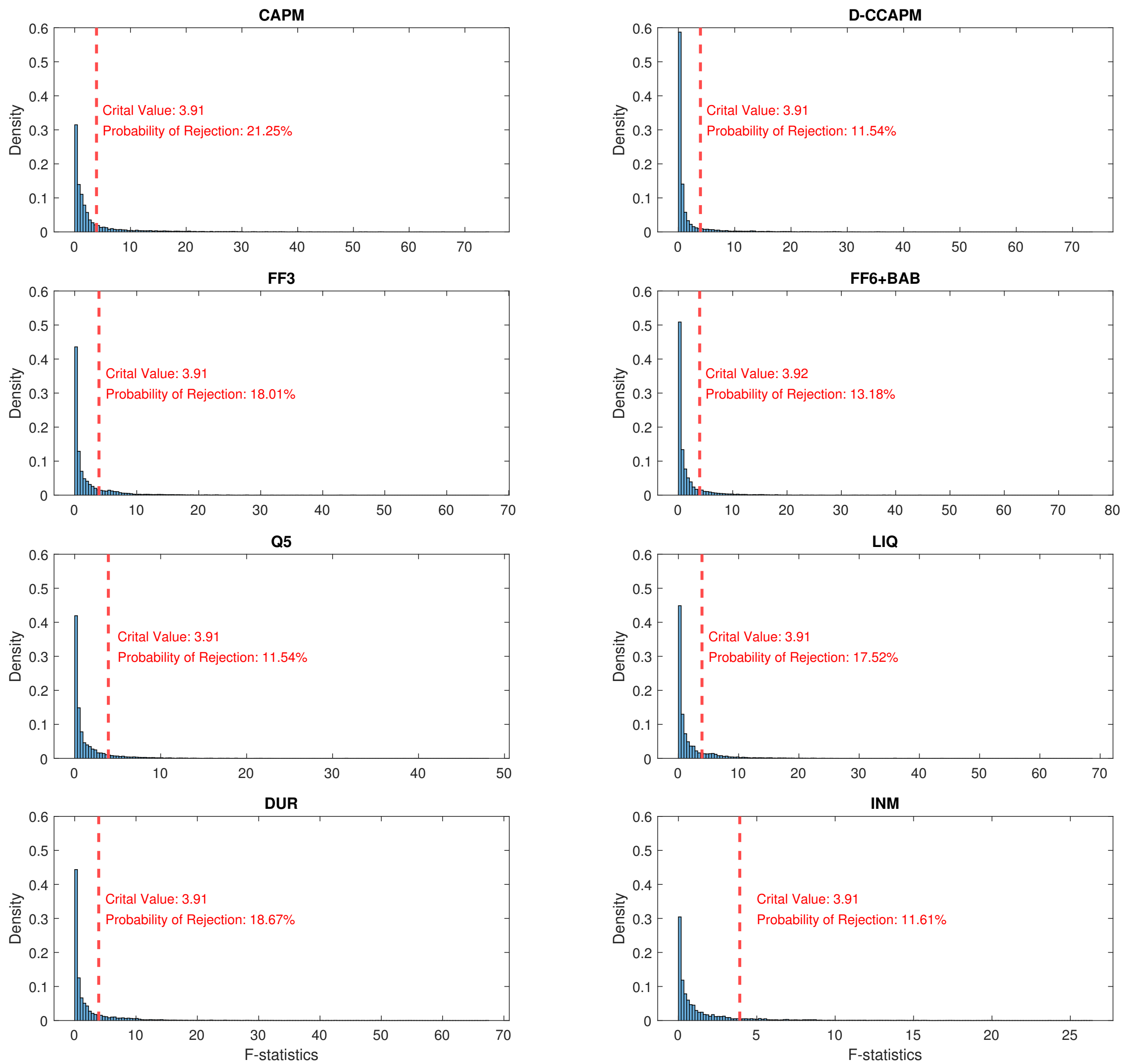
A.5. Prediction of the Minimum-Variance Zero-Beta Portfolio Returns

Table A.4 reports the point estimates and t-statistics for coefficients of the predictive regression in the zero-beta-portfolio approach from DHKW. This regression predicts the realized return of the minimum-variance zero-beta portfolio using 6 macroeconomic variables plus a constant term. Y_{1M}, $INFL$, and TMS are significantly different from zero. The signs of their coefficients are consistent with DHKW. The predictive R^2 is 2.5%.

