On the Tradeoffs between Interest Rates and Haircuts of Lending Facilities*

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Abstract

Banks facing funding stress often transform illiquid assets into liquidity by borrowing against them as collateral. In the United States, banks can obtain such funding either from the Federal Reserve's discount window (DW) or from the Federal Home Loan Banks (FHLBs). We study how differences in the terms offered by these facilities, specifically the interest rate and the haircut on collateral, affect banks' funding choices and run risk. We develop a Diamond–Dybvig style banking model with a global game and endogenous collateral values in which banks choose between borrowing from the DW and the FHLBs. Our main theoretical result is that runs can arise either because haircuts are too high, limiting collateral capacity, or because interest rates are too high, rendering banks insolvent. The model predicts that banks sensitive to borrowing costs prefer FHLBs, while banks constrained by collateral haircuts turn to the DW. We test these predictions using confidential data on DW and FHLB borrowing during recent stress episodes.

Keywords: Bank Runs, Discount Window, FHLBs
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1 Introduction

Banks engage in maturity transformation, i.e., funding long-term illiquid assets with short-term demandable liabilities, exposing them to run risk. Since the seminal work of Diamond and Dybvig (1983), the literature has emphasized that banks facing withdrawals may be forced to liquidate assets at distressed prices in order to meet liquidity needs. Yet micro-level evidence suggests that banks typically respond to stressed withdrawals not by selling assets but by pledging illiquid assets as collateral to obtain funding (Cipriani, Eisenbach, and Kovner, 2024). In practice, collateralized borrowing against illiquid assets has become a central mechanism through which banks manage liquidity stress.

In the United States, banks have access to two major standing sources of collateralized borrowing: the Federal Reserve's discount window (DW) and the Federal Home Loan Banks (FHLBs). Both facilities allow banks to pledge assets, including loans, to obtain liquidity on demand. Importantly, these facilities differ in the terms they offer. Borrowing contracts are multi-dimensional, with two key features: the interest rate charged on the loan and the haircut applied to pledged collateral. The interest rate determines the cost of funding, while the haircut determines how much liquidity can be obtained from a given amount of collateral.

The coexistence of these two facilities has attracted increasing attention in both policy discussions and the academic literature. The discount window is the central bank's traditional lender-of-last-resort facility designed to provide liquidity to solvent but illiquid banks. By contrast, the FHLBs are government-sponsored cooperatives originally created to support housing finance but that have increasingly become an important source of liquidity for banks. Because of this role, they have been described as a "lender of second-to-last resort." This institutional configuration raises several questions. What does the presence of FHLB lending imply for the way the discount window operates? Why do FHLBs provide this type of liquidity rather than private markets? Could a more active discount window crowd out the FHLBs, or can a system with multiple lending facilities be welfare improving? More broadly, what principles should guide the design of lending facilities and the pricing of collateralized liquidity support?

Our paper is motivated by three empirical facts about borrowing from the FHLBs and the discount window that existing theories of the discount window do not jointly explain. Much of the literature emphasizes stigma as a key reason banks avoid the discount window (Armantier, Ghysels, Sarkar, and Shrader, 2015). However, deriving stigma from microfoundations is challenging and most models abstract from it, with a few exceptions such as Ennis (2019). Moreover, recent experience—including the extensive use of the Bank Term Funding Program (BTFP), a facility similar to the discount window but with stricter disclosure requirements—raises questions about the broad applicability of stigma-based explanations.

The first empirical fact concerns how banks use the two facilities over time. In normal times banks rely heavily on FHLB advances for liquidity, whereas borrowing from the discount window is limited. However, banks *extensively* turn to both facilities during periods of financial stress. The second empirical fact concerns the collateral banks pledge to obtain liquidity. There is substantial heterogeneity in the types of assets pledged at the two facilities: mortgages account for the largest share of collateral pledged at the FHLBs, whereas the discount window receives a broader set of assets including commercial and industrial loans, credit card receivables, and other loans that are typically less liquid. The third empirical fact concerns the terms offered by the two facilities. FHLBs typically offer more competitive lending rates than the discount window, reflecting the penalty pricing traditionally associated with central bank lending. At the same time, the discount window often provides more favorable haircuts for several types of collateral, particularly less liquid loans.

Taken together, these facts suggest that understanding borrowing choices between the FHLBs and the discount window requires a framework that explicitly considers both the interest rate and the haircut applied to collateral. Existing theories of the discount window typically abstract from this distinction and therefore cannot account for the patterns observed in the data.

This paper develops a theory of collateralized borrowing that rationalizes these facts. We study how differences in interest rates and haircuts across lending facilities affect banks' borrowing decisions and the risk of runs. We build on a Diamond–Dybvig style banking model with a global game and endogenous collateral values. The bank-run framework is akin to Kashyap, Tsomocos, and Vardoulakis (2024), which allows us to characterize equilibrium run thresholds in the presence of strategic complementarities among depositors. But, contrary to them, our model is suitable to study collateralized borrowing and the important trade off between rates and haircuts.

In the model, banks hold a portfolio of liquid and illiquid assets and face stochastic withdrawals. When withdrawals exceed liquid reserves, banks can pledge illiquid assets as collateral to obtain funding from either the FHLBs or the central bank. These lenders offer different combinations of interest rates and haircuts. The haircut determines the fraction of the collateral value that can be borrowed against, while the interest rate determines the cost of replacing deposits with collateralized borrowing.

Our analysis highlights that these two contract terms play fundamentally different roles in determining borrowing behavior and financial stability. The haircut determines whether the bank has sufficient collateral capacity to meet withdrawals, while the interest rate determines whether the bank remains solvent once it replaces deposits with more expensive collateralized borrowing. As a result, a run can be triggered either because the haircut is too high—leaving the bank without sufficient collateral to pledge—or because the interest rate is too high—rendering the bank insolvent even when collateral capacity is sufficient.

The model yields two central theoretical results. In normal times, when withdrawals are limited and collateral is abundant, borrowing decisions are primarily driven by interest rates. Banks therefore prefer the lender offering the lower borrowing cost. In stressed times, when collateral values are uncertain and withdrawals increase, haircuts become critical. In that environment, banks may prefer a lender offering more favorable collateral terms even if the interest rate is higher. As a result, the relative attractiveness of lending facilities can change depending on the state of the financial system.

We test the model's predictions using a combination of confidential and public data. The empirical analysis relies on confidential daily data on collateral pledged at the discount window and discount window loans, combined with data on FHLB advances constructed from Fedwire transactions. These datasets are merged with bank-level information from Call Reports to analyze borrowing behavior across institutions. The data allow us to examine how borrowing from the two facilities evolves during periods of stress and how these choices depend on the composition of banks' collateral.

Consistent with the theory, we find that banks with collateral that receives relatively favorable haircuts at the discount window pledge this collateral at the window in normal times and increase their relative borrowing from the window during stress episodes. Moreover, the increase in discount window collateral during stress is concentrated in assets that receive relatively advantageous haircuts at the facility. These patterns support the view that differences in interest rates and haircuts play a central role in shaping borrowing choices.

Finally, we use the model to characterize the optimal design of lending facilities and their interaction with liquidity regulation. We show that when central bank lending involves operational or policy costs, it can be optimal for the discount window to maintain a positive penalty rate, allowing other institutions such as the FHLBs to provide routine funding in normal times. At the same time, when lenders differ in the collateral they accept and the terms they offer, a system with multiple facilities can improve welfare by allowing banks with different balance sheet characteristics to obtain liquidity under different conditions. These results provide a framework for evaluating the design of lender-of-last-resort policies and the role of institutions such as the FHLBs in the financial system.

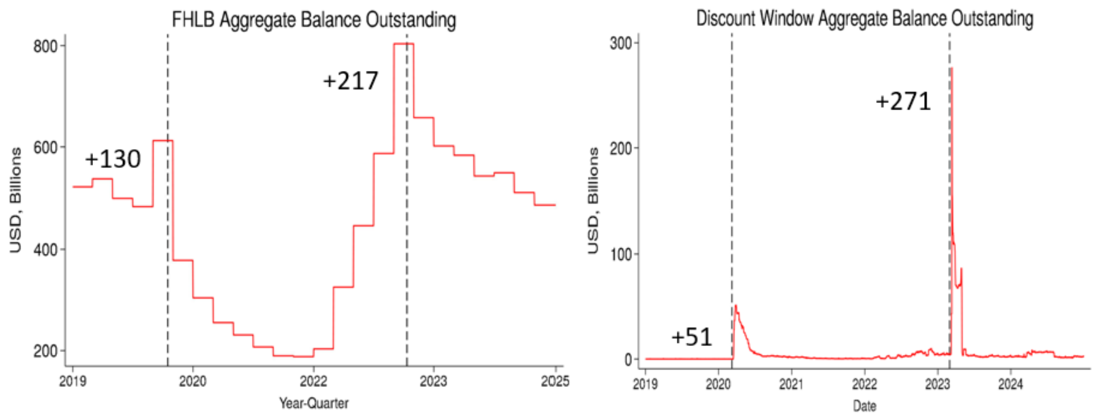
2 Stylized facts

Banks in the United States can obtain collateralized borrowing from both FHLBs and the central bank through discount window, which are both standing and available to banks to tap for any reasons under no-questions-asked.[1] One important element that distinguish these two sources of liquidity from other (standing) secured types of financing, such as borrowing in the repo market, is the broader collateral FHLBs and the discount accept, including loans.[2] At the same time, FHLBs and the discount window have been used markedly differently both across banks and across normal and stressed times. Past literature has argued that FHLBs ar preferred to the discount window due to the stigma associated with borrowing from the central bank—particularly due to the disclosure of borrowers identities after two years—despite the discount window reform that attempt to remove such stigma (Armantier et al., 2015).

Our purpose is not to critically access whether stigma is an important consideration for banks

[1] There is a distinction between primary and secondary credit at the Federal Reserve's discount window. Primary credit is available to generally sound banks and is intended to provide liquidity with minimal administrative burden and for any reason including expanding banks' balance sheets, often characterized as "no-questions-asked". Secondary credit is available to institutions that do not qualify for primary credit and is subject to greater oversight and a higher interest rate. Throughout our analysis we focus on solvent but potentially illiquid institutions and, thus, our reference to the discount window should be interpreted as primary credit.

[2] The Federal Reserve could also provide liquidity under Section 13(3) of the Federal Reserve Act. These facilities are not standing, require Treasury approval, and typically involve fiscal backing because they may extend credit more broadly to support financial institutions or markets during systemic stress.



Source: Call reports, quarter ends

Source: DW confidential data daily (disclosed)

Figure 1: Aggregate borrowing from the Discount Window and FHLBs.

between choosing to borrow from FHLBs or the discount window. Stigma could always be a possibility but deriving it from microfoundations is challenging and previous studies have typically abstracted from doing so with a few exceptions (Ennis, 2019). Moreover, recent experience with the extensive take-up form the Bank Term Funding Program (BTFP)—a discount-window-like facility with more favorable lending terms but harsher disclosure requirements at one year rather than two for the discount window—further raises questions about the broad applicability of stigma-based theories of the discount window. Our approach is, instead, to highlight important stylized facts for FHLB and discount window borrowing that may require developing a theory that goes beyond, but does not exclude, stigma. The rest of the sections is presenting the key stylized facts, derived to a large extend from micro-level high-quality data, and concludes by discussing the key theoretical element that one may need to capture and rationalize these facts.

The first set of stylized facts regards the overall borrowing from FHLBs and the discount window over time, both in normal and stressed times. Figure 1 shows the aggregate daily FHLB advances and discount window borrowing over time, across the vast majority of banks that constitute our sample; please note that the dailty FHLB advances chart is under clearance and we, thus, present the quarterly FHlB advances from Call reports in this draft.[3] While banks rely considerably on FHLBs for liquidity during normal times, they resort to both FHLBs and the discount window in times of stress—the COVID-19 and banking turmoil crises in March 2020 and March 2023. Importantly, both FHLB and discount window borrowing spiked in times of stress, consistent with a need for stressed-related liquidity to address runs, with the spike in discount window use being quantitatively meaningful compared to or even higher that the spike in FHLB borrowing. This set of stylized facts suggest a need for a theeory that differentiates between liquidity needs in normal

[3]We utilize two confidential databases to obtain these charts. FHLB advances are constructed from Fedwire data (see Appendix for details), while daily discount window borrowing is obtained from the Credit Risk Management Support Office (CRMSO) at the Federal Reserve Bank of Chicago.

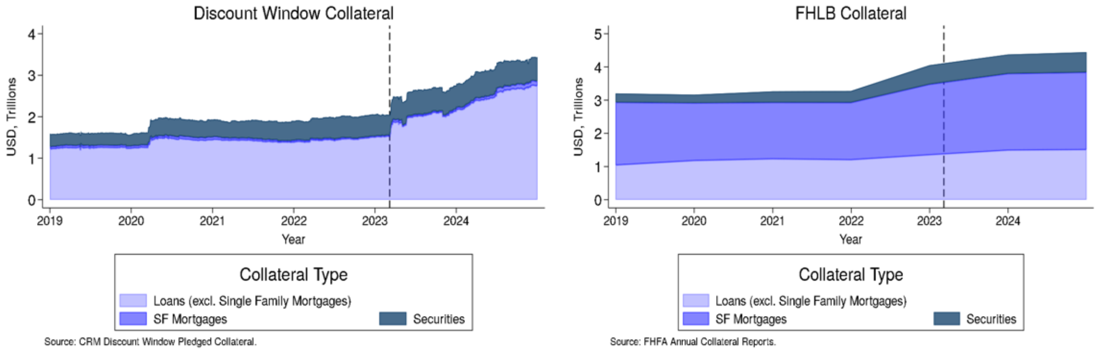


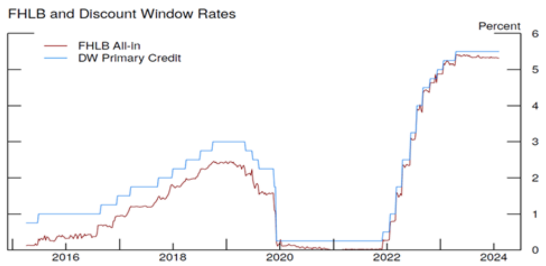
Figure 2: Aggregate collateral pledged at the Discount Window and FHLBs.

and stressed times. Moreover, is casts doubt on purely driven stigma-based theories of the discount window as one may expect that solvent, yet illiquid, banks may be more concerned about stigma in times of stress and, thus, avoid the discount window rather that using it.

