Scale-Dependent Returns
or
Dynamics of the Interest Rate?

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Preliminary Version.
Please do not circulate.

Abstract

Using historical U.S. data, I revisit the empirical evidence for scale-dependency of returns over a span of 70 years. Contrary to recent findings that suggest households, after controlling for portfolio shares, experience scale-dependent returns (i.e., higher returns as they become wealthier), I find that this has not always been the case. In fact, prior to 1980, we observed a negative scale-dependency of returns. I propose a potential explanation for this phenomenon: the observed scale-dependent returns are coming from within-asset class differences in the realization of interest rate risk, that are not captured in the cross-sectional regressions. The changes in interest rates affect the returns of different households differently based on the duration of their assets, which is the interest rate risk they bear. Since wealthier people tend to have assets for an average longer duration compared to less wealthy individuals, an increase in the real interest rate (as it was before 1980) resulted in lower returns for the wealthier people. Conversely, a decrease in the risk-free interest rate (as seen after 1980) led to higher returns for the wealthy. Finally, I developed a quantitative model to assess the extent to which this explanation accounts for the observed phenomenon.

JEL Classification: G51; D31; G11.

Keywords: scale-dependent returns, dynamics of inequality, returns to wealth, heterogeneous returns, duration of assets, portfolio choice.

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1 Introduction

There is growing empirical evidence that returns on wealth are increasing with wealth; That is, if you are richer, you have higher returns [1] on your wealth (e.g. see Fagereng, Guiso, Malacrino, and Pistaferri (2020); Bach et al. (2020) among many others). Moreover, there seems to be a scale premium. That is, even if we control for portfolio shares and many other factors, wealth still predicts a positive return in the cross-section of households' returns. This phenomenon is referred to as the scale-dependence property of returns.

As Gabaix, Lasry, Lions, Moll, and Qu (2016) argues, theoretically, scale-dependent returns can explain the high-speed dynamics of wealth inequality that we have observed in recent years (after the '80s), which is otherwise hard to explain by just relying on income inequality. Figure 1 plots wealth inequality in the U.S. across time measured as different top wealth percentiles wealth shares. Similar trends have been observed in many other developed countries, like the U.K. and France. (Greenwald, Leombroni, Lustig, and Van Nieuwerburgh (2023))

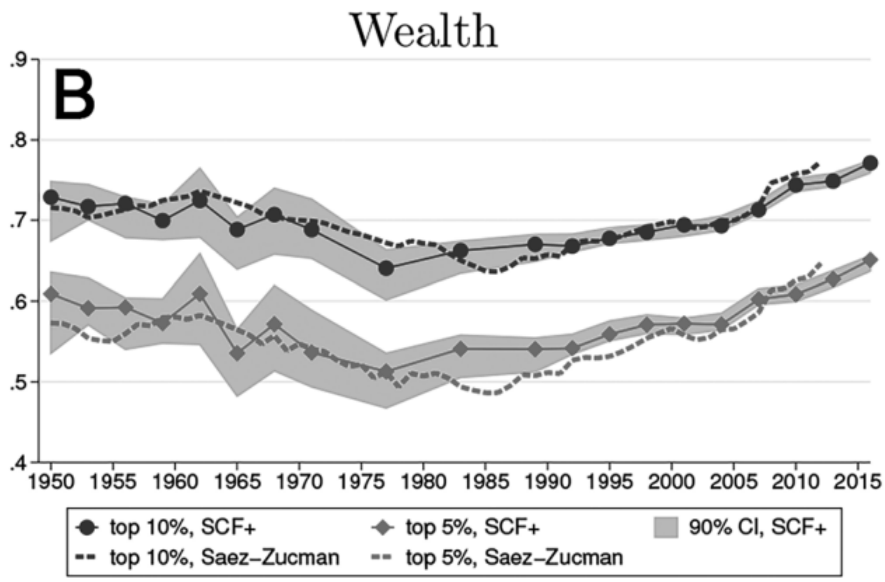

Figure 1: Wealth Inequality across time in terms of top percentiles wealth shares. Source: Kuhn et al. (2020)

As we can see, inequality was decreasing prior to 1980, and afterward, it started to increase. The existing evidence for increasing returns with scale from different countries (to the best of my knowledge) is only for the recent period when inequality was increasing fast. This is well in line with the arguments about the relation between scale-dependent returns and dynamics of inequality (Gabaix et al. (2016)). However,

[1] This paper and most of the other papers mentioned are about actual returns. However, there is evidence that it also extends to expected returns (Bach, Calvet, and Sodini (2020).

there is a lack of any evidence for scale-dependent returns for older years, say before 1980. Note that increasing wealth inequality is not necessarily equivalent to higher returns of the wealthy, and one needs to measure returns on wealth directly from household balance sheets.

In this paper, I go back before 1980 to see how the distribution of returns was when inequality was decreasing. It turns out that when inequality was decreasing, returns were actually decreasing with scale. That is, less wealthy households had better returns on their wealth compared to more wealthy.

The relation between changes in the direction of changes in inequality and the direction of scale-dependency of returns is very interesting. However, it does not help much (at least directly) to explain any of them. The changes in inequality are most likely the results of how returns are dependent on scale rather than the cause of it. On the other hand, they both might have an unknown confounding cause. One needs to come up with mechanisms that can explain both or each one separately. In this paper, I will argue that the changes in the long-term risk-free interest rate can explain the observed changes in scale dependency of returns.

Before going into details, let's recap why we care about scale-dependent returns. If we believe that returns are scale-dependent, then it has key implications for the dynamics of inequality, that is, how fast inequality changes (Gabaix et al. (2016)), tax system design (Gaillard and Wangner (2021)), and probably how inequality and asset prices interact. Depending on the source of this scale dependence, it might also have consequences for the welfare analysis of inequality dynamics.

Despite its importance and the broad consequences that scale-dependent returns might have, it has not been studied much, and we do not know the exact causes of that. Technically, it is difficult to identify the real cause of variation in returns among households with different wealth. It can be either from the demand or supply side. For instance, people might become less risk averse when they become rich (as in Gaillard and Wangner (2021)), so they demand riskier assets more and will be compensated with higher returns (demand side). On the other side, it can stem from the fact that there are some barriers to investing in some asset classes like hedge funds or indivisible assets like housing and private equity that may prevent poor people from enjoying higher returns (supply side).

In this paper, I will first explore whether the finding of richer people having higher returns is just a modern phenomenon or it has always been this way. It is well known that the 1980s marked the beginning of an increase in wealth inequality in many developed countries, which aligns with evidence that richer people have higher returns. However, if we look before the 1980s (back to the 1940s), the situation might be different, as inequality was not increasing like it did after the 1980s; in fact, it was decreasing.

Using older U.S. data, mostly historical waves of Survey of Consumer Finances (SCF), I revisit the empirical evidence for scale-dependency of returns in a longer horizon. I find that returns have not always been increasing with scale. Before the 80's (back to the 50's), it did not have this property, and actually it has been decreasing with scale. This makes the enigma of higher returns of richer people more puzzling.

Secondly, I connect this observed change in the direction of scale dependency of returns to the direction of changes in the real risk-free interest rate. As it is well known, risk-free interest rates have been increasing in the post-war period since 1980 when they started to decline. As we know, richer people have assets that are of a higher average duration, and their returns are more sensitive to changes in the interest rate. To show this mechanism formally, I build a parsimonious model to isolate this channel.

*** Talk about measuring duration

Finally, I try to build a quantitative model to see how much of the observed phenomenon can be explained by the interest rate mechanism. For doing so, I build a portfolio choice model, in which households choice of their duration relates to the cyclicality of their labor income (returns on human capital) and as the wealthiest people have a high cyclilacity of labor income, they choose a high duration. The rational behind that is that interest rate risk is counter cyclical and it is hedging strategy for them to do so. For the term structure part of the model, I bring some insights from the preferred habitat theory of the term structure to the household finance literature Vayanos and Vila (2021). I build on this literature by focusing on preferred habitat investors and reinterpreting them as households. I then study the effects of changes in the interest rate distribution of returns.

Related Literature This paper contributes to the rapidly growing literature on heterogeneous household behavior. It explores heterogeneous returns on wealth and the mechanisms underlying it. In recent years, there has been much evidence from different countries on the scale-dependency of returns: Fagereng et al. (2020) show the increasing with scale property of returns on wealth using administrative data in Norway. Bach et al. (2020) do similar using Sweden's administrative data. As Norway and Sweden have wealth tax data, measuring returns on wealth is easier and more precise using their data. Cao and Luo (2017), Gaillard and Wangner (2021), Snudden (2019), Snudden (2023), and Xavier (2021) use U.S. data for doing so. Brunner, Meier, and Naef (2020) use Swiss data for this task. As there is a wealth tax in Switzerland, the quality of Swiss data is also good. All these papers use data from recent years. Like all these papers, I measure returns on wealth. Unlike them, I will go back in the years till 1949. Of course, there is no reliable data for all the years till back then, but I succeeded in finding data for a handful of years.

Scale dependency of returns also relates to the dynamics of inequality, i.e., the speed at which inequality changes, and there is a bit of literature in economics on that Gabaix et al. (2016).

Like Greenwald et al. (2023) I build on interest rate mechanism and the facts related to increasing durations of portfolios of households across the wealth distribution. Unlike them, I directly measure returns on wealth using survey data and use the same mechanism to explain scale dependency of returns and changes in its direction.

This paper is also related to the Catherine, Miller, Paron, and Sarin (2023) which build a model based on the same mechanism as in Greenwald et al. (2023), but with human capital and social securities.

None of these two papers talk about returns distribution, neither theoretically nor empirically. Maybe among the earliest papers that talk putting scal-dependent returns in the macro models, are Gaillard and Wangner (2021) in which richere people are less risk averse and invest a higher share of their wealth in riskier assets, and hence enjoy higher returns for risk premium. Hubmer, Krusell, and Smith (2021) models a macro model with different features, including scale-dependent returns (see Fig. 6 in their paper) There are also other papers which feature some scale-dependent transaction costs, of which may the HANK model is the most famous one. Kaplan, Moll, and Violante (2018)

On the portfolio choice side, this paper connects to the idea of the counter-cyclical income risk and portfolio choice in Catherine, Sodini, and Zhang (2024) and Catherine (2021), but from the angle of duration risk selection. (also other papers in this strand are Azzalini, Kondziella, and Racz (2024))

** cite papers on background risk and also viceria papers ** cite papers about Gibrat law (eckhout)

Outline In section 2, I will describe the approach I take for measuring the returns, the data I use, and its challenges. Then, I compare empirical evidence on the scale dependency of returns before and after the 80's and establish our empirical findings. In section 3 I will discuss the interest rate channel and how it can explain the empirical findings. In section 4, I build a parsimonious model that captures the empirical findings and the discussed mechanisms. It contanis a portfolio choice model in which richer household who have more cyclical labor income risk, choose a higher interest rate risk exposure, which is counter cyclical to hedge their total wealth. Finally, I do the quantitative exercise in section 5.