A.6. Testing for the Uniqueness of the Zero-Beta Rate

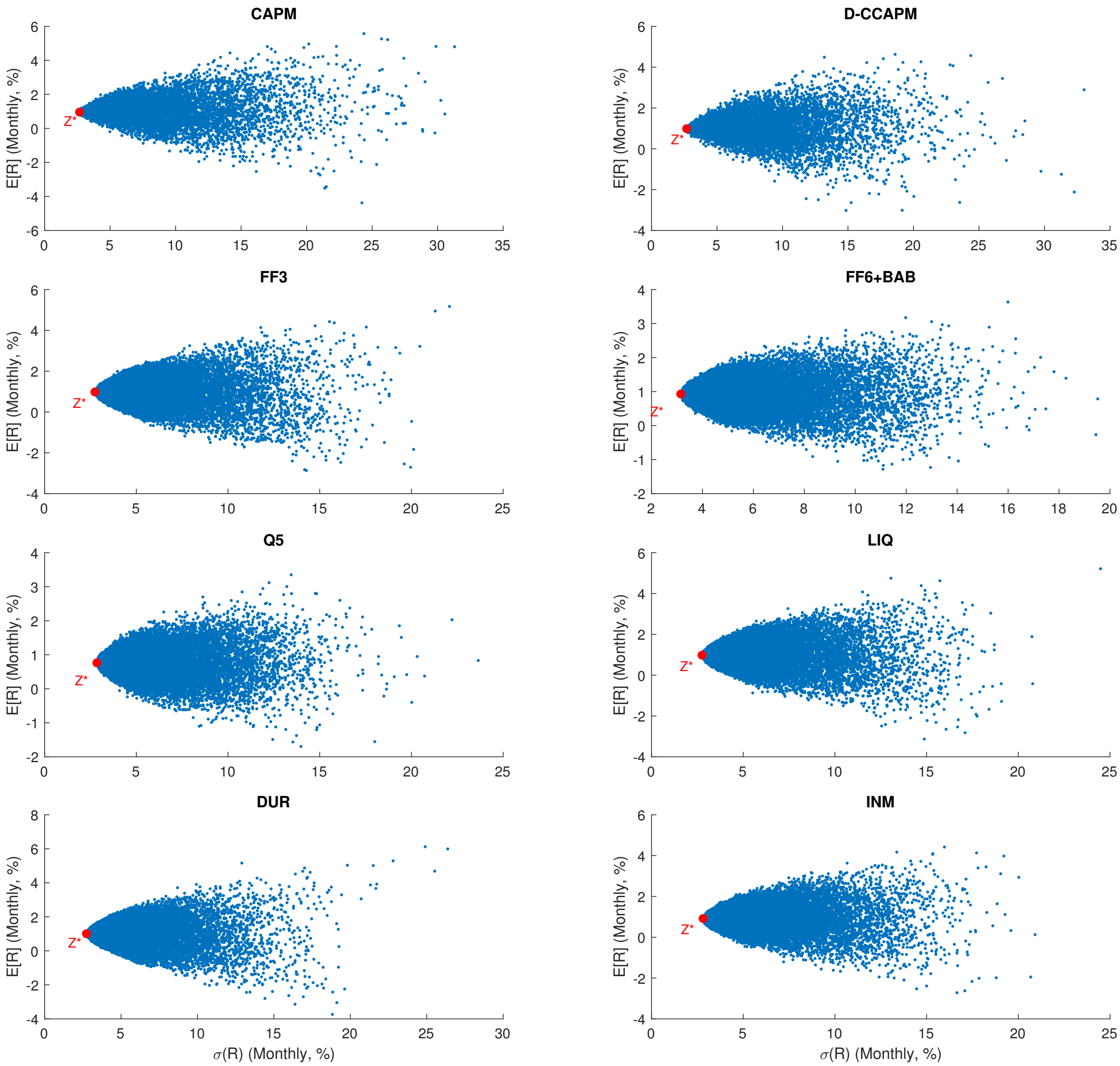
As explained in section 3.2, I propose two tests for the uniqueness of the zero-beta rate. The first test uses Fama-MacBeth time series regressions to generate $N \times (N-1)$ separate tests and F-statistics. Figure A.4 plots the histogram of $N \times (N-1)$ F-statistics for the 8 factor models. The red vertical dashed lines represent the critical values of the F distributions with degrees of freedom 1 and $N-K-1$ with a 5% significance level. The probabilities of rejection are all higher than the 5% significance threshold, indicating that all the 8 factor models are rejected based on this test. The second test constructs a large number of random zero-beta portfolios and checks whether their expected returns are statistcally equalized. Figure A.5 plots the constructed zero-beta portfolios around the minimum-variance zero-beta portfolio in the mean-standard deviation space for all 8 factor models. The red dots Z^* represent the minimum-variance zero-beta portfolios. The blue dots denote the random zero-beta portfolios. All tests have the ANOVA p-values of 0.00, indicating that all the 8 factor models are rejected based on this test.