The second set of stylized facts concerns the collateral that banks pledge at FHLBs and the discount window. Figure 2 shows the aggregate collateral pledged over time at FHLBs and the discount window by three broad categories: securities, single-family mortgages, and other loans. There are few takeaways. First, collateral pledged at both FHLBs and the discount window is substantial and of similar magnitude, and it is a multiple of actual borrowing especially for the discount-window case, signaling excess capacity on aggregate. Second, there is substantial heterogeneity between the type of collateral pledged at FHLBs and the discount window. In both cases securities are the smallest category, though more prevalent at the window. But, importantly, the plurality of collateral pledged at FHLBs consists of single family mortgages, which are minuscule at window. Instead, other loans, which are arguably less liquid, constitute the majority, something that we will discuss in further detail in Section 4. Third, collateral pledged at the discount window spikes during stressed times but there is not observable spike for collateral pledged at FHLBs, further casting doubt on purely driven stigma-based theories.

The third set of stylized facts concerns the terms for FHLB and discount window borrowing. There are several interesting differences but two that are striking, and that we focus on herein, are the interest rate charged and the haircut on collateral. Figure 3, left-panel, plots all-in rates offered by the Des Moines FHLB against the discount window rate, suggesting that FHLBs typically offer more competitive loan rates than the discount window.[4] With respect to collateral, FHLBs offer better haircuts for mortgages, while the discount is better for C&I loans, credit card receivables, mortgage-backed securities, and other more illiquid type of assets, as shown on the right panel, which summarizes our detailed construction of haircuts using confidential data (see Section 4 for details). In sum, FHLBs offer more competitive lending rates than the discount window, which offers better haircuts apart for mortgages.

[4]FHLBs are owned by the member banks, so borrowing banks may expect to receive part of their interest expenses back as dividend. The all-in rate account for this so as to reflect the actual borrowing cost.



Better Haircut	Loan Types (indicative and non-exhaustive)
FHLB	1-4 Family Mortgages (1st lien, 2nd, HELOC), Land Loans
DW	C&I Loans, Credit Card Receivables, Agency and Non-Agency MBS

Figure 3: Discount window primary credit rate and Des Moines FHLB all-in rates(left panel); haircuts on collateral at the discount window and FHLBs (right panel)

A theory of collateralized borrowing from FHLBs vis-a-vis the discount window should capture at minimum the aforementioned stylized facts. In particular, it should explain the choice of banks to obtain liquidity between FHLBs and the discount window using different assets as collateral, both in normal times and during stress when runs dynamics and coordination failures are at play. We present such a theory in Section 3.

3 Model

The model has three periods, $t = \{0,1,2\}$, features a single consumption good, and includes four types of agents: a continuum of depositors (D), a banker (B), a central bank (C), and an FHLB (F). We refer to individual depositors as the depositor, and we will use the singular or plural for depositors depending on the context. Each depositor is atomistic and takes equilibrium variables as given. Some depositors learn that they need to consume at date 1, while others do not have an immediate consumption need and can wait until date 2 to consume.

The banker owns and manages a bank, which is funded by deposits provided by depositors at $t = 0$. B internalizes how her choices affect the equilibrium variables that matter for her profitability. For simplicity, we assume that equity funding is zero and all deposits are uninsured.[5] B can invest the deposits at $t = 0$ in two assets that are in perfectly elastic supply at a price of one. The first asset is safe and liquid, paying one at $t = 1$ and $r \geq 1$ at $t = 2$, where r is the gross policy rate from $t = 1$ to $t = 2$; we have normalize the policy rate from $t = 0$ to $t = 1$ to one for simplicity. The second asset is risky and illiquid, paying ωR at $t = 2$. At the beginning of $t = 1$, the distribution from which ω is drawn is revealed. In particular, with probability π, $\omega = 1$ with certainty, while with probability $1-\pi$, ω is drawn from a uniform distribution $U[0,1]$. We call the former the "’certain state" and the latter the "uncertain state". At $t = 1$, depositors also receive private noisy signals about the true realization of ω, given by $x_i = \omega + \varepsilon_i$ with $\varepsilon \in U[-\varepsilon, \varepsilon]$. Note that the realization of the certain or uncertain state is common knowledge, but the precise realization of ω in the uncertain state is not.

Depositors are identical ex-ante and can choose to invest their endowment at $t = 0$ either in

[5] These assumptions, as well as several others below, can be relaxed as in Kashyap et al. (2024) at the cost of complicating the model. For the baseline analysis, we proceed with the simplest possible model.

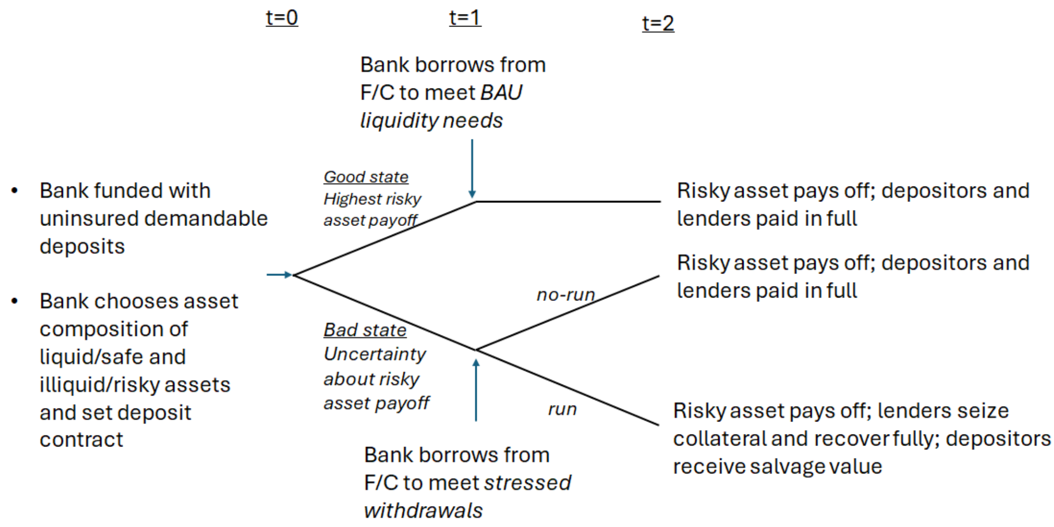


Figure 4: Timeline

the liquid asset or in bank deposits, but cannot directly invest in the illiquid asset. Deposits are redeemable on demand and the deposit contract specifies a deposit rate for withdrawals at $t = 1$ and $t = 2$. To pay deposit withdrawals at $t = 1$ the bank can use its liquid assets, borrow from F or C by pledging the illiquid asset as collateral, or could try to sell the illiquid asset in the market.

Given that the ability of the bank to raise funds at $t = 1$ depends on the value of the illiquid asset, the bank is at risk of self-fulfilling runs. If ω is not high enough, a patient depositor may demand her deposits early if she believes that other patient depositors will do the same. We will discuss in detail how the bank's balance sheet and lender choice matters for run incentives. To address the coordination problem and obtain a unique equilibrium, we employ global game techniques: the noisy signals depositors receive at $t = 1$ not only provide information about the fundamental ω, but also inform the beliefs of other depositors, serving to coordinate the patient depositors' decisions.

We show that there is a threshold strategy that patient depositors will follow such that when the signal is above a cutoff value they do not withdraw early, and when it is below the cutoff they run. Moreover, we show that when the noise goes to zero, there is a unique threshold for fundamentals ω_j^*, such that all patient depositors withdraw their deposits when the true realization of ω is below that value and they keep their deposits in the bank otherwise. We will refer to ω_j^* as the run threshold, and ω_j^* is the probability that a run occurs. Note the the threshold will differ depending on whether the bank chooses to borrow from F or C (or liquidate assets) to meet withdrawals. We add the subscript j to the threshold to identify which lender is assumed to be chosen.

Figure 4 presents the timeline of the model. Sections 3.1-3.3 describe the agents' optimization problems, section 3.4 derives the run threshold, and section 3.5 presents the optimal choice of lender. Finally, section 3.6 characterizes the private equilibrium.

3.1 Depositors

Depositors are identical ex-ante, are of mass one, and each of them is endowed with one unit of the consumption good at $t = 0$. We assume that depositors do not value consumption at $t = 0$ and can choose to invest their endowment either in the liquid asset, L_D, or in bank deposits, D, but do not have direct access to the illiquid asset. At $t = 1$ a portion of depositors, δ, receive a preference shock to consume immediately, while the rest, $1-\delta$, prefer to consume at $t = 2$. These shocks are private information, are independent and identically distributed, and are not contractible ex-ante akin to Diamond and Dybvig (1983). Thus, depositors are ex-post heterogeneous, and we will refer to them as impatient and patient, respectively. For simplicity, depositors have linear preferences at $t = 1$ and $t = 2$ over consumption and the other bank services specified below. To maintain the liquidity provision role that the bank plays for impatient depositors, we posit that impatient depositors suffer a disutility when their consumption is less than a certain level, which we set to one. This assumption is meant to capture in a reduced-form way the relative preference for (a certain level of) early consumption, which would be absent otherwise due to the assumption of linear preferences (Peck and Shell, 2010 and Matta and Perotti, 2023 make similar assumptions).[6]

Deposits are redeemable on demand and the deposit contract specifies gross deposit rates for early and late withdrawals, r_D and R_D, respectively. As typical in the global-games bank-run literature, we consider deposit rates that are uncontingent. Moreover, deposits carry an extra convenience yield, V, from providing payment services, which accrues only if the bank does not suffer a run at $t = 1$ or default at $t = 2$. Depositors enjoy this convenience yield if deposits are used as a means of payment at the time of consumption.

Deposit payoffs need to satisfy two conditions to attract depositors. First, patient depositors should not want to withdraw their deposits early if the bank does not suffer a run and is solvent at $t = 2$. Hence, the bank needs to respect the following *incentive compatibility constraint*:

$$IC: \quad R_D + V > r_D r, \tag{1}$$

which says that the total benefit to patient deposits, consisting of the late rate and the convenience yield, is higher than the return from withdrawing at $t = 1$ and investing in the liquid asset.

Second, depositors should prefer depositing at the bank at $t = 0$ to investing in the liquid asset. Because of risk-neutrality, depositors will invest all their endowment in deposits if the expected payoff, including the convenience yield, is higher than the payoff from investing in the liquid asset.

[6] This assumption simplifies the analysis. Alternatively, a sufficiently concave utility function with respect to consumption at $t = 1$ would yield similar results.

The following *participation constraint* needs to hold:

$$PC: \quad \overbrace{\left(\pi+(1-\pi)(1-\omega_j^*)\right)\left((1-\delta)R_D+\delta r_D+V\right)}^{\substack{\text{Expected deposit payoff}\\ \text{absent a run}}}+\overbrace{(1-\pi)\int_0^{\omega_j^*} L(\omega)\left((1-\delta)r+\delta\right)d\omega}^{\substack{\text{Expected deposit payoff}\\ \text{in a run}}}$$
$$\underbrace{-\delta P\mathbb{I}_{r_D<1}-\delta P(1-\mathbb{I}_{r_D<1})(1-\pi)\int_0^{\omega_j^*}(1-L(\omega)/r_D)d\omega}_{\substack{\text{Disutility to impatient depositors}\\ \text{when consumption is less than one}}} \quad \underbrace{-\left((1-\delta)r+\delta\right)}_{\text{Liquid asset payoff}} \geq 0. \qquad (2)$$

This expression is intuitive. With probability $\pi+(1-\pi)(1-\omega_j^*)$ a run does not occur, patient and impatient depositors receive the contractual payments, R_D and r_D, along with the convenience yield, V (the first term in first line in (2)).[7] In a run—occurring with probability $(1-\pi)\omega_j^*$—all depositors withdraw but receive r_D only with conditional probability $L(\omega)/r_D$, where $L(\omega)$ is the liquidation value of the bank's asset for realization ω (the second term in first line in (2)). Patient depositors invest the funds at r, while impatient consume them immediately. Moreover, impatient depositors suffer disutility P when their consumption is less than one, which occurs with certainty if $r_D<1$, as depicted by indicator function $\mathbb{I}_{r_D<1}$ that is equal to one if $r_D<1$ and zero otherwise (the first term in second line in (2)). If, instead, $r_D\geq 1$, impatient depositors suffer disutility P only if they are not served in a run, which occurs with conditional probability $1-L(\omega)/r_D$ (the second term in second line in (2)). The total expected deposit payoff should be higher than the payoff from investing in the liquid asset (the last term in second line in (2)).

To simplify the analysis, we focus on the case that the bank does not want to set $r_D<1$ and deposit-taking is viable for $r_D\geq 1$. The following Lemmas provide the sufficient conditions.

Lemma 1. *If $P\geq \underline{P}\equiv\left((1-\delta)(R_D-r)+V\right)/\delta$, setting $r_D<1$ violates depositors' PC in* (2).

Lemma 1 says that, for high enough P, depositors would prefer to hold liquid assets rather than invest in deposits if r_D is so low that it never allows them to consume the par value of their deposits.[8]

Lemma 2. *For $r_D\geq 1$, there exists a cutoff $\underline{\pi}$ such the deposit contract satisfies* (2) *for $\pi\geq\underline{\pi}$.*

Lemma 2 says that depositors willingness to invest in deposits depends on how likely the incidence of a bank run is. If that possibility is sufficiently low, then being sure that the deposit does not promise to pay less than the liquid asset is enough to induce them to invest in deposits.