2 The Cross-section of Households' Returns

I will first refresh the definition of returns on wealth and its components. I will then explain the data and how I tackle the empirical limitations in measuring returns. Then I will establish the results on the changes in scale dependency of returns over time.

2.1 Measurement of Returns

I follow a definition of returns based on return on assets (ROA) in accounting. Most other papers studying related questions use the same definition as well.[2] Thee gross ROA is defined as:

$$r_{it} = \frac{y_{it}^{div} + y_{it}^{kg}}{w_{it} + \frac{1}{2}f_{it}} \tag{1}$$

where,

- y_{it}^{div} is the (dividend) income from wealth, which includes all received interests, rents (inclusive of imputed rents of people who live in their own house), and stock dividends or income from private business.
- y_{it}^{kg} is the capital gains (both realized and unrealized) related to the period t.
- w_{it} is the wealth at the beginning of the period t.
- and finally, f_{it} is the net flow of (active) investment into wealth in period t. [3]

All variables are in real terms and measured before tax. If the ROA formula is applied to any subset of wealth or assets, then all the variables in the formula will be specific to that subset of wealth or assets. One can also measure net of debt-repayment ROA, but here I cannot do that for all the years because of data limitations.[4]

[2]Some papers use other definitions (like Xavier (2021), which subtracts debt in the denominator. However, the ROA is more natural and intuitive to be returns on wealth.)

[3]Note the difference between active saving and gross saving. The former does not include savings through capital gains on asset holdings: $w_{i,t+1} = w_{it} + y_{it}^{kg} + f_{it}$

[4]Another measure of returns of wealth is net of debt-repayment ROA:

$$r_{it} = \frac{y_{it}^{div} + y_{it}^{kg} - y_{it}^{b}}{w_{it} + \frac{1}{2}f_{it}} \tag{2}$$

Where y_{it}^{b} is the amount of money for loan repayment in period t. Due to data limitations, I do not observe y_{it}^{b} in my dataset. So, I only measure ROA. As $\frac{y_{it}^{b}}{w_{it}+\frac{1}{2}f_{it}}$ is the cost of debt, one expects that to be decreasing with wealth as richer people are more credible when it comes to getting loans. So, adding y_{it}^{b} strengthens the scale-dependency of returns for recent years (as is shown for Norwegian households in Fagereng et al. (2020), and for the U.S. household in Snudden (2019)). This channel could be true before or after the 80s, and I expect it to be observable in the data if implemented for the data before the 80s. As gross ROA is decreasing in wealth before 80's, subtracting a term that

Note that knowing flows to wealth is necessary for using this formula. (see Appendix A.1 for a discussion on the proper embedding of flow in this equation.) Knowing capital gains is also another challenge.

One can also derive and use other equivalent formulas for ROA dependent on the observed variables. For example, if we observe the end of the period t's wealth, we can rewrite ROA formula in terms of that. We know that $w_{i,t+1} = w_{it} + y_{it}^{kg} + f_{it}$, where $w_{i,t+1}$ is the end of the period t wealth or the wealth at the beginning of period $t+1$. Using this identity, we can get other equivalent formulas for ROA. (See Appendix A.6)

2.2 Data

I use the Survey of Consumer Finances (modern SCF) from 1983 to 2022, as well as its historical waves (old SCF) from 1949 till 1977, which will give us a span of 70 years from post-war period till now. The modern SCF is a triennial cross-sectional household survey of U.S. households created by the Board of Governors of the Federal Reserve Board. It covers many useful variables, such as asset holdings from different asset classes and the income generated by them. The old SCF is similar to the modern SCF, but with some differences like the variables included in the survey. Furthermore, it is done annually or biannually for most of the period, but it sometimes has larger gaps. SCF+ is the dataset which connects modern SCF with old SCF (Kuhn et al. (2020)). Old surveys and other databases are used to harmonize and reweight the historical data to create this extension in a way that represents US households. In this paper, I mostly use SCF+, as well as some data on active savings (flows) that I my-self extract from historical waves.

Assets The data includes liquid assets (checking, savings, call/money market accounts, and certificates of deposits.), housing and other real estate, bonds, stocks and business equity, mutual funds.

Income from Assets The data includes rental income (the imputed rental income of homeowners is separately added), interest and dividends income, as well as business and farm income.

Two things are quite difficult to measure in the data: Flows or active savings in each period (unless one has panel data) and Capital Gains. But, luckily, the historical waves included some questions about the flows (active saving) that help us solve these two challenges.

Flows: To solve the challenge with flows, I go back to historical waves of SCF

is decreasing in wealth might cause an overall net of debt-repayment ROA that is either decreasing, increasing, or neutral to scale. Having said that, it does not affect any of the results or arguments in this paper about gross ROA.

and look for years that the questionnaire includes questions about active savings in different asset classes. Fortunately, for four years, we have the flow data as well as the other needed variables in pre 80's data.[5] These data on active savings discover an important fact: active savings out of personal income (for people with positive wealth) is uncorrelated with their wealth. This finding is in lines with Fagereng, Blomhoff Holm, Moll, and Natvik (2019) who use administrative data for more recent year to show this fact using Norwegian households. Using these insights from the data, I approximate individual flows as a constant (to be the average saving rate of that year) times individual personal income.

Capital Gains: Capital gains are always difficult to measure, even using very high-quality administrative data. That is because it contains both realized and unrealized capital gains, and measuring things that are not realized can be tricky. Thanks to the data on flows, I am able to use an approximate measure for capital gain's returns across the wealth distribution using a pseudo panel technique and use it for calculating gross ROA. [6] There is another way that I use as a robustness check and that is using average asset class capital gains and using the portfolio shares to measure the capital gains.

Different data sets collect data on U.S. households' asset allocation and capital income like PSID or modern SCF. However, none of these two go before the 80's. The only known options are SFCC and historical waves of SCF. There are also imputed data sets in which they capitalize income to get wealth data. There are also ad-hoc data sets like university endowments, foundations and charities, and estate tax data which might be used. None of these known data sets (at least their public versions) give us full coverage for a long period like 1950 till 1980, however, I try to make the best use of the available years.

In Appendix A, I explain more about different data sets that I use.

2.3 Empirical Results

Pre and Post 1980 Returns across the wealth Let's review the findings of many recent papers. Figure 2 plots the average return on wealth for U.S. households with positive net wealth which is measured using survey data related to years after 1980.

[5] There is flow data for a few more years, but unfortunately, we do not have asset class holdings for those years. There is also a short panel survey for the years 1962 and 1963 that I have used.

[6] If we assume that the number of households that move between deciles of wealth in one year is negligible, we can aggregate households at the decile level and then treat the data as panel data and leverage the knowing of flows to measure return on capital gains:

$$r_{it}^{kg} = \frac{w_{i,t+1} - w_{it} - f_{it}}{w_{it} + \frac{1}{2} f_{it}} \tag{3}$$

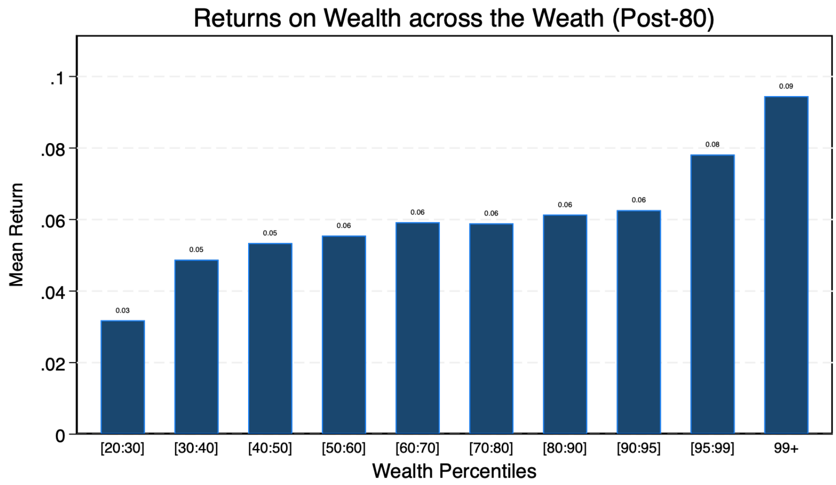

Figure 2: Average realized return on wealth for the years after 1980 using SCF

Now let's plot this graph using sample years before 1980. Figure 3 plots the same statistic for pre-1980 periods (which goes back till 1949).

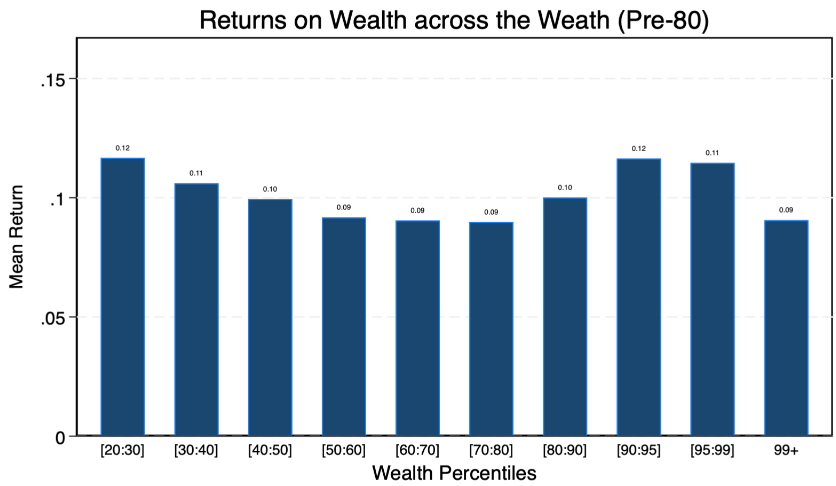

Figure 3: Average realized return on wealth for the years after 1980 using historical waves of SCF

This new evidence shows that returns have not always been scale-dependent in the way they are today. Specifically, if we look at the long-term average, before the 80s they are decreasing to scale and after the 80s they are increasing to scale.