Figure. A.4. Histogram of F-Statistics



Notes: This figure plots the histogram of $N \times (N-1)$ F-statistics in the Fama-MacBeth time series regression tests for all 8 factor models. The red vertical dashed lines represent the critical values of the F distributions with degrees of freedom 1 and $N-K-1$ with a 5% significance level.

Figure. A.5. Constructed Zero-Beta Portfolios



Notes: This figure plots the constructed zero-beta portfolios around the minimum-variance zero-beta portfolio in the mean-standard deviation space for all 8 factor models. The red dots Z^* represent the minimum-variance zero-beta portfolios. The blue dots denote the random zero-beta portfolios.

Appendix B. Proofs

B.1. Portfolio Weights of the Minimum-Variance Zero-Beta Portfolio

This section proves the analytical formula for the portfolio weights of the minimum-variance zero-beta portfolio in equation (4). After estimating the variance-covariance matrix of the asset returns, Σ_R, I solve the following variance minimization problem:

$$\begin{aligned} &\min_{\omega} \omega' \Sigma_R \omega \\ &s.t. \quad \omega' \beta = \vec{0} \\ &s.t. \quad \omega' \iota = 1 \end{aligned} \tag{B.1}$$

where ω is an $N \times 1$ vector of portfolio weights, β is an $N \times K$ matrix of estimated risk loading, $\vec{0}$ is a $1 \times K$ vector of zeros, and ι is an $N \times 1$ vector of ones. N is the number of asset returns and K is the number of factors.

Set up the Lagrangian equation with multipliers $\lambda_{1,(K \times 1)}$ and $\lambda_{2,(1 \times 1)}$:

$$\mathcal{L} = \omega' \Sigma_R \omega - 2\omega' \beta \lambda_1 - 2\omega' \iota \lambda_2 \tag{B.2}$$

The F.O.C. with respect to ω is:

$$\Sigma_R \omega = \beta \lambda_1 + \lambda_2 \iota = \begin{bmatrix} \iota & \beta \end{bmatrix} \begin{bmatrix} \lambda_2 \\ \lambda_1 \end{bmatrix} \tag{B.3}$$

Pre-multiply equation (B.3) by $\begin{bmatrix} \iota' \\ \beta' \end{bmatrix} \Sigma_R^{-1}$ and use the two constraints, I get:

$$\begin{bmatrix} \iota' \\ \beta' \end{bmatrix} \omega = \begin{bmatrix} 1 \\ \vec{0} \end{bmatrix} = \begin{bmatrix} \iota' \\ \beta' \end{bmatrix} \Sigma_R^{-1} \begin{bmatrix} \iota & \beta \end{bmatrix} \begin{bmatrix} \lambda_2 \\ \lambda_1 \end{bmatrix} \tag{B.4}$$

$$\Longrightarrow \quad \begin{bmatrix} \lambda_2 \\ \lambda_1 \end{bmatrix} = \left(\begin{bmatrix} \iota' \\ \beta' \end{bmatrix} \Sigma_R^{-1} \begin{bmatrix} \iota & \beta \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ \vec{0} \end{bmatrix} \tag{B.5}$$