Given the conditions in Lemma 2, we can derive a lax upper bound for r_D that satisfies (2) and that will be useful when we derive the bank's problem in Section 3.3. B can compensate depositors either with a higher R_D or a higher r_D. Thus, to derive the upper bound, we consider $R_D\to r_D r-V$, such that (1) is not violated, along with $\omega_j^*\to 1$ and $L(\omega)\to 0$. The latter two conditions suppose

[7] As it will become clear later, for $\omega\geq\omega_j^*$ the bank is also solvent when only impatient depositors withdraw, so V accrues in equilibrium to both impatient and patient depositors absent a run.

[8] For $R_D<r$—required for self-fulfilling runs as we show later—the sufficient condition reduces to $P>V/\delta$.

that there is always a run in the uncertain state for any realization of ω and depositors get zero in the run, justifying the derivation of a lax upper bound (lub) given by:

$$r_D^{lub} = \frac{1}{\pi} + \frac{\delta}{\pi} \frac{(1-\pi)P - \pi V}{(1-\delta)r + \delta}. \tag{3}$$

The bank will take into the account depositors' participation constraint when setting deposit rates to be sure to induce them to prefer deposits. This will lead everyone to invest in deposits so that the size of the bank equals the depositors' endowment, i.e., which totals to one in equilibrium.[9]

3.2 Lenders

By pledging collateral, the bank can borrow from F and C at $t = 1$ to meet deposit withdrawals. Both F and C set terms for collateralized borrowing, including the interest rate on the loan, the haircut on collateral, the frequency of haircut adjustments, and the seniority in bankruptcy. In what follows, we assume that lenders adjust their haircuts dynamically to reflect the current value of collateral, i.e., haircuts can be made contingent on the realization of ω. We also assume that F and C loans are full recourse and enjoy seniority in bankruptcy. All these choices reflect current practices, but our model could be extended to consider alternative configurations, which we leave for future work.[10]

These assumptions allow us to focus on how F and C set the interest rate and collateral haircut for their loans. We solve for a pair of interest rates and haircuts that protect the borrower from credit/default risk. Given that haircuts can be set to reflect the realization of ω, the no-default lending contract is feasible and optimal.[11]

Interest rates. Let R_j, $j \in \{F, C\}$, denote the lending rates that F and C offer. Under the no-default lending contract, F and C will set rates to at least cover their funding costs. F's debt is usually placed with money market funds and could enjoy an implicit government guarantee along with a series of regulatory exemptions and income tax exemptions that that make their debt attractive.[12] Hence, their funding cost is typically close to the policy rate, r. We assume that F act competitively and set R_F equal to their funding costs, i.e., R_F.[13,14] C funds itself at the policy rate but it typically charges a penalty $p > 0$ over the policy rate for discount window borrowing, i.e., $R_C = r + p$.[15] As elaborated

[9] The model can be easily extended to feature an endogenous bank size, but doing so does not add further insights for key results on the discount window design.

[10] Typically, F adjust their haircuts to market conditions and their loans can be callable. C also dynamically set haircuts, but may reset them once a month. Moreover, both F and C loans are full recourse and enjoy seniority in bankruptcy, with F loans being super senior.

[11] See, also Fostel and Geanakoplos, 2015 for the optimality of no-default collateralized loan contracts.

[12] Congressional Budget Office, The Role of Federal Home Loan Banks in the Financial System, 2024.

[13] R_F is analogous to the "all-in" rate, which is the lending rate that FHLBs offer adjusted for the fact that profits are returned back to FHLBs member, effectively reducing the lending rate a member pays to borrow.

[14] F funding costs may include a small premium for intermediation costs or for risk-compensation in times of aggregate illiquidity. Our analysis goes through even if we introduced such a positive premium.

[15] R_C is the actual borrowing rate on top of which we could have added an additional component to incorporate any potential stigma associated with discount window borrowing. As it will become clear, our analysis would remain the same even if we added such component.

in Section 2, the modeling choice for rates is consistent with the stylized facts.

Haircuts on collateral. Given a realization of ω, the value of the illiquid asset at $t = 2$ is $\omega R(1 - \ell)$, where ℓ is the percentage of the bank's assets that are liquid. Denote by ξ_j, $j = \{F, C\}$, the haircut applied on illiquid collateral for borrowing at $t = 1$, which is endogenously determined. The total lendable value of illiquid collateral is, then, $\xi_j \omega R(1 - \ell)$.

Seizing and liquidating the collateral can be costly and can differ for F and C. We denote by $\gamma_j \omega R(1 - \ell)$, the total recoverable value of illiquid collateral, with $\gamma_j \leq 1$. The parameter γ_j captures how *pledgeable* collateral is at each lender. In order to ensure full repayment of the loan obligation, the recoverable value of collateral must exceed the lendable value plus interest, i.e.,[16]

$$\underbrace{\xi_j \omega R(1-\ell)}_{\substack{\text{Principal} \\ (LendableValue)}} \underbrace{R_j}_{\text{Interest}} \leq \underbrace{\gamma_j \omega R(1-\ell)}_{\substack{\text{Recoverable Value} \\ \text{of collateral}}} \quad \Rightarrow \quad \xi_j \leq \gamma_j R_j^{-1}. \tag{4}$$

Assuming that haircuts are set competitively, (4) will hold with equality, yielding the equilibrium haircut as a function of the pledgeability and the interest rate:

$$\xi_j = \gamma_j R_j^{-1}. \tag{5}$$

Thus, haircuts across lenders can differ not only due to different interest rates, but also because of differences in pledgeability of collateral at each lender. We assume that the illiquid assets are less pledgeable at F than C, i.e., $\gamma_F < \gamma_C$. Without loss of generality and to reduce notation, we also assume that $\gamma_C = 1$. This modeling assumption is consistent with the stylized fact—presented in Section 2—that FHLBs and the discount window set different haircuts for the same type of assets, with the haircuts at the window being better excluding mortgages. [17]

In what follow, we examine how the interplay between the haircut size and the interest rate affect both the lender choice and the run dynamics. Before turning to that, it is worth noting that, in principle, banks could liquidate assets in the markets instead of borrowing from F or C pledging them as collateral. In practice, banks would do so if the market price was better than the collateralized borrowing terms and liquidating assets in secondary markets is not an option for C&I or residential loans, which are typically pledged at FHLBs and the discount window. The existing literature has extensively studied the run risk dynamics from liquidating assets at a loss. Given that our focus is on collateralized borrowing, which the bank-run literature has abstracted from, we will assume that borrowing from F or C dominates liquidating assets to meet withdrawals (see Online Appendix for a detailed proof). Our assumption is consistent with the empirical finding in Cipriani et al. (2024)

[16] Our model can be adjusted to consider haircuts other than those that guarantee full repayment. For example, the central bank may offer a haircut under which the recoverable value of collateral just covers the funding cost, r, but not the penalty, p, or even offers subsidized haircuts that would expose the central bank to losses. We leave such analysis to future work that could potentially study the optimal design of "subsidized" Emergency Liquidity Assistance programs.

[17] In practice, γ_C can be higher than γ_F for certain asset classes, such as mortgages. However, this nuance does not matter for our analytical results, which are qualitative in nature, and modeling it would just add unnecessary notational complexity. As we establish, as long as $\gamma_F \geq \gamma_C$, all banks will prefer F irrespective of how big the difference is.

show that during the March 2023 US Banking Turmoil banks that experienced runs borrowed either from FHLBs or the discount window, but did not liquidate assets.

3.3 Banker and Bank

The banker (B) is risk-neutral and makes all investment and funding decisions to maximize her own expected payoff. At $t = 0$, she receives deposits from depositors and makes investment decisions. As mentioned, depositors will place all their initial endowment at the bank as long as their participation constraint, (2), is satisfied. Hence, the bank receives funds equal to one and decides the portion ℓ that goes into the liquid asset, while the remaining funds are invested in the illiquid asset.

At $t = 1$, a portion of depositors $\lambda \in [\delta, 1]$ withdraw their deposits and the bank needs to decide how to serve these withdrawals. In equilibrium, there will either be a run on the bank or not. Absent a run, only impatient depositors withdraw and $\lambda = \delta$. In a run, all depositors will withdraw and $\lambda = 1$. However, we derive the bank's balance sheets not only for these corner cases but also for any intermediate $\lambda \in (\delta, 1)$ in order to specify all out-of-equilibrium outcomes for which patient depositors need to form expectation when contemplating whether to withdraw their deposits at $t = 1$.

Because the interest on liquid assets from $t = 1$ to $t = 2$, r, is (weakly) lower than the collateralized borrowing rates, R_j, the bank will first use the liquid asset to meet withdrawals and then borrow any additional amount from either F or C. To reduce the number of cases studied, we will focus on an equilibrium where $\ell < \delta r_D$, i.e., the bank need to borrow at $t = 1$ even if there is no run and only impatient depositors run in equilibrium. This case allows us to also study the choice of lender j in normal times when the probability of a run is zero.[18] Hence, the liquidity shortfall of the bank is $\lambda r_D - \ell$.

The portion $y_j(\omega, \lambda)$ of illiquid assets that needs to be pledged as collateral at lender j to cover the liquidity shortfall is, then, given by

$$\underbrace{y_j(\omega,\lambda)\xi_j\omega R(1-\ell)}_{\substack{\text{Lendable value}\\ \text{of pledged collateral}}} = \underbrace{\lambda r_D - \ell}_{\text{Loan to meet shortfall}} \quad \Rightarrow \quad y_j(\omega,\lambda) = \frac{(\lambda r_D - \ell)R_j}{\gamma_j \omega R(1-\ell)}, \tag{6}$$

using (5). In other words, $y_j(\omega, \lambda)$ is the portion of the illiquid assets that need to pledged such that the lendable value of collateral, after applying the haircut, is equal to the liquidity shortfall the bank wants to cover. The higher the lending rate R_j and the lower the pledgeability of collateral γ_j are, the larger is the amount of collateral that needs to be pledged to cover the same liquidity shortfall.

[18]The liquidity regulations that have been implemented since the global financial crisis, such as the liquidity coverage ratio (LCR), still allow for banks to choose $\ell < \delta r_D$. This is because these regulations pertain to banks' expected net liquidity needs. In particular, regulatory liquidity is calculated by summing the liquid assets that banks hold (ℓ in our model) plus some less liquid assets (such as corporate bonds or payments coming from all other banking activities) and subtracting expected fund outflows (δr_D in our model). If the realized level of inflows is unexpectedly small, then ℓ will be far below the level of expected deposit outflows. This condition is also relevant in practice, as evident by looking at LCR disclosures of the biggest banks.

Conditional on surviving a run at $t = 1$, the profits (equity) at $t = 2$ of the bank are given by

$$\Pi_j(\omega,\lambda) = \underbrace{\omega R(1-\ell)}_{\text{Loan Payoff}} - \underbrace{(1-\lambda)R_D}_{\text{Remaining deposits}} - \underbrace{(\lambda r_D - \ell)R_j}_{\text{Collateralized loan}}, \tag{7}$$

which depend on the choice of the lender j. The profits are equal to the realized loan payoff minus the payment on the remaining deposits and the repayment of the collateralized loan. Note that profits are decreasing in R_j but are not directly affected by pledgeability γ_j. This is intuitive. As long as the bank is solvent, i.e., profits are positive, the lender does not seize the collateral and the bank enjoys the payoff on the whole illiquid asset holdings after repaying the borrowed amount plus interest. Note that $\Pi_j(1,\delta) > 0$ is necessary for a banking equilibrium to exist, otherwise the bank would never intermediate.

We will focus on the realistic case that $R_D < R_j$, i.e., deposit funding is cheaper that collateralized borrowing due to the presence of payment services of deposits that generate a deposit franchise (though, we also derive results for $R_D > R_j$). Lemma 3 derives the sufficient condition on exogenous parameters such that $R_D < R_j$ in any equilibrium with run risk. Intuitively, the condition requires a high enough convenience yield from deposits to compensate for run risk.

Lemma 3. *If $\pi > \max\{\underline{\pi}, 0.5\}$, there exists a lower bound for the convenience yield for each π, $\underline{V}(\pi)$, such that $R_D < R_j$ for $V > \underline{V}(\pi)$.*

From Lemmas 2 and Lemma 3, we get that $d\Pi_j/d\lambda < 0$, because $R_D < r_D R_j$. In other words, the profits are decreasing in the amount of withdrawals λ. We make the following assumption such that the bank can never fail irrespective of the level of λ when the state realization is $\omega = 1$ with certainty.

Assumption 1. $R > \dfrac{r_D^{lub} - \ell}{1-\ell} R_j$ *and* $\gamma_j > \dfrac{r_D^{lub} - \ell}{1-\ell}\dfrac{R_j}{R}$.

This assumption is reasonable and realistic. It says that with the highest possible payoff and the maximum amount of pledgeability of the illiquid asset, obtained for $\omega = 1$, the bank is immune from runs, i.e., the bank profits are positive, $\Pi_j(1,\lambda) > 0$, and the bank has enough collateral to pledge to lender j, $y_j(1,\lambda) < 1$, even when $\lambda = 1$. However, it is a bit restrictive as we have assumed the lax upper bound for r_D to guarantee sufficiency. As we show below, the bank will optimally set $r_D = 1$ and, hence, the set of economies that a bank is immune to runs for $\omega = 1$ expands to include those with $R > R_j$ and $\gamma_j > R_j/R$.

In equilibrium, only impatient depositors withdraw in the certain state that $\omega = 1$ and the bank profits are $\Pi_j(1,\delta) = R(1-\ell) - (1-\delta)R_D - (\delta r_D - \ell)R_j$. The following proposition summarizes the choice of lender j by the banker in this state.

Proposition 1. *In the certain state, a run cannot occur, only impatient depositors withdraw, and the banker chooses lender j offering the lowest lending rate R_j to meet the liquidity shortfall $\delta r_D - \ell$.*

The proof follows directly from the above analysis. Proposition 1 says that only the lending rate matters when a run does not occur, so that the haircut is irrelevant; as long as both lenders offer haircuts under which the bank has sufficient collateral to meet any withdrawals (Assumption 1).

Note that this result is not true if the bank has enough collateral to meet impatient withdrawals, but not all possible withdrawals. In the next section, we turn to how both the interest rate and the haircut matter for the choice of lender when runs are possible. To do so, we need to derive the behavior of depositors precipitating a run and determine the probability that a run may occur.