Although, this finding seems a bit counter-intuitive in the beginning, but is in a way consistent with the observed changes in inequality measures in these two periods: before the 80s, top1-% wealth share decreased and after the 80s it increased (Saez and Zucman (2016); Greenwald et al. (2023)).

In the appendix, you can find plot the graphs for Decomposition of Returns across rkg and rdiv.

2.4 Scale-dependent Returns

Now let's go the main empirical findings. Back to the long standing question of "What explains the cross-section of households' returns?", we can run regressions like:

$$return_{i,t} = \beta_1 Wealth_{i,t} + \beta_2 Portfolio_{i,t} + \beta_3 X_{i,t} + \alpha_t + \epsilon_{i,t} \tag{4}$$

$Portfolio_{i,t}$ is a vector which includes asset shares for different asset classes (equity, housing, business, ...) and $X_{i,t}$ includes control varibles like age, education, race, etc. Table 1 summarizes the regression of returns on wealth for the pre-80's period. As we can see, the effect of wealth on returns is negative. That is, wealthier people had less returns on their wealth, which is the opposite of the findings for recent years.

	(1) Return	(2) Return	(3) Return
Log(Wealth)	0.00309***	0.00273***	0.0115***
	(0.000106)	(0.000114)	(0.000132)
leverage	0.00363**	0.00852***	
	(0.00116)	(0.00119)	
age of head		0.000142***	0.00000675
		(0.00000712)	(0.00000927)
head at least some college		0.000372	-0.00464***
		(0.000199)	(0.000306)
black or white head		-0.00227***	-0.00296***
		(0.000249)	(0.000400)
Portfolio Share	Yes	Yes	No
Year FE	Yes	Yes	No
Portfolio Share × Year FE	Yes	Yes	No
Observations	179661	179661	179661
R^2	0.699	0.700	0.102

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Table 1: Regression of households returns for the post-80 period

This is nothing surprising with what other people found. But, if we run the same regression for pre-80 period, we can a surprising result: the scale dependence coefficient gets negative. (Table 2)

	(1)	(2)	(3)
	return	return	return
Log(Wealth)	-0.000585*	$\boxed{\text{-0.00124***}}$	0.0156***
	(0.000281)	(0.000294)	(0.000255)
leverage	0.00430***	0.00912***	
	(0.000769)	(0.000862)	
age of head		0.000224***	0.0000359*
		(0.0000147)	(0.0000160)
head at least some college		0.00403***	-0.00612***
		(0.000520)	(0.000663)
black or white head		-0.00325***	0.00700***
		(0.000784)	(0.000949)
Portfolio Share	Yes	Yes	No
Year FE	Yes	Yes	No
Portfolio Share × Year FE	Yes	Yes	No
Observations	179837	179837	179857
R^2	0.524	0.526	0.073

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Table 2: Regression of households returns for the post-80 period

For the whole period, I get a non-significant beta. (Table 3)

	(1) Return	(2) Return	(3) Return
Log(Wealth)	0.000632***	-0.0000322	0.0140***
	(0.000191)	(0.000206)	(0.000197)
leverage	0.00470***	0.00924***	
	(0.000764)	(0.000828)	
age of head		0.000208***	-0.00000640
		(0.0000114)	(0.0000124)
head at least some college		0.00326***	-0.0102***
		(0.000381)	(0.000466)
black or white head		-0.00295***	0.00141*
		(0.000546)	(0.000669)
Portfolio Share	Yes	Yes	No
Year FE	Yes	Yes	No
Portfolio Share $\times$ Year FE	Yes	Yes	No
Observations	359498	359498	359518
R^2	0.543	0.545	0.068

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Table 3: Regression of households returns for whole period

*The time-series of yearly betas:

2.5 Robustness of the Empirical Results

I do a few robustness checks in the appendix. The first one is for the approximation of flows which we can see how well it works for the years that we know the flows. The second robustness check is for the way I measure the capital gains. The third one is about including other assets like pension funds, or vehicles. The results are robust to all of them.

2.5.1 Evidence from other data sets

For recent years, there are other data sets that one can cross-check the observed increasing with scale property of returns. However, for before 80's, it is a bit difficult to find other publicly accessible data sets. However, there some pieces of evidence for decreasing with scale property of returns for before 80's period.

* Us foundations data: could be uses for the post 80's. Make sure they measure the returns properly.

* matched Estate Tax

* Uni endowment data

* 80's Panel data
* Shiller housing data

2.6 Explaining the Evidence

So far, I uncovered the changes in the sign of scale dependence of returns. But, how can we explain that? In the next section, I will provide some empirical evidence that there is indeed no scale premium in returns and what we are detecting as scale-dependent returns is coming from the compensation for interest rate risk that households are taking. But, as it is very difficult to control for in our cross sectional regressions, we detect it as a scale-premium. If the realization of interest rate risk is good (interest rates going down), as it was for the post 80 period, it will be a positive coefficient and if it bad (interest rates going up) as it was for the pre 80 period, it will be a negative coefficient.

3 Interest Rate Risk and Returns

In this section, I will review the concept of the duration of an asset and a portfolio of assets. I will also provide insights into how changes in interest rates change the returns of assets based on their duration. Then, I will explain my empirical evidence on the heterogeneity of the duration of asset holdings of households across the wealth distribution.

3.1 Duration and Returns

The duration of an asset that generates future cash flows in the future is generally defined as the weighted average of the times until those cash flows are received. In different settings, different but quite similar measures of duration or price sensitivity to interest rate changes can be defined. Maybe, the most famous measure for duration is Macually duration defined for an asset with deterministic cashflows.

More formally, Macaulay duration (Macaulay (1938)) for an asset in a deterministic setting with future cash flows $\{x_t\}$ is defined as:

$$D := \frac{\sum_{t=0}^{\infty} t \times R^{-t} x_t}{P_0}. \tag{5}$$

where $R = 1 + r$ and r is the annualized discount rate at all maturities. Note that in this setting

$$P_0 = \sum_{t=0}^{\infty} R^{-t} x_t. \tag{6}$$

Duration is closely related to how sensitive an asset's price or valuation is to changes in interest rates or yields. One can prove that:

$$\frac{\partial \log P_0}{\partial \log R} = -D \tag{7}$$

Proof.

$$\frac{\partial P_0}{\partial R} = \sum_{t=0}^{\infty} -t \times R^{-t-1} x_t = \frac{-1}{R} \sum_{t=0}^{\infty} t \times R^{-t} x_t = \frac{P_0}{R} \times -D \tag{8}$$

which gives us:

$$\frac{\partial P_0}{\partial R} \times \frac{R}{P_0} = \frac{\partial \log P_0}{\partial \log R} = -D \tag{9}$$

□

A few papers (Greenwald et al. (2023), Catherine et al. (2023)) use this property of duration and the fact that richer people's portfolios have higher average durations to discuss the dynamics of inequality.

There is another aspect of duration, and that is related to returns. In the mentioned deterministic economy, all the assets, irrespective of their duration, have the same return R, which is because of the absence of arbitrage. However, assume a transition through which R unexpectedly changes to R'. During this transition period, the return on assets will be different based on their duration. The price of assets with a higher duration is more sensitive to this change in the interest rates.

For two assets A and B that have the same amount of cash flow in all the periods and are just different in their duration, in every period, they have the same return. If the economy is hit by an unexpected (MIT) shock, i.e, a permanent unexpected increase in R, then during the transition period:

$R^A > R^B$. Furthermore, it is also true along both dimensions of returns, that is: $R^{kg,A} > R^{kg,B}$ and $R^{div,A} > R^{div,B}$ in the new steady state, the returns are again the same.

Theorem 1. If the economy is hit by an unexpected (MIT) shock, i.e, a permanent unexpected increase in R, then during the transition period:

$$\frac{\partial \log return}{\partial \log R} = 1 - D \tag{10}$$

Proof.

$$Return_t(R') = \frac{x_t + P_{t+1}(R')}{P_t(R)} = \frac{R' P_t(R')}{P_t(R)} \tag{11}$$

$$Return_t(R) = R \tag{12}$$

$$\log return(R') - \log Return_t(R) = \log R' - \log R + \log P_t(R') - \log P_t(R) \tag{13}$$

$$\partial \log return = \log return(R') - \log Return_t(R) = \partial \log R - D\partial \log R \tag{14}$$

So,

$$\frac{\partial \log return}{\partial \log R} = 1 - D \tag{15}$$

. □

This theorem explains the interest rate mechanism for having scale-dependent returns. To see this more clearly, assume two simple risk-free assets:

- a one-period bond, paying 1 unit of consumption good next period
- an infinite-period bond, paying 1 unit of consumption good from the next period on (till infinity)

Price of these assets are:

- one-period bond: $\frac{1}{1+r}$
- infinite-period bond: $\frac{1}{r}$

Now assume that r increases unexpectedly and permanently to r' (MIT shock). "during" the transition period, returns will be:

- one-period bond: $1 + r'$
- infinite-period bond: $r' + \frac{r}{r'} < r' + 1$

* definition of duration for stochastic cash flows

* theorem with stochastic cash flows

3.2 Measuring Duration of Household's Portfolios

There are different ways to define and measure the duration of an asset. Greenwald et al. (2023) measure duration using Macualy duration for each asset class (they also use a simple Gordon growth model for asset class duration), Greenwald et al. (2023) (older draft), uses an affine asset pricing model to estimate the SDF. Catherine et al. (2023) uses sensitivity to interest rates. They also include social security.

As within asset class differences in duration is key in the cross sectional of returns and the observed scale-dependent returns, none of the methods mentioned above will work as the assume an average duration for each asset class. Instead, I directly measure duration of households total wealth using a smiple Gordon grwoth model. Unlike other methods, this approach can give us a rough idea of within asset class duration, and how it varies with wealth. On the down side, it is a rough approximation as it uses strong assumptions on the cash-flows and interest rate (and its term structure). Having said that, the approximated duration, and its cross-sectional differences across the wealth are comparable to those of Greenwald et al. (2023).