Substituting (B.4) into (B.3):

$$\omega = \Sigma_R^{-1} \begin{bmatrix} \iota & \beta \end{bmatrix} \left(\begin{bmatrix} \iota' \\ \beta' \end{bmatrix} \Sigma_R^{-1} \begin{bmatrix} \iota & \beta \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ \vec{0} \end{bmatrix} \tag{B.6}$$

□

B.2. Proof of Proposition 1

In this proof, I assume the factors are either returns or excess returns. This is without loss of generality because any non-return factors can be projected onto the span of returns and obtain the factor-mimicking portfolio returns.

Proof of *(i)*:

First, I look at the identities of the factor risk premiums λ. If a factor f_j is a return, then I plug f_j in equation (A.1) and get: $E[f_j]-r_z = Cov(f_j,F)\Sigma_F^{-1}\lambda$. Notice that f_j is one element in F, thus $Cov(f_j,F)\Sigma_F^{-1}$ turns out to be a vector with 1 on the jth location and 0 everywhere else. Hence, $Cov(f_j,F)\Sigma_F^{-1}\lambda = \lambda_j$ and the ith factor risk premium is: $\lambda_j = E[f_j]-r_z$. Alternatively, if a factor f_j is an excess return, I assume $f_j = R_1^j - R_2^j$. Substituting returns R_1^j and $R_1^j = R_2^j + f_j$ into equation (A.1) separately generates the following two equations: $E[R_2+f_j]-r_z = Cov(R_2+f_j,F)\Sigma_F^{-1}\cdot\lambda$ and $E[R_2]-r_z = Cov(R_2,F)\Sigma_F^{-1}\cdot\lambda$. Take the difference I get: $E[f_j] = Cov(f_j,F)\Sigma_F^{-1}\lambda = \lambda_j$. That is, the jth factor risk premium is: $\lambda_j = E[f_j]$. Suppose I have $\bar{k}$ return factors and $K-\bar{k}$ excess return factors, the factor model can be written as follows, for any return $R_i \in \mathcal{R}$:

$$E[R_i]-r_z = \sum_{j=1}^{\bar{k}} \beta_{j,i}\left(E[f_j]-r_z\right) + \sum_{j=\bar{k}+1}^{K} \beta_{j,i} E[f_j] \tag{B.7}$$

To prove *(i)*, suppose there are two zero-beta rates, r_{z_1} and r_{z_2}, that satisfy the factor model equation (B.7). Then for any return $R_i \in \mathcal{R}$:

$$E[R_i]-r_{z_1} = \sum_{j=1}^{\bar{k}} \beta_{j,i}\left(E[f_j]-r_{z_1}\right) + \sum_{j=\bar{k}+1}^{K} \beta_{j,i} E[f_j] \tag{B.8}$$

$$E[R_i]-r_{z_2} = \sum_{j=1}^{\bar{k}} \beta_{j,i}\left(E[f_j]-r_{z_2}\right) + \sum_{j=\bar{k}+1}^{K} \beta_{j,i} E[f_j] \tag{B.9}$$

Take the difference I get:

$$(r_{z_1}-r_{z_2}) = \sum_{j=1}^{\bar{k}} \beta_{j,i}\left(r_{z_1}-r_{z_2}\right) \quad \Leftrightarrow \quad (r_{z_1}-r_{z_2})\left(1-\sum_{j=1}^{\bar{k}}\beta_{j,i}\right) = 0 \tag{B.10}$$