Before that we present the maximization problem of the banker. Depending on the realization of ω, either only impatient or all depositors withdraw and, hence, the banker's expected payoff is

$$\overline{\Pi}_B = \max_{\ell, r_D, R_D, j} \left[\pi \cdot \Pi_j(1,\delta) + (1-\pi) \cdot \int_{\omega_j^*}^{1} \Pi_j(\omega,\delta) d\omega \right]. \tag{8}$$

The banker will choose liquid assets ℓ, the deposit rates r_D and R_D, and lender j that maximizes her expected profits. In doing so, the banker internalizes the participation constraint of depositors (2) and how her choices affect run risk ω_j^*. We present the optimality condition in Section 3.6.

It is important to note that the choice of $\{\ell, r_D, R_D\}$ takes place at $t = 0$, while the choice of lender happens at $t = 1$ after the resolution of uncertainty about the distribution of ω. This timing means that the banker knows whether $\omega = 1$ (occurring with probability π) or $\omega \sim U[0,1]$ (occurring with probability $1-\pi$) when choosing which lender j to use.

With respect to the deposit contract, the bank will trade off r_D against R_D. Increasing R_D allows the banker to decrease r_D, while satisfying depositors' participation constraint, and vice versa. Given Lemmas 1 and 2, the following Lemma establishes when the banker chooses $r_D = 1$.

Lemma 4. *For $P \geq \underline{P}$ and $\pi \geq \max(\underline{\pi}, 0.5)$, we have $r_D = 1$ if $\dfrac{d\omega_j^*}{dR_D}\dfrac{1}{1-\delta} < \dfrac{d\omega_j^*}{dr_D}\dfrac{1}{\delta}$.*

We show in the Online Appendix that the condition in Lemma 4 is always true in equilibrium. This result is intuitive. Because depositors have linear preferences, there is no motive to provide liquidity insurance to impatient depositors that would call for $r_D > 1$. Hence, the trade off between r_D and R_D is determined by how they affect run risk. A higher r_D depletes bank liquidity at $t = 1$ and unambiguously increases run risk. A higher R_D increases the incentives to wait, but also erodes bank profitability. While the total effect on run risk may be ambiguous, it is less severe than the effect of higher r_D. To simplify the notation and exposition, for the rest of the paper we set $r_D = 1$.

3.4 Depositors' withdrawal decision

This section derives depositors' withdrawal decisions in the uncertain state at $t = 1$, taking as given the decisions made at $t = 0$. We derive the conditions under which an individual depositor wants to withdraw and show what the aggregate withdrawals imply for bank solvency and liquidity.

We proceed in seven steps. First, we derive the range of realizations of ω where depositors act without regard to any others' actions. Second, we show that under general and realistic conditions, a region of ω's exists, where individual actions depend on beliefs about aggregate withdrawals. In this region self-fulfilling runs are possible. Third, we establish cutoffs for the level of withdrawals in this region, that if exceeded, the bank either becomes insolvent or runs out of collateral to pledge. We establish the relationship between these cutoffs and how they are affected by the pledgeability of collateral, which is a key and novel result in our paper. Fourth, equipped with these cutoffs, we compute the payoff differential for an individual patient depositor between maintaining and withdrawing their deposits given their beliefs about aggregate withdrawals. Fifth, focusing on threshold strategies, we show that if such a strategy is followed by all depositors, then individuals can form well-specified beliefs about aggregate withdrawals and compute unique cutoffs for the regions of ω where the bank is expected to have sufficient collateral to meet withdrawal requests and be solvent after doing so. Using these beliefs and cutoffs, individuals can compute the expected payoff from withdrawing given their private signals and posteriors about ω. Sixth, we show that following a strategy that supposes a common threshold signal is an equilibrium. Finally, we establish the existence and uniqueness of the threshold signal under different choices of lenders j. Having established the uniqueness of the threshold-strategy equilibrium, we compute the probability of a run.

Step 1. A bank is immune to runs if two conditions both hold. First, the bank should have enough collateral to pledge for borrowing funds to serve all potential withdrawals. Second, the bank should remain solvent after repaying the more expensive, relative to deposit funding, collateralized loan.[19] The bank has sufficient collateral to meet withdrawals λ when fundamentals are ω if $y_j(\omega,\lambda)$ in (6) less than one. Because $y_j(\omega,\lambda)$ is strictly decreasing in ω and strictly increasing in λ, the bank is immune to runs if ω is higher than a cutoff given by $y_j(\overline{\omega}''_j, 1)$, i.e.,

$$\omega \geq \overline{\omega}''_j \equiv \frac{R_j}{\gamma_j R}. \tag{9}$$

Turning to solvency, bank profits, given by (7) are decreasing in λ for the realistic case that $R_D < R_j$ (see Lemma 3), and are increasing in ω. Hence, the bank remains solvent and is immune to runs if ω is higher than a cutoff given by $\Pi_j(\overline{\omega}', 1)$, i.e.,

$$\omega \geq \overline{\omega}'_j \equiv \frac{R_j}{R}. \tag{10}$$

If $\omega \geq \overline{\omega}_j \equiv \max(\overline{\omega}'_j, \overline{\omega}''_j) = \overline{\omega}''_j$, the bank is always solvent and has sufficient collateral to pledge to raise any funds that are needed. Hence, runs are impossible. $\overline{\omega}_j$ is known as the upper dominance cutoff, which defines the region where no depositor withdraws, irrespective of beliefs about the actions of others. The following lemma establishes the existence of the upper dominance cutoffs.

Lemma 5. *Given Assumption 1, the upper dominance region always exists, i.e.,* $\overline{\omega}_j < 1$. *Moreover,* $\overline{\omega}_F \leq \overline{\omega}_C$ *if* $\gamma_F \geq R_F/R_C$, *and* $\overline{\omega}_F > \overline{\omega}_C$ *otherwise.*

[19] If the bank defaults at $t = 2$, the patient depositors do not enjoy the convenience yield and receive less than R_D, i.e., less that the outside option from withdrawing early r. As a result, patient depositors would have an incentive to withdraw.

It is also helpful to describe the conditions when a run is inevitable. This happens if the bank does not have enough collateral to pledge to even serve withdrawals by impatient depositors or if the realization of ω is so bad that the bank is insolvent irrespective of the level of withdrawals. The bank has insufficient collateral if ω_j is lower that a cutoff $\underline{\omega}''_j$ given by $y_j(\underline{\omega}''_j, \delta)$, i.e.,

$$\omega < \underline{\omega}''_j \equiv \frac{R_j(\delta - \ell)}{\gamma_j R(1-\ell)}. \tag{11}$$

The bank is always insolvent if ω is lower than a cutoff $\underline{\omega}'_j$ given by $\Pi_j(\underline{\omega}'_j, \delta) = 0$, i.e.,

$$\omega < \underline{\omega}'_j \equiv \frac{R_D(1-\delta) + R_j(\delta - \ell)}{R(1-\ell)}. \tag{12}$$

If $\omega < \underline{\omega}_j \equiv \max(\underline{\omega}'_j, \underline{\omega}''_j)$, the bank is either insolvent after serving impatient withdrawals or has insufficient collateral to borrow the funds it needs, and an individual patient depositor chooses to withdraw independently of what others do. The relationship between $\underline{\omega}'_j$ and $\underline{\omega}''_j$ depends on γ_j such that $\underline{\omega}'_j > \underline{\omega}''_j$ if and only if

$$\gamma_j > \underline{\gamma}_j \equiv \frac{(\delta - \ell)R_j}{(1-\delta)R_D + (\delta - \ell)R_j} < 1. \tag{13}$$

The following Lemma establishes the existence of $\underline{\omega}_j$ are known as the lower dominance cutoffs.

Lemma 6. *The lower dominance region always exists, i.e.,* $\underline{\omega}_j > 0$. *Moreover,* $\underline{\omega}_F < \underline{\omega}_C$ *if* $\gamma_F/R_F \geq \underline{\gamma}_C/R_C$, *and* $\underline{\omega}_F > \underline{\omega}_C$ *otherwise.*

These conditions are intuitive. Given the borrowing from F is cheaper, the lower dominance threshold is below than that for borrowing from C, if the pledgability of collateral with F is high enough. Under these conditions, it makes sense that the realizations of ω that are sufficient to trigger a unilateral decision to run would have to be worse if the bank borrows from F rather than C.

Step 2. For intermediate realizations of ω between the lower and upper dominance regions, self-fulfilling runs are possible whereby an individual patient depositor may decide to withdraw if they believe that other depositors may do the same. These runs are also driven by bad fundamentals but for which high-order beliefs also play an amplifying role. These runs are known in the literature as panic-driven runs but we will refer to them as *coordinated runs* to distinguish them from the *unilateral runs* described above and avoid any misinterpretation that they are not driven by bad fundamentals. Beliefs amplify the bad realization of fundamentals, making the probability of a coordinated run higher than the probability of a unilateral run. It follows that coordinated runs are possible if $\underline{\omega}_j < \overline{\omega}_j$. The following Lemma establishes the conditions under which this is true.

Lemma 7. *Coordinated runs are always possible if* $R_D < R_j$ *irrespective of lender* j. *They are also possible if* $R_F < R_D$ *and* $\gamma_F < \underline{\gamma}_F(1-\ell)/(\delta - \ell)$ *if the bank is borrowing from* F.

The first case is intuitive. If the bank needs to replace cheaper deposit funding with more expensive collateralized funding, then it may become insolvent or run out of collateral if enough depositors withdraw (unless the realization of ω is above the upper dominance cutoff). Lemma 3 derives the conditions for the convenience yield for this to be true. The second case describes the possibility of a run occurring even if the lending rate by the FHLB is lower than the cost of deposit funding. This can happen when the haircut on the borrowing is so steep that the bank cannot secure enough funding to meet withdrawals.

Step 3. In the intermediate region for ω, the bank can fail either due to insolvency or due to insufficient borrowing capacity. Given that profits $\Pi_j(\omega,\lambda)$ are decreasing in λ, the maximum level of withdrawals, $\hat{\lambda}_j$, that the bank can sustain while remaining solvent is given by $\Pi_j(\omega,\hat{\lambda}_j)$, i.e.,

$$\hat{\lambda}_j = \frac{\omega R(1-\ell) - R_D + \ell R_j}{R_j - R_D}. \tag{14}$$

Similarly, given the required collateral, $y_j(\omega,\lambda)$, for borrowing from j is increasing in λ, the maximum level of withdrawals, $\bar{\lambda}_j$, that the bank can sustain without running out of collateral to pledge is given by $y_j(\omega,\bar{\lambda})$, i.e.,

$$\bar{\lambda}_j = \ell + \gamma_j R_j^{-1}\omega R(1-\ell). \tag{15}$$

We call $\hat{\lambda}_j$ and $\bar{\lambda}_j$ the *insolvency cutoff* and the *insufficient collateral cutoff*, respectively.

Notice that $d\hat{\lambda}_j/dR_j = -(\omega R - R_D)(1-\ell)/(R_j - R_D)^2 < 0$ for intermediate values of ω. Hence, the lower is the lending rate R_j, the greater is the ability of the bank to withstand withdrawals while remaining solvent. Note that $\hat{\lambda}_j$ does not depend on pledgeability γ_j because $\hat{\lambda}_j$ is derived from the profit function. Moreover, $d\bar{\lambda}_j/dR_j = -\bar{\lambda}_j R_j^{-1} < 0$ and $d\bar{\lambda}_j/d\gamma_j = R_j^{-1}\omega R(1-\ell) > 0$, i.e., the bank can sustain fewer withdrawals before running out of collateral when the lending rate is higher and the pledgeability of collateral is lower.

A critical question is which of these two cutoffs is smaller. In other words, as withdrawals mount, does the bank first become insolvent, or does it run out of collateral to meet withdrawals? The answer to this question is of paramount importance to the design and effectiveness of the discount window and has not been studied in the literature.

For a given ω, the relationship between the two cutoffs depends on the pledgeability of collateral. In particular, $\hat{\lambda}_j > \bar{\lambda}_j$ requires that

$$\gamma_j < \gamma_j^P(\omega) \equiv \frac{1 - R_D/(\omega R)}{1 - R_D/R_j}. \tag{16}$$

In words, if the pledgeability of collateral is low enough, then the banks runs out of borrowing capacity before the interest cost of borrowing render the bank insolvent. The following Lemma establishes the relationship between the two cutoffs in the intermediate region of ω's.

Lemma 8. *If the bank borrows from C, then $\hat{\lambda}_C < \overline{\lambda}_C$ for all ω. If the bank borrows from F, then $\hat{\lambda}_F > \overline{\lambda}_F$ for $\gamma_F \leq \underline{\gamma}_F$. Otherwise, for $\gamma_F > \underline{\gamma}_F$, there exists an $\omega_F^P(\gamma_F) \in (\underline{\omega}_F, \overline{\omega}_F)$ such that $\hat{\lambda}_F < \overline{\lambda}_F$ for $\omega < \omega_F^P(\gamma_F)$ and $\hat{\lambda}_F \geq \overline{\lambda}_F$ for $\omega \geq \omega_F^P(\gamma_F)$. Equivalently, for any ω there exists $\gamma_F^P(\omega) \geq \underline{\gamma}_F$ such that $\hat{\lambda}_F < \overline{\lambda}_F$ for $\gamma_F > \gamma_F^P(\omega)$ and $\hat{\lambda}_F \geq \overline{\lambda}_F$ for $\gamma_F \leq \gamma_F^P(\omega)$.*

Lemma 8 is key to the analysis below of how collateralized borrowing terms matter for the choice of lender j when coordinated runs are possible. Before proceeding to the next step, we elaborate on this result to provide more intuition.