Gordon Duration: Like in the Gordon Growth Model, if we assume that the cash flows grow at a constant rate g and the interest rate is constant $R = 1 + r$:

$$D_t = \frac{1 + r}{r - g} = 1 + \frac{P_t}{Div_t}$$

which we can apply to households:

$$D_t \approx 1 + \frac{Wealth_t}{Dividend_t}$$

3.3 The Cross-section of duration

For showing that richer people, within asset class, have a higher duration, I run a regression of duration on wealth and portfolio controls. Portfolio controls take out the effect of across-asset-class differences, and the remaining part, which is coming from within asset class differences in duration, is increasing in wealth.

In Table 4 I run the regression:

$$Duration_{i,t} = \theta_1 Wealth_{i,t} + \theta_2 Portfolio_{i,t} + \theta_3 X_{i,t} + \gamma_t + \epsilon_{i,t} \tag{16}$$

	(1) Duration Pre-80	(2) Duration Post-80	(3) Duration Overal
Log(Wealth)	$\boxed{3.582^{***}}$	$\boxed{1.288^{***}}$	$\boxed{2.826^{***}}$
	(0.0974)	(0.0828)	(0.0714)
leverage	-6.568***	-3.917***	-6.685***
	(0.609)	(0.350)	(0.607)
age of head	-0.0238***	-0.0449***	-0.0318***
	(0.00438)	(0.00425)	(0.00346)
black or white head	2.096***	1.122***	1.698***
	(0.149)	(0.148)	(0.111)
head at least some college	-0.665***	-0.829***	-0.820***
	(0.166)	(0.109)	(0.122)
Constant	22.83***	31.50***	30.06***
	(1.430)	(1.313)	(1.254)
Portfolio Share	Yes	Yes	Yes
Year FE	Yes	Yes	Yes
Portfolio Share × Year FE	Yes	Yes	Yes
Observations	132205	142348	274553
R^2	0.280	0.393	0.326

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Table 4: Regression of Households Duration

Practical issues: Note that the Gordon duration is valid for the case where dividend income and asset's value are positive. So, I remove all the observations with negative values for their equivalents (it is about 2% of the total observations, which can be thought of as outliers.). I use the formula

$$Gordon\ Duration = \frac{w_t + \frac{1}{2} f_t}{y^{Div}} \tag{17}$$

as the dividend income is generated by the amount in the numerator. Another note is that, in this rough approximation, $D = 1 + \frac{1}{r^{Div}}$ and as for most of the cases $-1 < y^{Div} < 1$, duration will be highly non-linear (just like $1 + \frac{1}{x}$ for x close to zero. For this reason, I use a log transform, which helps with this issue.

Another note, is that regarding adding Duration or residual duration (after taking out portfolio controls) as a regressor to the duration regression will be misleading as it is just a rough approximation and it correlates with return (especially the dividend return) and wealth as well. But, ideally, if we have a proper measure of duration, it will be a great test to add that to the return regression.

* graph of cross-section of duration

* time series of duration

* Duration and life-cycle

When one looks at data, rather than wealth which positively correlates with duration, age also correlates negatively with duration. So, there is also a life-cycle component in duration heterogeneity across households.

Figure 3: Financial Duration by Net Wealth Percentiles and by Age

(a) By Net Wealth Bin

(b) By Age

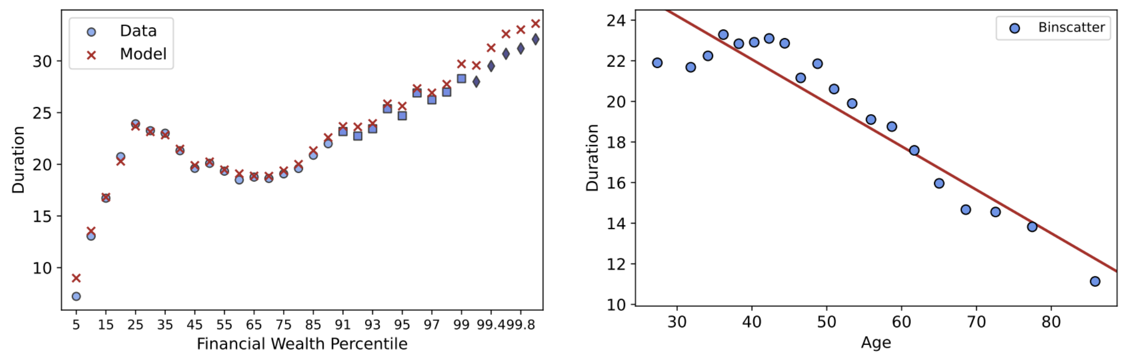

Figure 4: duration of net wealth across net wealth and age (after controlling for wealth) Source: Greenwald et al. (2023)

What we learn from this regression is that θ has always been positive; that is, richer people always have assets of higher (within asset class) duration.

3.4 Interest Rate Trends

An important change that coincides with the direction of changes in inequality and also the scale dependency of returns is the long-term interest rate. Figure 5 plots the close relation.

Figure 5: 5-year real risk-free interest rate based on the methodology in Greenwald et al. (2023)

Measuring real interest rates for older years is not straight forward. Because of absence of a clear measure of expected long term inflation (like TIPS, for recent years). I follow the methodology of Greenwald et al. (2023) to build an empirical asset pricing model and use that to estimate the price of a risk free bond (and hence the interest rate) using that. There are other ways to do that like Catherine et al. (2023) or JPayne. (see Appendix for a description of this methododology)

as you can see fro the figure, real interest rate was on an increasing trend from 1950 till 1980, when it start a decreasing return.

4 Interest rate risk and portfolio choice

In this section, I present a model to rationalize the findings that wealthy individuals exhibit a preference for high exposure to interest rate risk, and how this fact, combined with the realization of interest rate risk, has led to the observed positive and negative scale-dependence of returns.

I then provide empirical evidence on the cyclicality of labor income risk (returns on human capital) and how this risk is more cyclical for the super-rich. Then, I provide evidence that interest rate risk is countercyclical (short-term interest rates are cyclical themselves, but the risk is counter-cyclical).

Then I put these evidence into a potfolio choice model with risky human capital and establish the result that richer people choose higher duration because of their higher cyclycality of labor income. Finally, I feed in the realization of interest rate risk and get the observed phenomenan of scale-dependence of returns.

4.1 Model

In this section, I want to connect the choice of interest rate exposure (duration) to labor income risk. To get the idea, based on the Campbell and Viceira (2002) framework, I provide a simple formula for the optimal share of wealth invested in the risky asset (which can be thought of an asset with high duration risk).

4.2 Environment

A household has time-seperable CRRA utlity and live for ever. He or she receives some labor income, and should decide on his consumption and portfolio choice for his savings. He has access to two financial assets:

- Short-term bond (risk-free, 1-period duration)
- Long-term bond (n-period duration)

The assets are designed in a way so the household can choose any duration through its portfolio choice between 1 and n. The Risk-free rate, which will be the return on the short-term bond, follows an AR(1) process:

$$r_{f,t+1} = (1-\varphi)\bar{r}_f + \varphi r_{ft} + \sigma_r \epsilon_{r,t+1} \tag{18}$$

For the pricing of the long-term bond, I assume the expectation hypothesis holds, which pins down the term structure and the return of the long-term bond[7]:

$$r_{n,t+1} = r_{ft} + \mu_n - \sigma_n \epsilon_{r,t+1}, \tag{19}$$

(Where, $\sigma_n = \frac{1-\varphi^{n-1}}{1-\varphi}\sigma_r$)

The other feature of the model is that household's labor income is risky. Following Viceira (2001), I assume the income process:

$$L_{t+1} = L_t \exp(g + u_{t+1}) \tag{20}$$

Or in the log format:

$$l_{t+1} = l_t + g + u_t \quad ; \quad u_{t+1} \sim NIID\left(0, \sigma_l^2\right) \tag{21}$$

Furthermore, I assuume:

$$\mathrm{Cov}_t\left(u_{t+1}, \epsilon_{t+1}\right) = \sigma_{rl} \tag{22}$$

4.3 Duration choice

HHs optimization problem with constant relative risk aversion (CRRA) utility function, γ the coefficient of relative risk aversion:

$$\max_{\{C_t,\pi_t\}_{t=0}^{\infty}} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma} \tag{23}$$

$$s.t. \quad W_{t+1} = \left(W_t + L_t - C_t\right) R_{w,t+1} \tag{24}$$

where, $R_{w,t+1} = \pi_t\left(R_{n,t+1} - R_f\right) + R_f$.

[7]See Catherine et al. (2023) Appendix B.2 for the proof. Also, note that is shouldn't be mistaken with long-term ex ante interest rate. This is just a one-period expected return on an n-period bond and is quite different from the n-period ex ante yield or the expected long-term interest rate.

Proposition 1 The approximate portfolio choice in this model will be

$$\pi_{it} = \underbrace{\frac{1}{\gamma b_1}\frac{\mu_n + \frac{1}{2}\sigma_n^2}{\sigma_n^2}}_{\text{myopic demand}} + \underbrace{(1-\frac{1}{\gamma})\frac{\bar{b}_2}{\gamma b_1}\frac{\sigma_r}{\sigma_n}}_{\text{hedging demand}} + \underbrace{\frac{(1-b_1)}{\gamma b_1}\frac{\sigma_{rl}}{\sigma_n^2}}_{\substack{\text{human capital}\\ \text{substitution}}} \tag{25}$$

where $0 < b_1 < 1$ and $\bar{b}_2$ are constants defined in the appendix.

Proof. See appendix B.

As we can see from this equation, choice of duration has three components. The first one is myopic demand, which comes from the fact that the long-term asset has an expected return premium and the risk-averse household will demand this asset. The second term, the hedging demand, is coming from the fact that the return on long-term bond is time-variyng and so is the investment opportunity set. Any househould (with $\gamma \neq 1$) will try to take advantage of this change through his or her hedging demand. The last component, which is of key importance in this paper, is the human capital substitution demand.