Since this equality holds for all asset i, it is almost surely not possible that $1-\sum_{j=1}^{\bar{k}}\beta_{j,i} = 0$. Therefore, it must be that $r_{z_1} = r_{z_2}$. That is, the zero-beta rate r_z is unique with a factor model structure. Then, I can rearrange equation (B.7) to solve for r_z:

$$r_z = \frac{E\left[R_i\right] - \sum_{j=1}^{K} \beta_{j,i} E\left[f_j\right]}{1 - \sum_{j=1}^{\bar{k}} \beta_{j,i}} \tag{B.11}$$

If the factor model is correctly specified, equation (B.11) should hold for any return $R_i \in \mathcal{R}$. This is a strong structure imposed by the factor model on the data and the factors. That every asset should have this property can be used to test the factor model. Obviously, different factors imply different values of zero-beta rates.

Alternative proof of *(i)*:

Asset pricing theories are based on the stochastic discount factor (SDF) framework: for any asset return R_i, I have $1 = E\left[mR_i\right]$. When a risk-free asset is not traded, I usually augment the payoff space with a hypothetical unit payoff and assign an arbitrary price for this unit payoff (Hansen and Jagannathan, 1991). Define $\nu \equiv p(1) = E\left[m\right]$ to be the price of hypothetical unit payoff and also the mean of SDF. The zero-beta portfolios, R_z, are orthogonal to the SDF. Then, I have $1 = E\left[mR_z\right] = Cov(m, R_z) + E\left[m\right] E\left[R_z\right] = E\left[m\right] E\left[R_z\right]$. The zero-beta rate is defined to be the expected return of the zero-beta portfolios: $r_z = E\left[R_z\right] = 1/E\left[m\right] = 1/\nu$.

Once I define the mean of SDF, $1 = E\left[mR_i\right]$ implies $E\left[R_i\right] - r_z = -r_z Cov(m, R_i)$. This is the fundamental equation for risk premium with respect to r_z. When a risk-free asset exists, $r_z = r_f$. However, when the risk-free asset does not exist, this asset pricing equation holds for any value of ν. To put it in another way, for any arbitrarily assigned ν, I have a pair of the zero-beta rate, r_z, and the mean of SDF, $E\left[m\right]$, that can be used to price assets.

To find an SDF that prices all assets and the hypothetical unit-payoff asset, I project potential SDFs onto the span of returns[20] and a constant with the previous definition $E\left[m\right] = \nu$. Assume the unique projected SDF $m_\nu^* = \alpha + \beta R$ where R is the return vector of all risky assets. By the projection theory, $\alpha = E\left[m\right] - \beta E\left[R\right]$ and $\beta = Cov(m, R) Var(R)^{-1}$. Denote the mean vector of returns as μ, the covariance matrix of returns as Σ, and a unit vector ι. Also notice that $Cov(m, R) = E\left[mR'\right] - E\left[m\right] E\left[R'\right] = (\iota - \nu\mu)'$. I have:

$$\begin{aligned} m_\nu^* &= \nu - \beta\mu + \beta R = \nu + \beta\left(R - \mu\right) \\ &= \nu + (\iota - \nu\mu)'\Sigma^{-1}\left(R - \mu\right) \end{aligned} \tag{B.12}$$

In summary, for any arbitrarily assigned $\nu = 1/E\left[m\right] = 1/r_z$, I can find a unique SDF

[20] The span of returns is equivalent to the span of payoffs.

m_ν^* that satisfies $E[R_i] - r_z = -r_z Cov(m, R_i)$ and is able to price all risky assets and the hypothetical risk-free asset. It is written with a subscript ν to emphasize the point that it is defined up to an assigned value of ν. I think the main takeaway here is that the zero-beta rate is generally not uniquely identified. That is to say, it is not possible to compute "the" unique zero-beta rate without imposing further structures on the data in a world where no risk-free assets are traded. This argument is not surprising because I have not defined what risks are, hence I are probably not able to define a unique "risk-free" rate proxied by a unique zero-beta rate. Naturally, I move on to factor models where risks are clearly defined by the risk factors.