Both the insolvency and the insufficient collateral cutoffs are increasing in ω. But a higher ω has a larger effect on $\hat{\lambda}_j$ than on $\overline{\lambda}_j$, since $d\hat{\lambda}/d\omega = R(1-\ell)/(R_j - R_D)$ and $d\overline{\lambda}_j/d\omega = \gamma_j R(1-\ell)/R_j$. With respect to $\overline{\lambda}_j$, raising ω improves the ability to withstand withdrawals proportionally to the increase in the pledgeable value of assets discounted by the lending rate R_j. With respect to $\hat{\lambda}_j$, raising ω improves the ability to withstand withdrawals proportionally to the increase in the total value of assets discounted by the incremental cost of collateralized borrowing $R_j - R_D$. The reason is that the incremental cost rather than the lending rate matters for solvency. Hence, $\hat{\lambda}_j$ is more responsive than $\overline{\lambda}_j$ to changes in ω. Given that both are strictly decreasing in ω, the relative levels of the two cutoffs at $\overline{\omega}_j$, which is the highest ω for a coordinated run, determines which cutoff is reached first as withdrawals λ increase.

Suppose that the bank is borrowing from C. Notice that $\hat{\lambda}_C(\overline{\omega}_C) = \overline{\lambda}_C(\overline{\omega}_C)$, since the pledgeable value of assets at $t = 1$ is equal to the profit they deliver to the bank at $t = 2$ for perfect pledgeability, i.e., $\gamma_C = 1$. Since $\hat{\lambda}_C$ decreases faster than $\overline{\lambda}_C$ as ω decreases, the latter cutoff will always be higher than the former cutoff and, hence, the bank will become insolvent before it runs out of collateral to pledge as withdrawals increase. Figure 5 shows this diagrammatically.

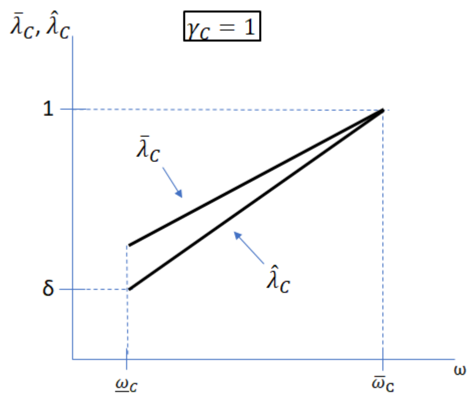


Figure 5: Insolvency and inefficient collateral cutoffs for lender C as a function of ω

Suppose alternatively that the bank is borrowing from F. Since $\gamma_F < 1$, the pledgeable value of assets at $t = 1$ is less than the profits they deliver to the bank at $t = 2$ and $\hat{\lambda}_F(\overline{\omega}_F) > \overline{\lambda}_F(\overline{\omega}_F) = 1$. Given that $\hat{\lambda}_F$ decreases with ω faster than $\overline{\lambda}_F$ the two cutoffs may cross, reversing their relationship. Lemma 8 shows that whether a crossing occurs in the intermediate region of ω's depends on pledgeability γ_F. Figure 6 shows this diagrammatically for different levels of γ_F.

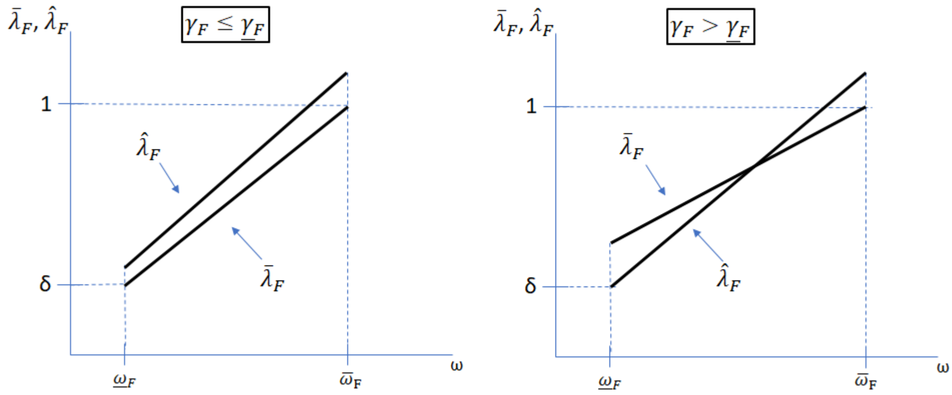


Figure 6: Insolvency and inefficient collateral cutoffs for lender F as a function of ω

Step 4. The withdrawal decision of an individual patient depositor depends on the perceived payoff differential between waiting until $t = 3$ and withdrawing at $t = 2$. We denote this differential by $\nu(\omega, \lambda)$ and report in (17) the value it takes for different beliefs about the level of withdrawals λ when the realization of fundamentals is equal to ω:

$$\nu_j(\omega,\lambda) = \begin{cases} R_D + V - r & \text{if} \quad \delta \leq \lambda \leq \min(\hat{\lambda}_j, \bar{\lambda}_j) \\ \dfrac{\omega R(1-\ell) - (\lambda - \ell)R_j}{1-\lambda} - r & \text{if} \quad \min(\hat{\lambda}_j, \bar{\lambda}_j) < \lambda \leq \bar{\lambda}_j \\ -\dfrac{\gamma_j R_j^{-1} \omega R(1-\ell) + \ell}{\lambda} r & \text{if} \quad \bar{\lambda}_j < \lambda \leq 1 \end{cases} . \tag{17}$$

We discuss the three regions in (17) in turn. The first case covers beliefs for λ when the banker remains solvent so that full repayment at $t = 3$ is possible. The payoff differential in this region is $R_D + V - r$, which is strictly positive, otherwise no patient savers would ever wish to wait. The second case is when the bank has sufficient collateral to meet withdrawals but has become insolvent. A patient depositor who waits will not receive the full payment R_D, but instead receives the pro-rata share of any remaining salvage value of the bank after collateralized lenders are repaid; deposits also lose their convenience yield in bankruptcy. Finally, in the third region, the bank runs out of collateral and is liquidated. A deposit gets zero if she waits, while she is repaid with probability $(\gamma_j R_j^{-1} \omega R(1-\ell) + \ell)/\lambda$ if she withdraws. The expected payoff differential is equal to $-(\gamma_j R_j^{-1} \omega R(1-\ell) + \ell)/\lambda r$.

Figure 7 plots the payoff differential for $\hat{\lambda}_j < \bar{\lambda}_j$ and $\hat{\lambda}_j \geq \bar{\lambda}_j$. From Lemma 8, $\nu_C(\omega, \lambda)$ is always depicted by the chart on the left, while for $\nu_F(\omega, \lambda)$ it depends on the level of γ and ω. The first discontinuity in the left chart at $\hat{\lambda}_j$ arises from the fact that depositors lose V when the bank becomes insolvent. The second discontinuity at $\bar{\lambda}_j$ arises because just before liquidation there is still some uncollateralized assets that can distributed in bankruptcy but the pledgeability of these assets is not enough to serve the additional withdrawals if $\gamma_j < 1$, leading to liquidation.[20] The

[20] To see more clearly, take the limits of $\nu_j(\omega, \lambda)$ as λ approaches $\bar{\lambda}_j$ from the left and from the right: $\lim\limits_{\lambda \to \bar{\lambda}_j^-} \nu_j(\omega, \lambda) =$

discontinuity in the right chart is the extreme manifestation of the aforementioned outcome: The bank has enough assets to be solvent for withdrawals $\lambda = \overline{\lambda}_j$, but the pledgeability of its collateral is so low that the bank cannot raise the funds it needs to serve fully more than $\overline{\lambda}_j$ withdrawals.

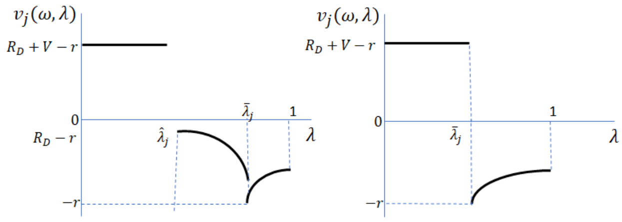


Figure 7: Payoff differential for different λ given ω.

Step 5. Because the sign of $\nu(\omega,\lambda)$ in (17) depends on beliefs about λ, a coordination problem among patient depositors exists and it can lead to multiple equilibria. To resolve the multiplicity, we model a global game whereby each patient depositor i receives a private noisy signal $x_i = \omega + \varepsilon_i$ at $t = 1$; ε_i are small error terms that are independent and uniformly distributed over $[-\varepsilon, \varepsilon]$. These signals not only provide information about ω, but also about other depositors' signals, so that inference about their actions is possible. We seek a symmetric equilibrium characterized by a threshold signal x_j^* such that an individual patient depositor will withdraw her deposits if $x_i < x_j^*$. Under such a threshold strategy, the number of depositors that withdraw at a given level of fundamentals ξ is

$$\lambda(\omega, x_j^*) = \begin{cases} 1 & \text{if } \omega < x_j^* - \varepsilon \\ \delta + (1-\delta) Prob\left(x_i \leq x_j^*\right) & \text{if } x_j^* - \varepsilon \leq \omega \leq x_j^* + \varepsilon \\ \delta & \text{if } \omega > x_j^* + \varepsilon \end{cases}. \tag{18}$$

If ω is lower than $x_j^* - \varepsilon$, all patient depositors receive signals $x_i < x_j^*$ and withdraw, i.e., $\lambda(\omega, x_j^*) = 1$. The opposite is true for $\omega > x_j^* + \varepsilon$. In this case, all patient depositors receive signals $x_i > x_j^*$ and keep their deposits in the bank, while only the impatient ones withdraw, i.e., $\lambda(\omega, x_j^*) = \delta$. Finally, if $\omega \in [x_j^* - \varepsilon, x_j^* + \varepsilon]$, some patient depositors receive signals that are lower than x_j^* and will withdraw their deposits; others will receive a signal higher than x_j^* and will keep their deposits in the bank. Because ε_i is independently and identically distributed, the law of large numbers obtains and the number of savers withdrawing for a given level of ω is $\lambda(\omega, x_j^*) = \delta + (1-\delta) Prob\left(x_i \leq x_j^*\right) = \delta + (1-\delta)\left(x_j^* - \omega + \varepsilon\right)/2\varepsilon$, which is increasing in x_j^* and decreasing in ω.

Given x_j^*, there are two cutoffs for fundamentals, $\omega_j^{\hat{\lambda}}(x_j^*)$ and $\omega_j^{\overline{\lambda}}(x_j^*)$, such that the bank is solvent for realizations $\omega \geq \omega_j^{\hat{\lambda}}(x_j^*)$ and has sufficient collateral for realizations $\omega \geq \omega_j^{\overline{\lambda}}(x_j^*)$. These two cutoffs are, respectively, the solutions to $\lambda(\omega_j^{\hat{\lambda}}(x_j^*), x_j^*) = \hat{\lambda}_j(\omega_j^{\hat{\lambda}}(x_j^*))$ and $\lambda(\omega_j^{\overline{\lambda}}(x_j^*), x_j^*) = \overline{\lambda}(\omega_j^{\overline{\lambda}}(x_j^*))$, the properties of which are established in Lemma 9.

$\frac{\omega R(1-\ell) - (\overline{\lambda}_j - \ell) R_j}{1-\lambda} - r = \frac{\omega R(1-\ell)(1-\gamma_j)}{1-\lambda} - r$ and $\lim_{\lambda \to \overline{\lambda}^+} \nu_j(\omega,\lambda) = -r$.

Lemma 9. Regions of ω with insolvency and insufficient collateral. *Consider signal threshold x_j^*. Then, there exist unique cutoffs $\omega_j^{\hat{\lambda}}(x_j^*)$ and $\omega_j^{\bar{\lambda}}(x_j^*)$, with $\omega_j^{\hat{\lambda}}(x_j^*) > \omega_j^{\bar{\lambda}}(x_j^*)$ if and only if $\hat{\lambda}_j(\omega_j^{\hat{\lambda}}(x_j^*)) < \bar{\lambda}(\omega_j^{\bar{\lambda}}(x_j^*))$.*

Under threshold strategy x_j^*, the beliefs of an individual depositor about aggregate withdrawals are well-defined, allowing her to compute the expected payoff differential between waiting and withdrawing given her private signal. The saver uses the signal to update her beliefs about the realization of ω. Given that both ω and ε_i are uniformly distributed, the posterior distribution of ω given x_i is $\omega|x_i \sim U[x_i - \varepsilon, x_i + \varepsilon]$. Thus, the expected payoff differential is

$$\Delta(x_i, x^*) = \frac{1}{2\varepsilon}\int_{x_i-\varepsilon}^{x_i+\varepsilon} \nu(\omega, \lambda(\omega, x^*))d\omega. \tag{19}$$

A patient depositor prefers to withdraw if $\Delta(x_i, x_j^*)$<0, for all $x_i < x_j^*$, and prefers to wait if $\Delta(x_i, x_j^*) > 0$, for all $x_i > x_j^*$. $\Delta(x_i, x_j^*)$ is continuous in x_i, because a change in the signal only changes the limits of integration $[x_i - \varepsilon, x_i + \varepsilon]$ and the integrand is bounded. Hence, a patient depositor who receives signal $x_i = x_j^*$ is indifferent between waiting and withdrawing if

$$\bar{\Delta}(x_j^*) \equiv \Delta(x_j^*, x^*) = \frac{1}{2\varepsilon}\int_{x_j^*-\varepsilon}^{x_j^*+\varepsilon} \nu(\omega, \lambda(\omega, x^*))d\xi = 0. \tag{20}$$

We need to establish two results. First, that a threshold strategy is an equilibrium, i.e., $\Delta(x_i, x_j^*) < 0$ for $x_i < x_j^*$ and $\Delta(x_i, x_j^*) > 0$ for $x_i > x_j^*$. Second, that there exists a unique x_j^* satisfying $\bar{\Delta}(x_j^*) = 0$. The proof of the first result is given in the next step.