Human capital substitution term in the duration choice is telling us that the household will take into account his or her human capital asset (which is a non-tradable asset) when choosing for duration. Especially, if his labor income is in a way that is very much correlated whith the return on the short therm asset, he will choose a higher duration to hedge that risk.This term

$$\frac{(1-b_1)}{\gamma b_1}\frac{\sigma_{rl}}{\sigma_n^2} \tag{26}$$

is proportional to the regression hedge ratio of labor income ($\frac{\sigma_{rl}}{\sigma_r^2}$), which is the slope in the regression of labor income shocks onto unexpected interest rate shocks.[8] I call this value as *interest rate beta of labor income* β_r, which can be measured for people in each earnings percentile g :

$$\Delta l_{i,t} = \alpha_g + \beta_{r,g}\Delta r_t + \xi_{i,t} \tag{27}$$

where $l_{i,t}$ is the log of average (labor) income of households in percentile g, and r_t is the log of gross short-term interest rate.[9]

[8]Note that since in this model, log labor income is an AR(1) process with fully persistent shock (random walk), for the empirical measurement of σ_{rl} we simply have $Cov(u_t, \epsilon_t) = Cov(\Delta l_t, \Delta r_t) = \sigma_{rl}$.

[9]Note that taking log and then the first difference of labor income and gross interest rate is making them stationary. The results are robust to using band pass filters like Christiano-Fitzgerald filter (Christiano and Fitzgerald (2003)) for extracting the cycle component of interest rate, ot later GDP.

4.4 β_r across the earning percentiles

Note that before 1951 (mainly during World War II and the short period after it, interest rates were not allowed to move freely, so it is not a good idea to think of them as the return on short-term investment. After 1951, when the treasury-fed accord happened, interest rates where allowed to move independent of government financial needs. (See Mueller (1952) for more information.)

(a) Interest rate beta of labor income (b) Interest rate beta of total income

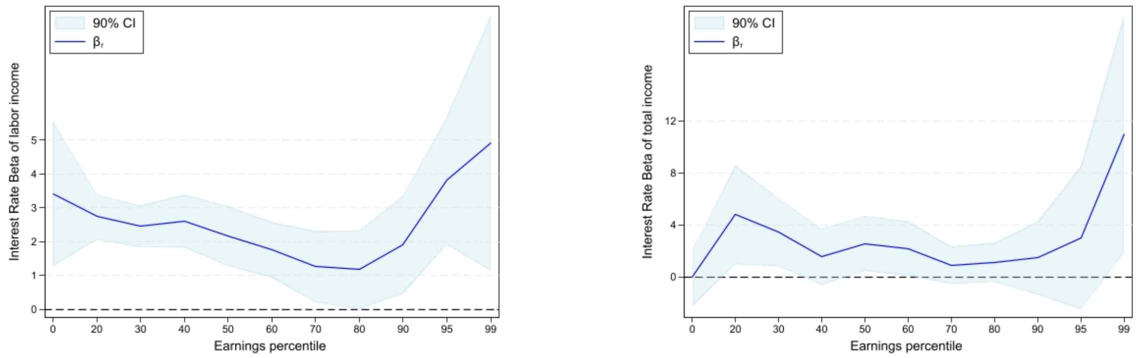

Figure 6: The figures plot the interest rate sensitivity of income across earning percentiles. The shaded area is a 90% confidence interval.

4.5 Intuition for β_r

For getting a better intution of the notion of interest rate beta of income, it is better to think of it as GDP beta of income, introduced by Guvenen, Schulhofer-Wohl, Song, and Yogo (2017):

$$\Delta l_{i,t} = \alpha_g + \beta_{y,g} \Delta GDP_t + \zeta_{i,t} \tag{28}$$

Figure **??** plots the GDP beta of households earnings growth across the earning percentiles. It basically captures how much income change correlates with changes in GDP. If it is higher, it means the cyclicality of the labor income in higher. As we can see, it is a U-shaped curve.

Although my data set for this task is a pseudo panel, the results are quite comparable to Guvenen et al. (2017), who uses admin panel data of social security. And also to Amberg, Jansson, Klein, and Picco (2022) (Appendix C), who uses admin data in Sweden. [10]

[10] Another seemingly similar to the GDP beta results, is Parker and Vissing-Jorgensen (2009), but note they regress income fluctuation on aggregate income fluctuation and not GDP, which is a different thing, and is more related to the question of which groups' fluctuations explain more of the aggregate fluctuations. They also have different results for before and after 1980, but the GDP beta looks almost the same before and after 1980.

(a) GDP beta of labor income

(b) GDP beta of total income

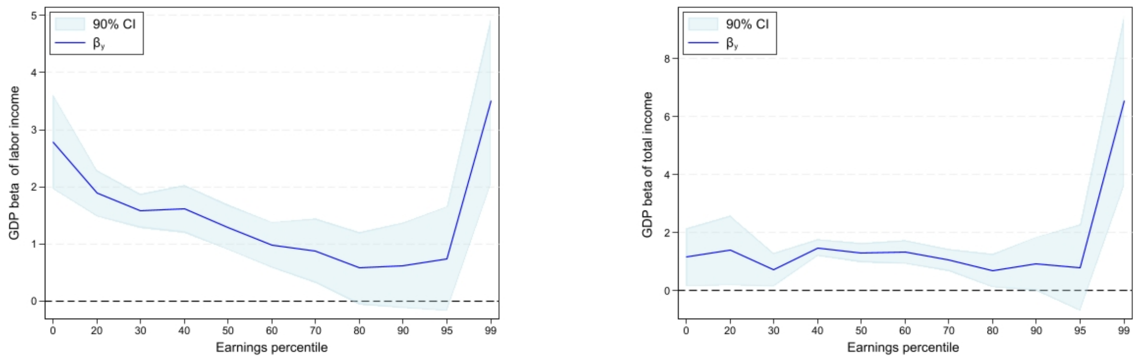

Figure 7: The figures plot the GDP sensitivity of income across earning percentiles. The shaded area is a 90% confidence interval.

The close connection between GDP beta and interest rate beta of income is through the cyclical behavior of the economic variables. GDP and interest rate are highly correlated and co-move with each other across the cycles.

4.6 β_r across other dimensions

Talk about differences across age, gender, race, and college education.

4.7 Calibration

Calibrate the model to see the implied choice of duration across the wealth distribution

5 Conclusions

In this paper, I try to provide evidence that the observed scale-dependent property of returns on wealth is actually comming from the fact the in our cross-sectional regressions, we cannot control for the interest rate risk. This fact, as well as the other fact which I provide evidence for, that richer people choose assets (within asset class) with higher duration can explain the observed posiitive scale-dependent returns for the post-80 period observed in the US and many other countries.

With the help of some survey data from historical waves of SCF, I managed to measure returns on wealth for households for some years before 1980, a period with increasing interest rates. The measurements reveal that the observed scale-dependent returns become negative for that period.

I connect this evidence with the literature on the dynamics of inequality and changes in interest rates. Richer people have assets with higher durations, so they experience valuation losses if the interest rate goes down.

I build a model to try to explain this phenomenon based on other macro trends in the long run.

In future research, one might experience the welfare consequences of changes in returns and implications for tax system design.

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Appendix

A Data

A.1 SCF

Survey of Consumer Finances (SCF) is a triennial cross-sectional household survey of U.S. households created by the Board of Governors of the Federal Reserve Board from 1983 to 2022. It covers many useful variables like asset holdings from different asset classes and the income generated by those. SCF+ is an extension of SCF going back to 1949 (Kuhn et al. (2020)). Old surveys and some other databases are used to create this extension.

A.2 Data Cleaning

Outlier Removal I trim the distribution of returns in each year and for each wealth decile at the top and the bottom by 3%. This ensures that there are no outliers polluting the estimates of the regression of returns and aims to reduce measurement errors.

Adjusting the Weights adjusted weights from SCF+ and divide by the total number of yearly observations

Approximating the flows I use the approximation of:

$$Individual\ flow_i \approx Saving\ Rate \times Disposable\ Income_i \tag{A.1}$$

which is in sprit of the findings of Fagereng et al. (2019), and in lines with the flow data that I extract from historical waves of SCF.

Saving rate out of disposable income: For measuring the aggregate active saving rate, I use the ratio of aggregate personal savings to aggregate disposable income. More precisely, I use Households and Nonprofit Organizations; Personal Saving Excluding Consumer Durables and Federal Government Life Insurance Reserves and Railroad Retirement Board and National Railroad Retirement Investment Trust Pension Fund Reserves (NIPA), Transactions (BOGZ1FA156007015Q) devided by Households and Nonprofit Organizations; Disposable Income, Net (IMA), Transactions (BOGZ1FU156012095Q) as saving rate, which matches quite well the Personal Saving Rate (PSAVERT), but has a longer duration as I need it here.

Owner-occupied housing rent I use the time series of housing yield from Jordà-Schularick-Taylor Macrohistory Database (Jord‘a, Knoll, Kuvshinov, Schularick, and Taylor (2019)). I also extend it for the post-2020 period, using the dynamics of the housing price to rent ratio index.

Tax SCF does not include data on how much tax each household pays. To approximate the tax payment, I use Distributional National Accounts (DINA), which is a synthetic data based on Internal Revenue Service (IRS) data introduced in Piketty, Saez, and Zucman (2017) and develop linear models based on observed income variables in SCF, to predict each households paid tax based on their observed income. I use these models to approximate household tax payments.

Disposable Income Household disposable income, following Fagereng et al. (2019), is defined as the sum of labor income, business income, capital income, transfers, and housing service flows, minus taxes.[11]

Average Asset-class Capital Gains For stocks (public equity), real estate, businesses (private equity), bonds, and mutual funds, I approximate the average capital gains to use in robustness checks. For stocks and real estate, I use changes in price indeces, for businesses, I use data from flow of funds to back out capital gains[12], for bonds, I assume and average duration of 4 years and use Duration formula to back out the capital gains, and finally for mutual funds, I use the average of stocks and bonds capital gains with respective weights of 60 and 40 percent.

Interest Rate Data For the nominal short-term interest rate, I use the series of 3-Month Treasury Bill Secondary Market Rate, Discount Basis (TB3MS) from Fred. For backing out the real short-term interest rate, I use the 1-year inflation expectation series estimated by Hall, Payne, Sargent, and Szőke (2019) (and the series 1-Year Expected Inflation (EXPINF1YR) from Fred for the few recent years that are not included in the former). For the long-term real interest rate, I use the results of the empirical estimation of 5-year real interest rate of Greenwald et al. (2023) (and for hte few recent years that are not inclued there, I use the series Market Yield on U.S. Treasury Securities at 5-Year Constant Maturity, Quoted on an Investment Basis, Inflation-Indexed (DFII5) from Fred). For the nominal long-term interest rate, I use the yield on 10-year US treasury bond from Damodaran's dataset.