According to equation (B.12), there is a unique SDF in the span of returns and a constant: $m_\nu^* = \nu + (\iota - \nu\mu)'\Sigma^{-1}(R-\mu)$. In the meantime, I know that a factor model is equivalent to a linear specification of the SDF[21]: $\tilde{m} = 1/r_z \cdot \left(1 - \lambda'\Sigma_F^{-1}(F - \mu_F)\right)$ where μ_F denotes the mean of factors. Because the factors are either returns or excess returns, $\tilde{m}$ is also a linear function of returns, excess returns, and a constant. That is, $\tilde{m}$ is in the span of payoffs and a constant. Finally, since the span of returns is equivalent to the span of payoffs, it must be that $m_\nu^* = \tilde{m}$. Hence, $\nu + (\iota - \nu\mu)'\Sigma^{-1}(R-\mu) = 1/r_z \cdot \left(1 - \lambda'\Sigma_F^{-1}(F - \mu_F)\right)$. In the previous proof of *(i)*, I looked at the identities of the factor risk premiums. Our results allow us to write $\lambda = \mu_F - r_z\eta$ where η is a vector of 1's and 0's. $\eta_j = 1$ when f_j is a return factor; $\eta_j = 0$ when f_j is an excess return factor. With $\nu = 1/r_z$, I can solve for r_z:

$$r_z = \frac{\mu'\Sigma^{-1}(R-\mu) - \mu_F'\Sigma_F^{-1}(F-\mu_F)}{\iota'\Sigma^{-1}(R-\mu) - \eta'\Sigma_F^{-1}(F-\mu_F)} \tag{B.13}$$

Clearly, this is valid if the denominator $\iota'\Sigma^{-1}(R-\mu) - \eta'\Sigma_F^{-1}(F-\mu_F) \neq 0$, which is true almost surely. Equation (B.13) delivers the same implications as equation (B.11), the zero-beta rate is fixed in a factor model structure, and different sets of factors produce different values of zero-beta rates. Intuitively, risks are well defined by the risk factors in a factor model, which helps uniquely identify the zero-beta rate.

<u>Proof of *(i) and (ii)* following Roll (1980):</u>

to be added

□

[21] The proof can be found in Cochrane (2009) and Back (2010).

B.3. Proof of Equation (6)

For a general factor model, suppose that $E[R] - r_z = \beta \cdot \lambda = Cov(R, F)\,\Sigma_F^{-1} \cdot \lambda$ where $\beta \equiv Cov(R, F)\,\Sigma_F^{-1}$ is the risk loading. Define $\tilde{R} = \lambda' \Sigma_F^{-1} F$, then,

$$Cov(R, \tilde{R}) = Cov(R, \lambda' \Sigma_F^{-1} F) = Cov(R, F)\,\Sigma_F^{-1} \lambda = E[R] - r_z \tag{B.14}$$

For portfolio $\tilde{R}$,

$$Cov(\tilde{R}, \tilde{R}) = E[\tilde{R}] - r_z = Var(\tilde{R}) \quad \Rightarrow \quad \frac{E[\tilde{R}] - r_z}{Var(\tilde{R})} = 1 \tag{B.15}$$

Substituting (B.15) into (B.14) I get:

$$E[R] - r_z = Cov(R, \tilde{R}) = \frac{Cov(R, \tilde{R})}{Var(\tilde{R})} \left(E[\tilde{R}] - r_z \right) \equiv \tilde{\beta} \tilde{\lambda} \tag{B.16}$$

Hence, any factor model can be written as a single-factor model with factor $\tilde{R}$. This completes the proof of equation (6).

The single factor return $\tilde{R}$ is on the mean-variance frontier because for any asset R_i s.t. $E[R_i] = E\left[\tilde{R}\right]$, I will have:

$$\sigma(\tilde{R})^2 = Var(\tilde{R}) = Cov(R_i, \tilde{R}) \le \sigma(R_i)\sigma(\tilde{R}) \quad \Rightarrow \quad \sigma(\tilde{R}) \le \sigma(R_i) \tag{B.17}$$

□