Step 6. For signal $x_i < x_j^*$, decompose the intervals $[x_i - \varepsilon, x_i + \varepsilon]$ and $[x_j^* - \varepsilon, x_j^* + \varepsilon]$ into a common part $c = [x_i - \varepsilon, x_i + \varepsilon] \cap [x_j^* - \varepsilon, x_j^* + \varepsilon]$, and two disjoint parts $d^i = [x_i - \varepsilon, x_i + \varepsilon] \backslash c$ and $d^* = [x_j^* - \varepsilon, x_j^* + \varepsilon] \backslash c$. Under a common threshold strategy, all depositors form beliefs according to (18). Individual signals matter for depositors' posterior beliefs about ω and the possible realizations of λ. Hence, (19) and (20) can be also decomposed to common and disjoint parts, respectively, as $\Delta(x_i, x_j^*) = \Delta_{\omega\in c} + \Delta_{\omega\in d^i}$ and $\bar{\Delta}(x_j^*) = \Delta_{\omega\in c} + \Delta_{\omega\in d^*}$. From (18), λ is always one over d^i, thus $\Delta_{\omega\in d^i} = \int_{\omega\in d^i} \nu(\omega, 1)d\omega = -\int_{\omega\in d^i}(\gamma_j R_j^{-1}\omega R(1-\ell) + \ell)d\omega < 0$ from (17). Hence, it suffices to show that $\Delta_{\omega\in c} < 0$. We will use the facts that $\bar{\Delta}(x_j^*) = 0$, and that ν changes sign ("crosses zero") only once, while being positive and negative for higher and lower values of $\omega \in [x_j^* - \varepsilon, x_j^* + \varepsilon]$. Hence, $\Delta_{\omega\in d^*} > 0$ and $\Delta_{\omega\in c} < 0$, since the fundamentals are higher over d^* than c. The fact that ν may be increasing in ω in the lower segment of c does not matter, because ν is still negative in that segment. Essentially, observing a signal x_i below x_j^* shifts probability from positive values of ν to negative values of ν –recall that noise is uniformly distributed– and $\Delta(x_i, x_j^*) < \bar{\Delta}(x_j^*) = 0$. The argument holds trivially if the interval c is empty. The proof for $x_i > x_j^*$ is similar. As in **?**, single crossing of ν at zero suffices for a threshold strategy, should it exist, to be an equilibrium.

Step 7. To prove existence of x_j^*, note that $\bar{\Delta}(x_j^*) < 0$ for $x_j^* < \underline{\omega}_j - \varepsilon$, because this level of signal x_j^* implies that $\omega < \underline{\omega}_j$ and, thus, $v(\omega, \lambda(\omega, x_j^*)) < 0$ always. Similarly, $\bar{\Delta}(x_j^*) > 0$ for $x_j^* \geq \overline{\omega}_j + \varepsilon$. Moreover, all the integrands and the limits of integration in $\bar{\Delta}(x_j^*)$ are bounded and continuous in x_j^*, and the discontinuity in ν_j appears only in discrete points. Hence, $\bar{\Delta}(x_j^*)$ is continuous in x_j^*, which implies there are solutions to $\bar{\Delta}(x_j^*) = 0$.

Proving uniqueness is more cumbersome because of the perverse state monotonicity property analyzed in **?**. In particular, $\bar{\Delta}(x_j^*)$ may not be always increasing in x_j^*, which has been the typical property in many global-game frameworks (Morris and Shin, 2003; Goldstein and Pauzner, 2005). As can be easily seen from (17), ν_j is decreasing in ω for $\lambda > \bar{\lambda}_j$, which also implies that $\bar{\Delta}$ is decreasing in x_j^* for the region of fundamentals that define this region, i.e., $\omega < \omega_j^{\bar{\lambda}}(x_j^*)$ (see Kashyap et al. 2024 for a detailed exposition).[21]

Moreover, our framework introduces additional difficulties in proving the uniqueness of the equilibrium because the relationship between the insolvency and insufficient collateral cutoffs can vary (Lemmas 8 and 9). To establish uniqueness of x_j^* we will use a combination of the arguments put forward in Kashyap et al. 2024 along with strict convexity properties of $\bar{\Delta}$. As is typical in the literature, we will focus on the equilibrium as the noise shrinks to zero, $\varepsilon \to 0$. Proposition 2 below establishes the uniqueness of the signal threshold for limiting noise. We also present diagrammatically the logic underlying the approach we take in the proof. The general proof for non-vanishing noise is presented in the Online Appendix.

Note that taking the limit $\varepsilon \to 0$ and using the expressions in Lemma 9, the thresholds for fundamentals $\omega_j^{\hat{\lambda}}(x_j^*)$ and $\omega_j^{\bar{\lambda}}(x_j^*)$ converge to the same threshold ω_j^* equal to x_j^*. When working with $\varepsilon \to 0$, it is convenient to change the variable of integration in $\bar{\Delta}(x_j^*)$ from ω to λ. Recall that a depositor who receives signal x_j^* believes that λ is given by $\lambda(\omega, x_j^*) = \delta + (1-\delta)(x_j^* - \omega + \varepsilon)/2\varepsilon$ and that $\omega \sim U[x_j^* - \varepsilon, x_j^* + \varepsilon]$. Thus, as ω increases uniformly from $x_j^* - \varepsilon$ to $x_j^* + \varepsilon$, λ decreases uniformly from 1 to δ. Rewrite $\omega = x_j^* + \varepsilon - 2\varepsilon(\lambda - \delta)/(1-\delta)$ as a function of $\lambda \sim U[\delta, 1]$. Per change of variable of integration from ω to λ and using $\nu(\omega, \lambda)$ in (17), the indifference condition in (20) at the equilibrium $x_j^* \to \omega_j^*$ becomes:

$$\bar{\Delta}_j^* \equiv \lim_{\varepsilon \to 0} \bar{\Delta}(x_j^*) = \int_{\delta}^{\min(\hat{\lambda}_j, \bar{\lambda}_j)} (R_D + V - r) \frac{d\lambda}{1-\delta} + \int_{\min(\hat{\lambda}_j, \bar{\lambda}_j)}^{\bar{\lambda}_j} \left[\frac{\omega_j^* R(1-\ell) - (\lambda - \ell) R_j}{1-\lambda} - r \right] \frac{d\lambda}{1-\delta}$$
$$- \int_{\bar{\lambda}_j}^{1} \frac{\gamma_j R_j^{-1} \omega_j^* R(1-\ell) + \ell}{\lambda} r \frac{d\lambda}{1-\delta}. \qquad (21)$$

Proposition 2. Threshold-strategy equilibrium. *Assume that the model parameters are such that Lemmas 1-7 obtain. For $\varepsilon \to 0$, there exists a unique ω_j^* such that, when the bank borrows from lender j, a patient depositor does not withdraw at $t = 1$ if $x_i > \omega_j^*$, and withdraws if $x_i < \omega_j^*$.*

[21] This property implies that depositors have higher incentives to run on a stronger bank than a weaker bank, conditional on a run occurring, because their chances of getting paid are higher.In bank-run global games, the liquidation value of an asset is generically linked to the market value of the asset, something that the literature had largely abstracted from due to the technical difficulties this realism introduces.

Therefore, a run does not occur if $\omega \geq \omega_j^*$*, and the bank is fully liquidated if* $\omega_j < \omega_j^*$*.*

The following Proposition shows how these unique run thresholds ω_F^* and ω_C^* change with R_F, R_C, and γ_F, which are useful to derive the optimal lender choice in the following section.

Proposition 3. *The run threshold* ω_F^* *is increasing in* R_F *and decreasing in* γ_F*, while the run threshold* ω_C^* *is increasing in* R_C*.*

Intuitively, when the lending rates R_F or R_C then the run probability when borrowing from F or C is also higher. On the other hand, the run probability when borrowing from F is lower the higher the pledgeability of collateral is.

3.5 Bank's choice of lender

Having established the uniqueness of the run threshold for each lender-choice and, thus, computed the probability of a run, we proceed to derive the optimal choice of lender by the bank. Recall that the bank will choose its lender before the realization of ω. That is the bank will choose the lender j that maximizes its expected profits given by

$$\max_j \int_{\omega_j^*}^1 \Pi_j(\omega,\delta)d\omega = \max_j \int_{\omega_j^*}^1 [\omega R(1-\ell)-(1-\delta)R_D-(\delta-\ell)R_j]\,d\omega, \tag{22}$$

since only impatient depositors withdraw if $\omega \geq \omega_j^*$, and the bank is liquidated in a run, otherwise.

Because $R_F < R_C$, if it is also the case that $\omega_F^* < \omega_C^*$, i.e., the lower interest rate also translates into lower run risk, then borrowing from F is unambiguously preferred. However, we show that $\omega_F^* \lesseqgtr \omega_C^*$ depending on γ_F even if $R_F < R_C$. In particular, the bank may choose to borrow from C for low enough γ_F if the reduction in run risk is high enough to compensate for the higher interest rate. This is one the key result of our paper that we prove in Proposition 4.

Proposition 4. *For* $R_D < R_F < R_C$ *and any penalty rate p, there exists a cutoff* $\gamma^+(p) < 1$ *such that the bank chooses lender F for* $\gamma_F \geq \gamma^+(p)$*, and chooses lender C otherwise.*

In sum, Proposition 4 establishes that banks with sufficiently highly pledgeable assets will go to F to take advantage of its lower interest rate. Conversely, banks with sufficiently low levels of what can be pledged will go to C despite its higher lending rate.[22]

We conclude by showing in Proposition 5 that the aforementioned result does not carry over to the case of unilateral runs, in which depositors do not need to form beliefs about actions of others. It is the nature of coordinated runs and the underlying beliefs about the actions of others that make the lending rate and the haircut, matter differently depending on bank characteristics. We should reiterate that both unilateral and coordinated runs are caused by bad realizations of fundamentals, yet beliefs amplify the initial fundamental shock for the latter. Proposition 4 showed that the different

[22] This property would carry over also for a bank portfolio consisting of assets with different degrees of pledgeability. The bank would use the more pledgeable assets to borrow from F and the less pledgeable assets to borrow from C.

combinations of lending rate and haircut address this amplification differently depending on the pledgeability of the banks' assets. Proposition 5 establishes that only the lending rate is the key term for unilateral runs that lack such amplification.

Proposition 5. *Suppose $R_D > R_j$, $j \in \{F, C\}$, and $\gamma_F \geq \underline{\gamma}_F(1-\ell)/(\delta-\ell)$ such that only unilateral runs are possible. Then, the banker chooses F due to the lower lending rate R_F irrespective of γ_F.*

3.6 Private equilibrium

This section is only for completeness and a reader may feel free to jump direectly to Section 4. We now consider the optimal choice of R_D and ℓ at $t=0$.[23] With probability π, the banker will choose the lender with the lowest lending rate, i.e., F, independent of R_D and ℓ (Proposition 1). With probability $1-\pi$, the choice of lender depends not only on the relative R_j's, but also on the level of γ_F relative to thresholds that depend on R_D and ℓ (Proposition 4). Given that the banker can choose the lender after the choice of R_D and ℓ have been made, it is important that we solve for a time-consistent choice such that the banker takes into account how R_D and ℓ matter for the future choice of lender j taking the decision rule of its future self as given. Then, the first-order condition of (8) with respect to $y = \{R_D, \ell\}$ are:

$$\overbrace{\pi\frac{d\Pi_F(1,\delta)}{dy} + (1-\pi)\left[\int_{\omega_k^*}^{1}\frac{d\Pi_k(\omega,\delta)}{dy}d\omega - \frac{d\omega_k^*}{dy}\Pi_k(\omega_k^*,\delta)\right]}^{\substack{\text{Effect of higher } y=\{R_D,\ell\} \\ \text{on bank profits}}} \quad \overbrace{+\psi\frac{dPC}{dy}}^{\substack{\text{Effect of higher } y=\{R_D,\ell\} \\ \text{on depositors' participation}}} = 0, \tag{23}$$

where $k = F$ if $\int_{\omega_F^*}^{1}\Pi_F(\omega,\delta)d\omega > \int_{\omega_C^*}^{1}\Pi_C(\omega,\delta)d\omega$ evaluated at the R_D and ℓ that solve (23), and $k = C$ otherwise. ψ is the Lagrange multiplier on the participation constraint (2), $d\omega_k^*/dR_D = -(d\bar{\Delta}_k^*/dR_D)/(d\bar{\Delta}_k^*/d\omega_k^*)$, $d\omega_k^*/d\ell = -(d\bar{\Delta}_k^*/d\ell)/(d\bar{\Delta}_k^*/d\omega_k^*)$—both obtained by implicitly differentiating (21).

The first two terms in (23) capture how increasing R_D or ℓ affects bank profits both directly and indirectly via the probability of a run. The third term captures the effect of a higher R_D or ℓ on depositors' payoff and the tightness of the participation constraint. By combining the first-order conditions for R_D and ℓ, we can eliminate ψ and obtain the following optimality condition:

$$\frac{\pi\frac{d\Pi_F(1,\delta)}{dR_D} + (1-\pi)\left[\int_{\omega_k^*}^{1}\frac{d\Pi_k(\omega,\delta)}{dR_D}d\omega - \frac{d\omega_k^*}{dR_D}\Pi_k(\omega_k^*,\delta)\right]}{\pi\frac{d\Pi_F(1,\delta)}{d\ell} + (1-\pi)\left[\int_{\omega_k^*}^{1}\frac{d\Pi_k(\omega,\delta)}{d\ell}d\omega - \frac{d\omega_k^*}{d\ell}\Pi_k(\omega_k^*,\delta)\right]} = \frac{\frac{dPC}{dR_D}}{\frac{dPC}{d\ell}} \tag{24}$$

Definition 1. *For conditions that satisfy Lemmas 1-4, the private equilibrium is defined as the share of liquid assets ℓ, the deposit rate R_D, the run thresholds, ω_F^* and ω_C^*, and a choice of lender j that satisfy: (i) the optimality condition* (24)*, (ii) the depositors' incentive compatibility constraint* (1)*, (iii) the depositors participation constraint* (2)*, and (iv) the determination of the run thresholds in* (21) *for $j = \{F, C\}$.*

[23] Recall that we have set $r_D = 1$ given that in equilibrium the conditions of Lemmas 1, 2, and 4 are satisfied.