Pseudo-panel Capital Gains For the approximation of the rate of return of capital gains, I assume the Pseudo-panel assumption on wealth percentiles. That is, treating the average observations of each percentile (or decile) as a single observation in a panel setting. Then, it is possible to approximate the rate of return on capital

[11]For some years when DINA is not available, I use data for close years to do the approximation of the linear model.

[12]I use the series availible on FRED for Households and Nonprofit Organizations; Corporate Equities; Asset (Level and Tranactions: HNOCEAQ027S and HNOCESQ027S) and Households and Nonprofit Organizations; Proprietors' Equity in Noncorporate Business (Level and Transactions: HNOPEBA027N and HNOPEUQ027S). I use the formula $r^{KG} = \frac{w_{t+1} - w_t - f_t}{w_t + \frac{1}{2} f_t} = \frac{w_{t+1} - \frac{1}{2} f_t}{w_t + \frac{1}{2} f_t} - 1$, where w_t is the level of holdings at time t and f_t is the flow (transactions) at time t

gains for each percentile (or decile) using my approximation of flows.[13]

A.3 Variance of Returns

A.4 Importance of Flows in ROA Formula

Fagereng et al. (2020) show (in their appendix) that if the multiplier of flows in the denominator is not 0.5, our measurement of returns will be wrong. This issue makes the use of non-panel datasets (like SCF) difficult.

$$r_{it} = \frac{R(w_{it}) * (w_{it} + \frac{1}{2}F_{it})}{w_{it} + \lambda F_{it}} = R(w_{it}) + \frac{(\frac{1}{2} - \lambda) * F_{it} R(w_{it})}{w_{it} + \lambda F_{it}} = R(w_{it}) + (\frac{1}{2} - \lambda)\frac{R(w_{it})}{\frac{w_{it}}{F_{it}} + \lambda} \tag{A.2}$$

A.5 Other Formulas for Returns

One can also derive and use other equivalent formulas for ROA dependent on the observed variables. For example, if we observe the end of the period t's wealth, we can rewrite ROA formula in terms of that. We know that $w_{i,t+1} = w_{it} + y_{it}^{kg} + f_{it}$, where $w_{i,t+1}$ is the end of the period t wealth or the wealth at the beginning of period $t+1$. Using this identity, we can have:

$$r_{it} = \frac{y_{it}^{div} + y_{it}^{kg}}{w_{i,t+1} - \frac{1}{2}f_{it} - y_{it}^{kg}} \tag{A.3}$$

Or if one has panel data, then there is no necessary need to know flows directly. we can use this equivalent definition:

$$r_{it} = \frac{y_{it}^{div} + y_{it}^{kg}}{\frac{w_{it} + w_{i,t+1}}{2} - \frac{1}{2}y_{it}^{kg}} \tag{A.4}$$

Decomposition of Returns: To analyze which components are serving more the scale dependency of returns, it is a good idea to decompose the returns to their components:

- a) Dividend Income Return

$$r_{it}^{div} = \frac{y_{it}^{div}}{w_{it} + \frac{1}{2}F_{it}} \tag{A.5}$$

[13] As the data is not annual for most of the years, I use linear approximation for the flows of the years in between.

- b) Capital Gain's Return

$$r_{it}^{kg} = \frac{y_{it}^{kg}}{w_{it} + \frac{1}{2}F_{it}} \tag{A.6}$$

- c) Cost of debt

$$c_{it} = \frac{y_{it}^{b}}{w_{it} + \frac{1}{2}F_{it}} \tag{A.7}$$

From $w_{i,t+1} = w_{it} + y_{it}^{kg} + f_{it}$ we know that $y_{it}^{kg} = w_{i,t+1} - w_{it} - f_{it}$. We can also have another representation of y_{it}^{kg}, using this formula $y_{it}^{kg} = w_{it}r_t^{kg} + \frac{1}{2}f_{it}r_t^{kg}$ which is based on our definition of ROA, specified to capital gains:

$$r_{it}^{kg} = \frac{y_{it}^{kg}}{w_{it} + \frac{1}{2}f_{it}} \tag{A.8}$$

so, by mixing these two recent equations we will get:

$$r_{it}^{kg} = \frac{w_{i,t+1} - w_{it} - f_{it}}{w_{it} + \frac{1}{2}f_{it}} \tag{A.9}$$

if we have r_{it}^{kg}, then we can calculate the aggregate ROA using this formula:

$$r_{it} = \frac{y_{it}^{div} + y_{it}^{kg}}{w_{it} + \frac{1}{2}f_{it}} = \frac{y_{it}^{div}}{w_{it} + \frac{1}{2}f_{it}} + \frac{y_{it}^{kg}}{w_{it} + \frac{1}{2}f_{it}} = \frac{y_{it}^{div}}{w_{it} + \frac{1}{2}f_{it}} + r_{it}^{kg} \tag{A.10}$$

$$= \frac{y_{it}^{div}}{\frac{1}{1+r_{it}^{kg}}(w_{i,t+1} - \frac{1}{2}f_{it})} + r_{it}^{kg}$$

where I have replaced $w_{it} = \frac{1}{1+r_{it}^{kg}}(w_{i,t+1} - f_t(1 + \frac{1}{2}r_{it}^{kg}))$

A.6 DINA

- use it just for a cross-check of the results.
- the good this is that using this capitalization method, we do not need to know the flows as it already estimates $(w_t + w(t+1)/2)$
- issue with capital gains: I assume it only takes into account realized capital gains. You maybe can add it on your own approximately
- use it to study post-tax returns
- note that returns within asset classes are the same for everybody, mechanically by the capitalization method that they use.

A.7 PSID

A.8 Robustness Checks

A.8.1 How much flows matter?

Using the available flows data for a few years, it seems that flows might affect the level of returns, but the slope of cross section of returns is not effected much.

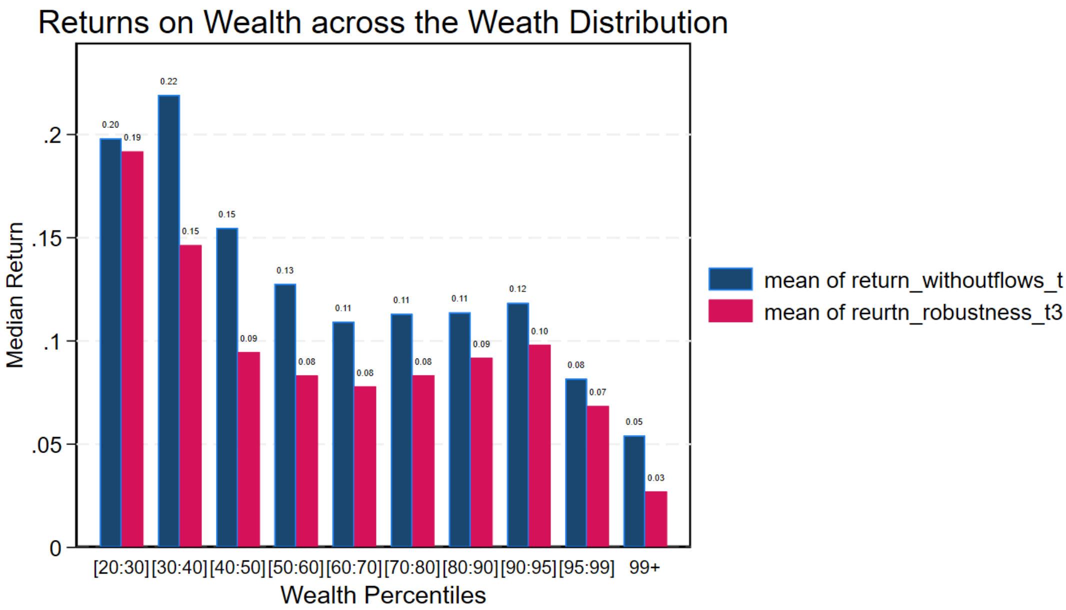

Figure A.1: Robustness check for removing flows

A.8.2 What about pensions or social security?

"Social Security and Trends in Wealth Inequality" SYLVAIN CATHERINE, MAX MILLER, and NATASHA SARIN (JF2024) argue that adding social security to wealth while measuring wealth dynamics might change the results. Fagereng et al. (2020) argues that including pension wealth in measuring returns does not affect return inequality for people above the median wealth. (section 3.3.4 in their paper. They conclude: As expected, the adjustment reduces inequality in returns (and wealth) by increasing the return at the bottom of the distribution (where pension wealth is a quantitatively important wealth component), but it has virtually no effect above median wealth.) SCF+ dataset has the pension variable from 1983 onward. Kuhn et al. (2020) argues (in section 2.1, footnote 8) that according to the financial accounts of the United States, this variable makes up a small part of household wealth before the 1980s, so missing information before 1983 is unlikely to change the picture meaningfully.

Try to follow Fagereng et al. (2020) METHODOLOGY in the mentioned section and use SCF+ data to implement this robustness check. In case, you can use the methods in "Social Security and Trends in Wealth Inequality" SYLVAIN CATHERINE, MAX MILLER, and NATASHA SARIN (JF2024).

A.8.3 Confidence intervals for weights

Plot confidence intervals like how they do it in Kuhn et al. (2020).

B Model Solution

HHs optimization problem with constant relative risk aversion (CRRA) utility function, γ the coefficient of relative risk aversion:

$$\max_{\{C_t,\pi_t\}_{t=0}^{\infty}} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma} \tag{B.1}$$

$$\begin{aligned} s.t. \quad & W_{t+1} = \left(W_t + L_t - C_t\right) R_{w,t+1}, \\ & W_{t+1} > 0. \end{aligned} \tag{B.2}$$

where, the gross return on wealth is $R_{w,t+1} = \pi_t \left(R_{n,t+1} - R_f\right) + R_f$. I will denote the logarithm of gross returns with small letter: $r_{w,t} = \log(R_{w,t})$ So, for the return of the short-term bond, we have:

$$r_{f,t+1} = (1-\varphi)\bar{r}_f + \varphi r_{ft} + \sigma_r \epsilon_{r,t+1} \tag{B.3}$$

The return of the long-term bond :

$$r_{n,t+1} = r_{ft} + \mu_n - \sigma_n \epsilon_{r,t+1}, \tag{B.4}$$

(Where, $\sigma_n = \frac{1-\varphi^{n-1}}{1-\varphi}\sigma_r$). And, for the return of the wealth, we will have:

$$r_{w,t+1} \approx r_{f,t} + \pi_t \left(r_{n,t+1} - r_{f,t}\right) + \frac{1}{2}\pi_t \left(1-\pi_t\right) \mathrm{Var}_t\left(r_n\right) \tag{B.5}$$

which will be precise if time is continuous. (See Campbell and Viceira (2002), Appendix, pages 2-5.)