4 Empirical analysis

This section studies whether cross-sectional differences in collateral composition help explain how banks allocate borrowing between the DW and the FHLBs, especially in stress. The empirical motivation follows directly from the model. In the framework, the relevant terms of secured borrowing are not only the interest rate charged on the loan, but also the haircut applied to pledged collateral. These two margins jointly shape facility choice. Banks that are relatively more sensitive to funding costs tilt toward FHLBs, which typically offer lower borrowing rates, whereas banks that are more constrained by collateral pledgeability tilt toward the DW when it applies more favorable haircuts to the assets they hold.

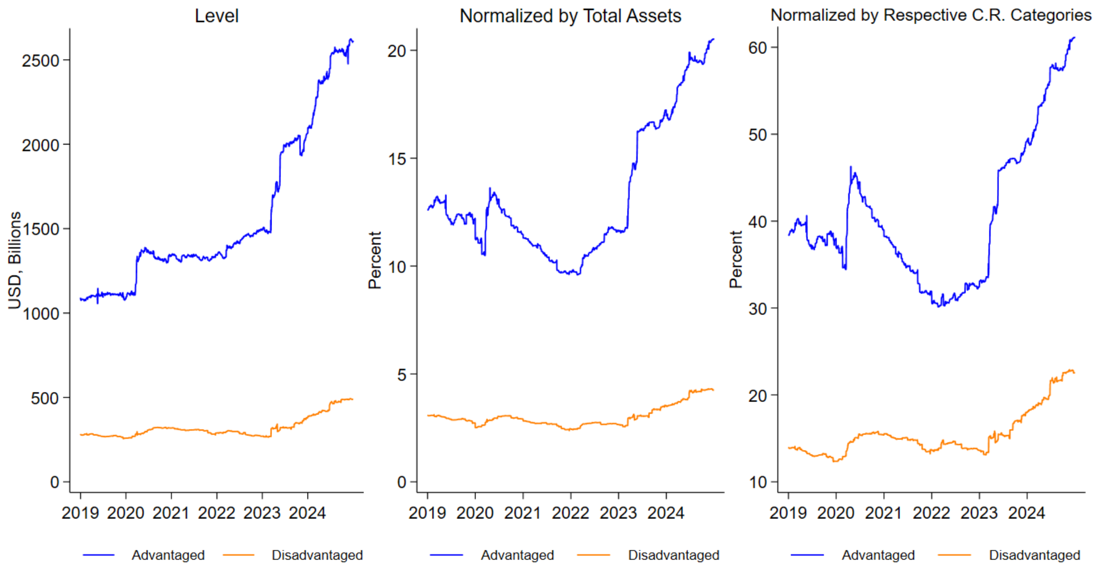
The institutional and aggregate evidence makes this mechanism plausible. In both March 2020 and March 2023, borrowing rose sharply at both facilities, with the increase in DW borrowing quantitatively meaningful relative to the increase in FHLB borrowing. At the same time, pledged collateral differs systematically across the two lenders. Mortgages account for a much larger share of FHLB collateral, while the DW is used more heavily against other loans and securities. These differences line up with the pricing of contracts: FHLBs generally offer more competitive all-in rates, while the DW applies more favorable haircuts for several non-mortgage asset classes, including C&I loans, credit card receivables, mortgage-backed securities, and other less-liquid collateral.

Table 5 in the Appendix provides the institutional foundations. The table shows that haircut schedules differ systematically across the DW and FHLBs. The DW does not dominate uniformly, nor do the FHLBs. Rather, the relative advantage depends on the asset being pledged. FHLBs tend to offer better treatment for mortgage collateral, especially first-lien and certain junior-lien residential mortgages. By contrast, the DW applies better haircuts to several securities and to less-liquid loan categories. This heterogeneity is precisely what gives empirical content to the model's prediction that banks with different balance-sheet composition will make different facility choices.

Figure 8 provides a direct bridge between the institutional haircut differences and banks' actual collateral use at the DW. The figure decomposes DW collateral by whether the asset is relatively advantaged at the window relative to the FHLBs. The pattern is consistent with the mechanism emphasized in the model: when banks turn to the DW, they do not post collateral indiscriminately, but instead tilt toward collateral types for which the window offers relatively more favorable terms. In that sense, the figure provides a reduced-form visualization of the contract dimension emphasized by the theory. The relevant distinction across facilities is not only the borrowing rate, but also how each facility values the collateral that banks can mobilize.[24]

[24]Figure 11 in the Appendix shows DW collateral changes by asset category in 2020 and 2023.

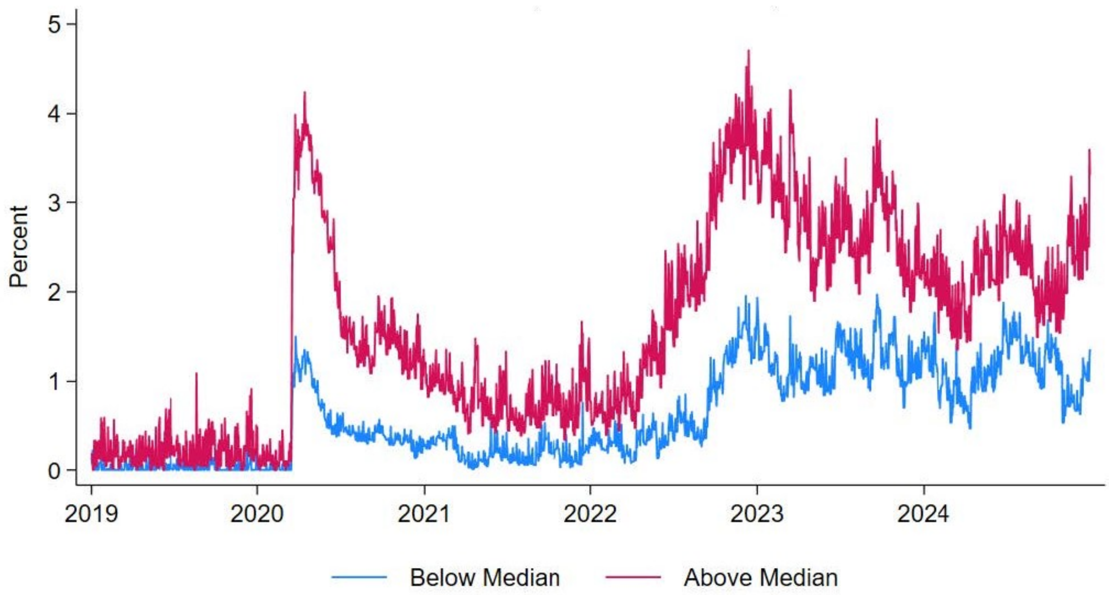
Figure 8: Discount Window Collateral by Advantage Status



Notes: The chart plots DW collateral separately by advantage status. "Advantaged" collateral refers to collateral types for which the DW applies more favorable terms than the FHLBs, based on the haircut comparison described in Table XX. The figure is constructed using confidential DW collateral data.

Figure 9 brings the same point to the bank level. It plots the share of DW borrowing in total DW plus FHLB borrowing separately for banks with high and low values of the ratio of on-balance-sheet DW-advantaged collateral to total assets, where the grouping is defined relative to the cross-sectional median at the end of the previous year. The figure shows that banks with more ex-ante DW-advantaged collateral on their balance sheets rely more heavily on the DW relative to the FHLBs once funding pressures emerge.

Figure 9: Discount Window Borrowing by Bank and Advantage Status of Collateral



Notes: The figure plots the ratio DW Borrowing to total borrowing (DW + FHLB borrowing) separately for banks with low and high values of the ratio of on-balance-sheet DW-advantaged collateral to total assets. For each bank, the ratio is computed as of December 31 of year t-1. For each year, "high-ratio" banks are those with values above the cross-sectional median ratio on December 31 of year t-1 and "low-ratio" banks are those below that median. DW-advantaged collateral is defined using the haircut comparison in Table XX.

Taken together, Figures 8 and 9 sharpen the empirical interpretation of the results that follow. Figure 8 shows that the composition of collateral actually mobilized at the DW is consistent with the haircut advantage of the facility for certain asset classes. Figure 9 shows that, across banks, institutions with a larger stock of such collateral on balance sheet tilt more strongly toward the DW in their borrowing mix. These patterns motivate the regression evidence in Tables 1 through 4, which tests the same mechanism more formally.

In particular, Table 1 considers the COVID episode and asks whether on-balance-sheet DW-advantaged collateral predicts subsequent pledging at the window. It does. Banks with a larger stock of such collateral pledge more of it at the DW, and this relationship strengthens during COVID. The result is important because it ties ex-ante balance-sheet composition to ex-post operational use of the DW.[25]

[25] Table 6 presents summary statistics for the main variables used in the analysis.

Table 1: COVID Crisis: On-Balance Sheet and Discount Window Advantaged Collateral

	DW Pledged Advantaged Collateral / Assets			
	(1)	(2)	(3)	(4)
$\frac{\text{On-BS Adv. Collateral}}{\text{Assets}}$	0.054***		0.054***	
	(0.006)		(0.006)	
COVID	0.001***	0.001***		
	(0.000)	(0.000)		
$\frac{\text{On-BS Adv. Collateral}}{\text{Assets}} \times$ COVID	0.003***	0.003***	0.003***	0.003***
	(0.000)	(0.001)	(0.000)	(0.001)
Observations	214,950	214,950	214,950	214,950
R-squared	0.026	0.987	0.026	0.987
Bank FE	no	yes	no	yes
Day FE	no	no	yes	yes

*Notes: The table shows regression results from bank-day regressions in the period between February 01, 2020, and March 31, 2020. The dependent variable is DW Pledged Advantaged Collateral/Assets, which is a bank's daily share of advantaged collateral pledged at the discount window to its total assets (measured as of the fourth quarter of 2019). On-BS Adv. Collateral/Assets is a bank's share of on-balance sheet advantaged collateral to its total assets. Both on-balance sheet advantaged collateral and total assets are measured as of the fourth quarter of 2019. The variable COVID takes the value of one on and after March 09, 2020 (Black Monday). Standard errors are double clustered at the bank and day level. *,**,*** indicate significance at 10%, 5%, and 1%, respectively.*

Table 2 turns from pledging to borrowing choice. The dependent variable is the share of DW borrowing in total borrowing (DW and FHLB). Here again, the evidence is consistent with the model. Banks pledging more DW-advantaged collateral borrow relatively more from the DW, and the effect becomes materially stronger during the COVID turmoil. This key result shows that collateral composition predicts not only whether banks prepare to use the DW, but also whether they substitute toward it when liquidity demand rises sharply.

Table 2: COVID Crisis: Discount Window Advantaged Collateral and Borrowing

	$\frac{\text{DW Borrowing}}{\text{DW Borrowing + FHLB Borrowing}}$			
	(1)	(2)	(3)	(4)
$\frac{\text{DW Pledged Adv. Collateral}}{\text{Assets}}$	0.006**	0.847***	0.006**	0.823***
	(0.002)	(0.172)	(0.002)	(0.172)
COVID	0.004***	0.003***		
	(0.001)	(0.001)		
$\frac{\text{DW Pledged Adv. Collateral}}{\text{Assets}} \times$ COVID	0.248***	0.149***	0.246***	0.149***
	(0.063)	(0.041)	(0.063)	(0.042)
Observations	127,897	127,897	127,897	127,897
R-squared	0.039	0.488	0.044	0.491
Bank FE	no	yes	no	yes
Day FE	no	no	yes	yes

*Notes: The table shows regression results from bank-day regressions in the period between February 01, 2020, and March 31, 2020. The dependent variable is DW Borrowing/(DW Borrowing + FHLB Borrowing), which is a bank's daily share of DW borrowing to total (DW and FHLB) borrowing. DW Pledged Adv. Collateral/Assets is a bank's daily share of advantaged collateral pledged at the discount window to its total assets (measured as of the fourth quarter of 2019). The variable COVID takes the value of one on and after March 09, 2020 (Black Monday). Standard errors are double clustered at the bank and day level. *,**,*** indicate significance at 10%, 5%, and 1%, respectively.*

Tables 3 and 4 show that the same pattern reappears in the March 2023 banking turmoil. In Table 3, banks with more on-balance-sheet DW-advantaged collateral pledge more such collateral at the window, with a stronger relationship during the stress period. In Table 4, those same banks rely more heavily on DW borrowing relative to FHLB borrowing once stress hits.[26]

[26] We designate March 10 — the date of SVB's collapse — as the crisis onset. Our findings remain robust whether we mark the crisis beginning as March 8 or March 9 instead. Additionally, the results hold up even when we apply a stricter filter: removing zero-value observations and focusing solely on institutions that actually pledged collateral at the DW.