B.1 HHs FOCs

I start by writing the Lagrangian:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \left\{ \beta^t \frac{C_t^{1-\gamma}}{1-\gamma} + \lambda_{t+1} [(W_t - C_t) R_{w,t+1} - W_{t+1}] \right\} \tag{B.6}$$

First order conditions (FOCs) will be: [14]

$$\begin{aligned} [C_t]: &\qquad \beta^t U'(C_t) - E_t [\lambda_{t+1} R_{w,t+1}] = 0 \\ [\pi_t]: &\qquad (W_t - C_t) E_t [\lambda_{t+1} (R_{n,t+1} - R_{f,t})] = 0 \\ [W_{t+1}]: &\qquad -\lambda_{t+1} + E_{t+1} (\lambda_{t+2} R_{w,t+2}) = 0 \end{aligned} \tag{B.7}$$

Simplifying the FOCs will give us three Euler Equations (for consumption and for asset holdings)[15]:

$$1 = \beta E_t \left[\frac{U'(C_{t+1})}{U'(C_t)} R_{j,t+1} \right] \quad ; \quad \textit{for } j \in w, n, f \tag{B.8}$$

B.2 Approximating the FOCs and Budget Constraint

General note on linearizing logarithms of expectation: There are two approaches. Either we rely on the normality of the distribution of random variables and use the fact that for a Normally distributed random variable x, we have $\log E_t [e^x] = \mu_t + \frac{1}{2}\sigma_t^2$. Or more generally, one can use a second-order Taylor approximation around the mean, if x is close to its mean (Var(x) is close to zero):

$$\begin{aligned} \log E_t (e^x) &\approx \log E_t \left(e^{\bar{x}} + e^{\bar{x}}(x - \bar{x}) + \frac{1}{2} e^{\bar{x}} (x - \bar{x})^2 \right) \\ &\approx \log \left(e^{\bar{x}} + \frac{1}{2} e^{\bar{x}} \operatorname{Var}_t(x) \right) \\ &\approx \log \left(e^{\bar{x}} \left(1 + \frac{1}{2} \operatorname{Var}_t(x) \right) \right) \\ &\approx \bar{x} + \frac{1}{2} \operatorname{Var}_t(x) \end{aligned} \tag{B.9}$$

Approximating EEs: For the EEs, we have the approximation (taking the

[14] Note: have we to take derivative with respect to W_{t+1} as well, since it is a function of controls (C_t, π_t) and state variable (W_t).

[15] Note that two of the three equations above will give the third one as a result, and any two of them are enough for finding the solution.

logarithm of both sides and then a second-order Taylor expansion) will give us:

$$0 = \log \beta + E_t \left[-\gamma \Delta c_{t+1} + r_{j,t+1} \right] + \frac{1}{2} \mathrm{Var}_t \left(-\gamma \Delta c_{t+1} + r_{j,t+1} \right); \quad \textit{for} \quad j = w, n, f \tag{B.10}$$

(Note that when j=f, it should be $r_{f,t}$) If we subtract the above equations for n and f (n minus f), we will get:

$$E_t \left[r_{n,t+1} - r_{f,t} \right] + \frac{1}{2} \mathrm{Var}_t \left(r_{n,t+1} \right) = \gamma \,\mathrm{Cov} \left(r_{n,t+1}, \Delta c_{t+1} \right) \tag{B.11}$$

Approximating the budget constraint: The budget constraint is:

$$W_{t+1} = \left(W_t + L_t - C_t \right) R_{w,t+1} \tag{B.12}$$

divide both sides by L_{t+1}, to get:

$$\frac{W_{t+1}}{L_{t+1}} = \left(\frac{W_t}{L_t} + 1 - \frac{C_t}{L_t} \right) \frac{L_t}{L_{t+1}} R_{w,t+1} \tag{B.13}$$

Then I take log of both sides, denoting log variables with small letters ($w_t = \log(W_t)$):

$$w_{t+1} - l_{t+1} = \log \left(1 + \exp \left(w_t - l_t \right) - \exp \left(c_t - l_t \right) \right) - \Delta l_{t+1} + r_{w,t+1} \tag{B.14}$$

We can linearize the above equation by taking applying first-order Taylor expansion around $\mathrm{E}\left[c_t - l_t \right]$ and $\mathrm{E}\left[w_t - l_t \right]$. This gives:

$$w_{t+1} - l_{t+1} \approx \kappa + \rho_w \left(w_t - l_t \right) - \rho_c \left(c_t - l_t \right) - \Delta l_{t+1} + r_{w,t+1} \tag{B.15}$$

where

$$\begin{aligned} \rho_w &= \frac{\exp \left\{ \mathrm{E} \left[w_t - l_t \right] \right\}}{1 + \exp \left\{ \mathrm{E} \left[w_t - l_t \right] \right\} - \exp \left\{ \mathrm{E} \left[c_t - l_t \right] \right\}} \\ \rho_c &= \frac{\exp \left\{ \mathrm{E} \left[c_t - l_t \right] \right\}}{1 + \exp \left\{ \mathrm{E} \left[w_t - l_t \right] \right\} - \exp \left\{ \mathrm{E} \left[c_t - l_t \right] \right\}} \end{aligned} \tag{B.16}$$

and

$$\kappa = -\left(1 - \rho_w + \rho_c \right) \log \left(1 - \rho_w + \rho_c \right) - \rho_w \log \left(\rho_w \right) + \rho_c \log \left(\rho_c \right). \tag{B.17}$$

Note that $\rho_w, \rho_c > 0$. (*Proof*: Since along the optimal path we need to have $W_{t+1} >$ and so $W_t + L_t - C_t > 0$. This is equivalent to $1 + \frac{W_t}{L_t} - \frac{C_t}{L_t} > 0$ or $1 + \exp(w_t - l_t) - \exp(c_t - l_t) > 0$. And as our variables here are continuous, we will have $1 + \exp(E(w_t - l_t)) - \exp(E(c_t - l_t)) > 0$, and this immediately results in $\rho_w, \rho_c > 0$.)

Also, note that the definition of ρ_w and $\rho_c > 0$ depend on the values of w_t and l_t, which should be added to the final system of equations to be solved simultaneously. (Viceira (2001) section IV.A)

B.3 Solving the approximated system

Proof of proposition 1: The system of equations that we should solve now is: I use the EE for $j = w$ and substitute for π_t.

$$E_t\left[\Delta c_{t+1}\right] = \frac{1}{\gamma}\log\beta + \frac{1}{\gamma}E_t\left[r_{w,t+1}\right] + \frac{\gamma^2}{2\gamma}\operatorname{Var}_t\left[\Delta c_{t+1} - \frac{1}{\gamma}r_{w,t+1}\right] \tag{B.18}$$

an another equation:

If we subtract the above equations for n and f (n minus f), we will get:

$$E_t\left[r_{n,t+1} - r_{f,t}\right] + \frac{1}{2}\operatorname{Var}_t\left(r_{n,t+1}\right) = \gamma\operatorname{Cov}_t\left(r_{n,t+1}, \Delta c_{t+1}\right) \tag{B.19}$$

These are the two equilibrium conditions for finding our two unknowns c_t and π_t.

For using the first equation, we should first find the values of $E_t\left[\Delta c_{t+1}\right]$ and $\operatorname{Var}_t\left[\Delta c_{t+1} - \frac{1}{\gamma}r_{w,t+1}\right]$, and for the second equation, we need $\operatorname{Cov}_t\left(r_{n,t+1}, \Delta c_{t+1}\right)$.

$$\begin{aligned} E_t\left[\Delta c_{t+1}\right] &= E_t\left[c_{t+1} - l_{t+1} - (c_t - l_t) + \Delta l_t\right] \\ &= E_t[c_{t+1} - l_{t+1}] - (c_t - l_t) + E_t\left(\Delta l_t\right) \\ &= b_0 + b_1 E_t[w_{t+1} - l_{t+1}] + b_2 E_t\left(r_{f,t+1}\right) - (c_t - l_t) + E_t\left(\Delta l_t\right) \end{aligned} \tag{B.20}$$

One need to calculate $\operatorname{Cov}_t\left(r_{n,t+1}, \Delta c_{t+1}\right)$ for EEs. I use the identity

$$\Delta c_{t+1} = c_{t+1} - l_{t+1} - (c_t - l_t) + \Delta l_t \tag{B.21}$$

and guess and verify

$$c_{t+1} - l_{t+1} = b_0 + b_1\left(w_{t+1} - l_{t+1}\right) + b_2 r_{f,t+1} \tag{B.22}$$

Replacing these two amount step by step, we will have:

$$\begin{aligned}
\mathrm{Cov}_t\left(r_{n,t+1}, \Delta c_{t_{+1}}\right) &= \mathrm{Cov}_t\left(r_{n,t+1}, c_{t+1} - l_{t+1} - (c_t - l_t) + \Delta l_t\right) \\
&= \mathrm{Cov}_t\left(r_{n,t+1}, c_{t+1} - l_{t+1}\right) + \mathrm{Cov}_t\left(r_{n,t+1}, l_{t+1}\right) \\
&= \mathrm{Cov}_t(r_{n,t+1}, b_0 + b_1\left(w_{t+1} - l_{t+1}\right) + b_2 r_{f,t+1}) - \sigma_n\sigma_{rl} \\
&= \mathrm{Cov}_t(r_{n,t+1}, b_1\left(w_{t+1} - l_{t+1}\right)) + \mathrm{Cov}_t(r_{n,t+1}, +b_2 r_{f,t+1}) - \sigma_n\sigma_{rl} \\
&= -b_1\,\mathrm{Cov}_t\left(r_{n,t+1}, \Delta l_t\right) + b_1\,\mathrm{Cov}_t\left(r_{n,t+1}, r_{w_1 t+1}\right) - b_2\sigma_n\sigma_r - \sigma_n\sigma_{rl} \\
&= -\left(1 - b_1\right)\sigma_n\sigma_{rl} + b_1\pi_t\sigma_n^2 - b_2\sigma_n\sigma_r
\end{aligned} \tag{B.23}$$

Now, I replace this in the second EE to solve for π_t. We will have:

$$\pi_t = \frac{E_t\left[r_{n,t+1} - r_{f,t}\right] + \frac{1}{2}\mathrm{Var}_t\left(r_{n,t1}\right)}{\gamma b_1\sigma_n^2} + \frac{b_2\sigma_r\sigma_n}{\gamma b_1\sigma_n^2} + \frac{\left(1 - b_1\right)\sigma_{rl}}{\gamma b_1\sigma_n^2} \tag{B.24}$$

Note that π_t is time invariant as expected, since we do not have life-cycle or anything time-varying parameters in this model.

Now, we have found the solution for π_t. Now, we need to pin down the coefficients b_0, b_1, and b_2. For doing so, I use the first EE (for $j = w$) and substitute for π_t.

$$E_t\left[\Delta c_{t+1}\right] = \frac{1}{\gamma}\log\beta + \frac{1}{\gamma}E_t\left[r_{w,t+1}\right] + \frac{\gamma^2}{2\gamma}\mathrm{Var}_t\left[\Delta c_{t+1} - \frac{1}{\gamma}r_{w,t+1}\right] \tag{B.25}$$

For using this equation, we should first find the values of $E_t\left[\Delta c_{t+1}\right]$ and $\mathrm{Var}_t\left[\Delta c_{t+1} - \frac{1}{\gamma}r_{w,t+1}\right]$.

$$\begin{aligned}
E_t\left[\Delta c_{t+1}\right] &= E_t\left[c_{t+1} - l_{t+1} - (c_t - l_t) + \Delta l_t\right] \\
&= E_t[c_{t+1} - l_{t+1}] - (c_t - l_t) + E_t\left(\Delta l_t\right) \\
&= b_0 + b_1 E_t[w_{t+1} - l_{t+1}] + b_2 E_t\left(r_{f,1+t}\right) - (c_t - l_t) + E_t\left(\Delta l_t\right) \\
&= b_0 + b_1\kappa + b_1\rho_w\left(w_t - l_t\right) + \left(-b_1\rho_c - 1\right)\left(c_t - l_t\right) + b_2\left((1 - \varphi)\bar{r}_f + \varphi r_{f,t}\right) \\
&\quad + r_{f,t} + \pi_t\mu_n + \frac{1}{2}\pi_t(1 - \pi_t)\sigma_n^2
\end{aligned} \tag{B.26}$$

and

$$
\begin{aligned}
\operatorname{Var}_t\left[\Delta c_{t+1}-\frac{1}{\gamma} r_{w,t+1}\right] &= \operatorname{Var}_t\left[c_{t+1}-l_{t+1}-(c_t-l_t)+\Delta l_t-\frac{1}{\gamma} r_{w,t+1}\right] \\
&= \operatorname{Var}_t\left[(c_{t+1}-l_{t+1})+l_{t+1}-\frac{1}{\gamma} r_{w,t+1}\right] \\
&= \operatorname{Var}_t\left[b_0+b_1\left(w_{t+1}-l_{t+1}\right)+b_2 r_{f,t+1}+l_{t+1}-\frac{1}{\gamma} r_{w,t+1}\right] \\
&= \operatorname{Var}_t\left[\left(1-b_1\right) l_{t+1}+\left(1-\frac{1}{\gamma}\right) r_{w,t+1}+b_2 r_{f,t+1}\right] \\
&= \operatorname{Var}_t\left[\left(1-b_1\right) l_{t+1}\right]+\operatorname{Var}_t\left[\left(1-\frac{1}{\gamma}\right) r_{w,t+1}\right]+\operatorname{Var}_t\left[b_2 r_{f,t+1}\right] \\
&+2 \operatorname{Cov}_t\left(\left(1-b_1\right) l_{t+1},\left(1-\frac{1}{\gamma}\right) r_{w_1 t_{+1}}\right)+2 \operatorname{Cov}_t\left(\left(1-b_1\right) l_{t+1}, b_2 r_{f,t+1}\right) \\
&+2 \operatorname{Cov}_t\left(\left(1-\frac{1}{\gamma}\right) r_{n,t+1}, b_2 r_{f,t+1}\right) \\
&=\left(1-b_1\right)^2 \sigma_l^2+\left(1-\frac{1}{\gamma}\right)^2 \pi^2 \sigma_n^2+b_2^2 \sigma_r^2 \\
&+2\left(1-b_1\right)\left(1-\frac{1}{\gamma}\right) \pi\left(-\sigma_n \sigma_{r_l}\right) \\
&+2\left(1-b_1\right) b_2 \sigma_r \sigma_{rl}+2\left(1-\frac{1}{\gamma}\right) b_2\left(-\sigma_r \sigma_n\right)=V
\end{aligned} \tag{B.27}
$$

As the above term does not depend on the state variables of the model, I have denoted with V, which is a function of parameters.

Now, back to the first EE, we replace the equivalents:

$$
\begin{aligned}
& b_0+b_1 \kappa+b_1 \rho_w\left(w_t-l_t\right)+\left(-b_1 \rho_c-1\right)\left(c_t-l_t\right)+b_2\left((1-\varphi) \bar{r}_f+\varphi r_{f,t}\right) \\
+ & r_{f,t}+\pi_t \mu_n+\frac{1}{2} \pi_t(1-\pi_t) \sigma_n^2=\frac{1}{\gamma} \log \beta+\frac{1}{\gamma}\left(r_{f_1 t}+\pi \mu_n+\pi(1-\pi) \sigma_n^2\right)+\frac{\gamma}{2} V
\end{aligned} \tag{B.28}
$$

With a bit of rearranging, we will have:

$$
\begin{aligned}
& b_0+b_1 \kappa+b_2(1-\varphi) \bar{r}_f+\pi_t \mu_n+\frac{1}{2} \pi_t(1-\pi_t) \sigma_n^2-\frac{1}{\gamma} \log \beta-\frac{1}{\gamma}\left(\pi \mu_n+\frac{1}{2} \pi(1-\pi) {\sigma_n}^2\right)-\frac{\gamma}{2} V \\
& +b_1 \rho_w\left(w_t-l_t\right)+\left(-b_1 \rho_c-1\right)\left(c_t-l_t\right)+b_2\left(\varphi r_{f,t}\right)+r_{f,t}-\frac{1}{\gamma} r_{f,t}=0
\end{aligned} \tag{B.29}
$$

Now, I replace $(c_t - l_t)$ from the guess:

$$\begin{aligned}
&b_0 + b_1\kappa + b_2(1-\varphi)\bar{r}_f + \pi_t\mu_n + \frac{1}{2}\pi_t(1-\pi_t)\sigma_n^2 - \frac{1}{\gamma}\log\beta \\
&- \frac{1}{\gamma}\left(\pi\mu_n + \frac{1}{2}\pi\left(1-\pi\right){\sigma_n}^2\right) - \frac{\gamma}{2}V + \left(-b_1\rho_c - 1\right)b_0 \\
&+ \left\{b_1\rho_w + \left(-b_1\rho_c - 1\right)b_1\right\}\left(w_t - l_t\right) \\
&+ \left\{b_2\left(-b_1\rho_c - 1\right) + b_2\varphi - \frac{1}{\gamma} + 1\right\} r_{f,t} = 0
\end{aligned} \tag{B.30}$$

This equation should hold for all t. Equating the coefficients to zero, we get[16]:

$$b_0 = \frac{1}{b_1\rho_c}\{b_1\kappa + b_2(1-\varphi)\bar{r}_f - \frac{1}{\gamma}\log\beta + (1-\frac{1}{\gamma})\left(\pi\mu_n + \frac{1}{2}\pi\left(1-\pi\right){\sigma_n}^2\right) - \frac{\gamma}{2}V\} \tag{B.31}$$

$$b_1 = \frac{\rho_w - 1}{\rho_c} \tag{B.32}$$

$$b_2 = \frac{1-\frac{1}{\gamma}}{\rho_w - \varphi} = (1-\frac{1}{\gamma})\bar{b}_2 \tag{B.33}$$

Note that the definitions of ρ_w and ρ_c and the fact that $W_t + L_t - c_t > 0$ imply that they are both positive and $0 < 1 - \rho_w + \rho_c$, which implies $\frac{\rho_w - 1}{\rho_c} < 1$. For provint that $b_2 > 0$, note that from the solution to for optimal consumption:

$$c_{t+1} - l_{t+1} = b_0 + b_1\left(w_{t+1} - l_{t+1}\right) + b_2 r_{f,t+1} \tag{B.34}$$

If $b_2 < 0$, it implies that consumption is a decreasing function of wealth for all income levels. That is, the individual is better off with less wealth, which is a contradiction.

This completes the derivation of the approximate solution. □

B.4 Useful equalities for simplification

For which the following equations have been used:

$$E_t\left[r_{w,t+1}\right] = r_{ft} + \pi\mu_n + \frac{1}{2}\pi(1-\pi)\sigma_n^2 \tag{B.35}$$

$$\mathrm{Var}_t\left[r_{n,t+1}\right] = \pi^2\,\mathrm{Var}_t\left[r_{n,t+1}\right] = \pi^2\sigma_n^2 \tag{B.36}$$

$$\mathrm{Var}_t\left[(1-b_1)\Delta l_1\right] = \left(1-b_1\right)^2\sigma_u^2 \tag{B.37}$$

[16] One can rule out the case where $b_1 = 0$, for which b_0 gets arbitrary, and b_3 will have no solution.

$$\mathrm{Var}_t \left[r_{f_{t+1}} \right] = \sigma_r^2 \tag{B.38}$$

$$\mathrm{Var}_t = (r_{w,t+1}, \Delta\ell_t) = \mathrm{Cov}_t \left(\pi r_{n,t+1}, \Delta\ell_t \right) \tag{B.39}$$

$$\mathrm{Cov}_t \left(r_{n,t+1}, r_{f,t+1} \right) = -\sigma_r \sigma_n \tag{B.40}$$

$$\mathrm{Cov}_t \left(\Delta l_t, r_{f_1 t+1} \right) = \sigma_{rl} \sigma_r \tag{B.41}$$

$$\mathrm{Cov} \left(\Delta l_t, r_{n,t+1} \right) = -\sigma_n \sigma_{r_a} \tag{B.42}$$

$$\mathrm{Cov}_t \left(r_{w,t+1}, r_{n,t+1} \right) = \pi_t \sigma_n^2 \tag{B.43}$$