Table 3: 2023 Banking Turmoil: On-Balance Sheet and Discount Window Advantaged Collateral

	DW Pledged Advantaged Collateral / Assets			
	(1)	(2)	(3)	(4)
$\frac{\text{On-BS Adv. Collateral}}{\text{Assets}}$	0.066***		0.066***	
	(0.010)		(0.010)	
Banking Turmoil	0.003***	0.003***		
	(0.000)	(0.001)		
$\frac{\text{On-BS Adv. Collateral}}{\text{Assets}}$ × Banking Turmoil	0.007***	0.007*	0.007***	0.007*
	(0.002)	(0.004)	(0.002)	(0.004)
Observations	127,641	127,641	127,641	127,641
R-squared	0.029	0.947	0.029	0.947
Bank FE	no	yes	no	yes
Day FE	no	no	yes	yes

*Notes: The table shows regression results from bank-day regressions in the period between February 01, 2023, and March 31, 2023. The dependent variable is DW Pledged Advantaged Collateral/Assets, which is a bank's daily share of advantaged collateral pledged at the discount window to its total assets (measured as of the fourth quarter of 2022). On-BS Adv. Collateral/Assets is a bank's share of on-balance sheet advantaged collateral to its total assets. Both on-balance sheet advantaged collateral and total assets are measured as of the fourth quarter of 2022. The variable Banking Turmoil takes the value of one on and after March 10, 2023 (SVB collapse). Standard errors are double clustered at the bank and day level. *,**,*** indicate significance at 10%, 5%, and 1%, respectively.*

Table 4: 2023 Banking Turmoil: Discount Window Advantaged Collateral and Borrowing

	DW Borrowing / (DW Borrowing + FHLB Borrowing)			
	(1)	(2)	(3)	(4)
$\frac{\text{DW Pledged Adv. Collateral}}{\text{Assets}}$	0.350***	0.149	0.350***	0.151
	(0.078)	(0.150)	(0.078)	(0.151)
Banking Turmoil	0.000	0.001		
	(0.001)	(0.001)		
$\frac{\text{DW Pledged Adv. Collateral}}{\text{Assets}} \times$ Banking Turmoil	0.102*	0.078*	0.102*	0.078*
	(0.062)	(0.045)	(0.062)	(0.045)
Observations	83,868	83,868	83,868	83,868
R-squared	0.065	0.750	0.065	0.751
Bank FE	no	yes	no	yes
Day FE	no	no	yes	yes

*Notes: The table shows regression results from bank-day regressions in the period between February 01, 2023, and March 31, 2023. The dependent variable is DW Borrowing/(DW Borrowing + FHLB Borrowing), which is a bank's daily share of DW borrowing to total (DW and FHLB) borrowing. DW Pledged Adv. Collateral/Assets is a bank's daily share of advantaged collateral pledged at the discount window to its total assets (measured as of the fourth quarter of 2022). The variable Banking Turmoil takes the value of one on and after March 10, 2023 (SVB collapse). Standard errors are double clustered at the bank and day level. *,**,*** indicate significance at 10%, 5%, and 1%, respectively.*

The replication across two distinct episodes matters. It suggests that the mechanism is not specific to the pandemic environment, but instead reflects a more general feature of the dual-facility arrangement. When banks come under liquidity pressure, the composition of pledgeable assets helps determine which source of funding they use. If the relevant distinction between the DW and FHLBs were only the borrowing rate, then there would be less reason for cross-sectional differences in collateral composition to carry such strong explanatory power. This interpretation is exactly what the model delivers.

5 Optimal penalty rate

This section discusses the optimal choice of the discount window penalty rate p. Our purpose is to establish the conditions under which the social planner/central bank set a positive p, so we allow more generally for $p \geq 0$. We derive the time-consistent optimal policy under which C can reoptimize p after bank decisions about the deposit rate and liquid holdings have been made.[27] In the Online Appendix we also derive the optimal p when C can credible commit to keep that level even if that would be suboptimal in the wake of crises. The trade-offs we highlight below continue to be present but there are additional considerations, which complicate the analysis and relate to the fact that C internalizes in this case how the choice of p affects B's ex ante choices of R_D and ℓ.

To better highlight the trade-offs that C faces and derive an interior solution for p, we consider that there is a continuum of banks, z, uniformly distributed in $U[R_F/R, 1]$, which differ in the pledgeability of their risky assets at F. Their respective pledgeability, $\gamma_{F,z}$, is given by their index z, i.e., $\gamma_{F,z} = z$, which means that there is a continuum of pledgeability levels from R_F/R, the lower bound from Assumption 1, to one.

Finally, we posit that that discount window lending may impose direct and indirect social costs. Recall that the pledgeability of collateral is higher at C than F. One possible micro-foundation is that C can combine information for its supervisory function to better evaluate collateral. Another possible micro-foundation is that C is more capable of taking collateral through an orderly resolution process, which is lengthy process allowing for higher recovery. During resolution both the C and F would need to pay their funding costs and, thus, book losses in the short-run. Because C can operate with short-term losses/negative equity it is more willing to lend against the long-term "recoverable" value rather than the short-term "market" value that encompasses an illiquidity discount. Hence, the costs of discount window lending could accrue from the direct labor costs and infrastructure required to evaluate collateral as well as from indirect costs from a large central bank footprint on financial markets including the consequences of financial repression or potentially higher inflation risk from monetizing illiquid assets.[28] We model these costs in a reduced-form way by assuming that lending to a bank through the discount window entails a cost $c \geq 0$.

As we show in Section 5.1, introducing costs to discount window usage is not required to establish the optimality of a positive p, and under loose sufficient condition we can obtain $p > 0$ for $c = 0$. However, $c > 0$ adds to the desire for a higher penalty rate. Moreover, in Section 5.2, we differentiate between the costs from borrowing against more pledgeable and less pledgeable collateral to examine the optimality of introducing multiple discount window facilities with different pricing schedules.

[27] Indeed, the Federal Reserve has cut the penalty rate at the outbreak of major crises. Before the Federal Reserve shifted to a target rate range in December 2008, the penalty rate—measured by the difference between the discount window primary credit rate and the target rate—was reduced from 100 to 50 basis points in August 2007, and further down to 25 basis points in March 2008 during the Bear Sterns collapse. After the shift to a target rate range, the penalty rate—measured by the difference between the primary credit rate and the upper limit for the target rate range—was was set 25 basis points until February 2010 when it was raised to 50 basis points. The Federal Reserve reduced the penalty rate to 0 basis points at the outbreak of COVID-19 in March 2020 and it has remained at that level since then.

[28] Other direct costs could be accrue from the facilitation of arbitrage by banks. FHLB rates are typically lower than the interest rate on reserve balance than banks can earn if they deposit at the Federal Reserve. Setting a discount window penalty rate to zero would equate the discount window and FHLB rate, and enable banks to engage in arbitrage by borrowing at the discount window and depositing at the Federal Reserve to earn a risk free spread.

5.1 Optimal p

For simplicity, we assume that C cares about maximizing a utilitarian welfare function that is the equally-weighted sum of the utilities/profits of banks, depositors, FHLBs, and the central bank. Recall that the depositors' participation constraint binds and their expected utility is always equal to their (exogenously given) outside option. Moreover, FHLBs set their lending rate competitively, set their collateral to avoid losses in default, and enjoy zero profits in equilibrium. Hence, maximizing a utilitarian welfare function is equivalent to maximizing the consumption of a representative agent that receive the profits from all banks and the central bank.

Then, the social welfare is given by:

$$\begin{aligned}
\mathbb{SW} = \pi \int_{R_F/R}^{1} \Pi_{F,z}(1,\delta)\frac{dz}{1-R_F/R} \\
+(1-\pi)\int_{\gamma^+(p)}^{1}\int_{\omega^*_{F,z}}^{1} \Pi_{F,z}(\omega,\delta)d\omega\frac{dz}{1-R_F/R} + (1-\pi)\int_{R_F/R}^{\gamma^+(p)}\int_{\omega^*_{C,z}}^{1} \Pi_{C,z}(\omega,\delta)d\omega\frac{dz}{1-R_F/R} \\
+(1-\pi)\int_{R_F/R}^{\gamma^+(p)}\left[\int_{\omega^*_{C,z}}^{1}(\delta-\ell_z)pd\omega + \int_{0}^{\omega^*_{C,z}} R_C^{-1}\omega R(1-\ell_z)pd\omega - c\right]\frac{dz}{1-R_F/R},
\end{aligned} \tag{25}$$

where $\Pi_{j,z}$ is given by (7) for bank z. C chooses p to maximize $\mathbb{SW}$ in (25). The deposit rate $R_{D,z}$ and liquid holdings ℓ_z bank z are determined by the participation constraint (2) and the private optimality condition (24); see Definition (1) of the private equilibrium. In other words, C does not have additional tools to distort the private equilibrium conditions and can only choose p (see Online Appendix for an extension).

The first line in (25) shows aggregate bank profits in the certain state of the world. As shown in Proposition 1, all banks will borrow from F. By contrast, in the bad state materializing with probability $1-\pi$, banks with $\gamma_{F,z} \geq \gamma^+(p)$ borrow from F and banks with $\gamma_{F,z} < \gamma^+(p)$ borrow from C, as established in Proposition (4). The two terms in the second line in (25) show the expected bank profits for banks that borrow from F and C, respectively, in the uncertain state of the world conditional that a run does not materialize. The third line in (25) shows the expected profits to the central bank from lending at a penalty rate p minus the social costs, c, from discount window use, i.e., for $\gamma_{F,z} < \gamma^+(p)$. Conditional that a run does not occur, the central bank lends $\delta - \ell_z$ to bank z with a profit margin of $R_C - r = p$. If a run occurs, bank z borrows against its full collateral resulting in central bank lending of $R_C^{-1}\omega R(1-\ell_z)$ at the same profit margin; recall that discount window lending is fully collateralized. Note the social costs accrue independent of whether the run materializes or not.

Because C can choose p after B's decisions about the deposit rate and liquid holdings have been made, i.e., C takes $R_{D,z}$ and ℓ_z as pre-determined, the optimality condition is

$$
\begin{aligned}
& -\int_{R_F/R}^{\gamma^+(p)} \frac{d\omega^*_{C,z}}{dp}\left[\Pi_{C,z}(\omega^*_{C,z},\delta)+(\delta-\ell_z)p\right]dz+\int_{R_F/R}^{\gamma^+(p)} \frac{d\omega^*_{C,z}}{dp} R_C^{-1}\omega^*_{C,z}R(1-\ell_z)pdz \\
& +\frac{d\gamma^+}{dp}\left[\int_{\omega^*_{C,z^+}}^{1}(\delta-\ell_{z^+})pd\omega+\int_0^{\omega^*_{C,z^+}} R_C^{-1}\omega R(1-\ell_{z^+})pd\omega\right] \\
& +\int_{R_F/R}^{\gamma^+(p)}\left[\int_{\omega^*_{C,z}}^{1}(\delta-\ell_z)d\omega+\int_0^{\omega^*_{C,z}}\omega R(1-\ell_z)\frac{r}{(r+p)^2}d\omega\right]dz-\frac{d\gamma^+}{dp}c+\phi\frac{1-R_F/R}{1-\pi}=0, \qquad (26)
\end{aligned}
$$

where ϕ is the multiplier on $p \geq 0$ and z^+ is the marginal bank with $\gamma_{F,z^+}=\gamma^+(p)$.[29]

The first term in (26) captures the decrease in bank and central bank profits absent a run due to an increase in the run probability from a higher p for those banks that choose to borrow from C; recall that $d\omega^*_{C,z}/dp > 0$ from Proposition 3. This term is unambiguously negative pushing for a lower p. By contrast, the second term captures the increase in central bank profits conditional on a run due to an increase in the run probability. The third term captures the decrease in central bank profits decrease in the number of banks borrowing from C, $d\gamma^+/dp < 0$. The fourth term captures the increase in central bank profits due to higher profit margin per unit of lending. The combined effects of these three effects is ambiguous. The following Proposition derives the sufficient condition under which it is optimal to set a positive p.

Proposition 6. *For $\pi \in (\tilde{\pi}, 1)$, with $\tilde{\pi} = \max\{\underline{\pi}, 0.5, \underline{V}^{-1}((1-\delta)r/2)\} < 1$, C sets $p > 0$ optimally.*

The intuition behind this result is as follows. Suppose that $c = 0$ and also assume that $p = 0$. Then, C trades-off the decrease in bank profits from a higher run probability $\omega^*_{C,z}$—first term in the first line in (26)—with the incremental increase in central bank profits from a marginally higher penalty rate p—first term in the third line in (26). The lower is the deposit rate R_D, i.e., the higher is the deposit franchise of the bank, the stronger the former force is. The reason is simple. A bank that already borrows from C has much more to lose from borrowing at a higher penalty rate compared to what the central bank can gain. If the former force dominates, then $\phi > 0$ and, hence, $p = 0$. Now, recall that the source of the deposit franchise is the convenience yield V. Thus, a sufficient condition for $p > 0$ is that R_D is not very low, or equivalently, V is not very high. However, $R_D < R_C$ requires that V is above a certain threshold that depends on π; see Lemma 3. The lower bound $\tilde{\pi}$ for the range of π that it is optimal to set $p > 0$.

As we show in the proof in the Appendix it suffices that the gross deposit rate R_D is higher than 0.5, which should realistically always hold, because $R_D < 0.5$ means that late depositors willingly receive less than half of the face value of their deposits, despite positive real interest rates. Though theoretically plausible, V would need to be unrealistically high for that to be the case.

Finally, note that for $c > 0$, the incentive to set $p > 0$ increases further because even fewer banks will choose to borrow from C decreasing the costs from doing so ($d\gamma^+(p)/dp < 0$).

[29] For the marginal bank z^+, it holds that $\int_{\omega^*_{F,z^+}}^1 \Pi_{F,z^+}(\omega,\delta)d\omega = \int_{\omega^*_{C,z^+}}^1 \Pi_{C,z^+}(\omega,\delta)d\omega$. Hence, the terms that include the derivative $d\gamma^+(p)/dp$ of the limit of integration in the second and third lines cancel out.

5.2 Multiple central bank lending facilities

Thus far we have assumed a uniform social cost, c, from the use of the discount window irrespective of the type of collateral pledged. It is instructive to consider the case that lending against collateral that has lower pledgeability in private markets, entails higher costs. This assumption could be justified by higher direct costs from evaluating more opaque collateral or by higher indirect costs from monetizing assets with low pledgeability. Suppose that there is a $\tilde{\gamma}$ such that the social cost is $\underline{c}$ for $\gamma_{F,z} \geq \tilde{\gamma}$ and $\bar{c}$ for $\gamma_{F,z} < \tilde{\gamma}$, with $\underline{c} < \bar{c}$.

We are interested in deriving conditions under which the central bank would choose to "price discriminate" and offer different penalty rates for borrowers with collateral pledgeability below and above $\tilde{\gamma}$. Intuitively, the planner may want to charge a higher penalty rate for borrowers that impose a higher social cost such that to more targetedly dissuade discount window usage.[30] Denote by $\tilde{p}$ the extra penalty on top of p for borrowers with $\gamma < \tilde{\gamma}$. Following the logic described in the previous section, the central bank could separately choose p and $\tilde{p}$ to set cutoffs $\gamma^{+}(p)$ and $\gamma^{++}(p+\tilde{p})$ such that borrowers with $\gamma_{F,z} \in [\tilde{\gamma}, \gamma^{+}(p))$ borrow from C at rate $r+p$, while borrowers with $\gamma_{F,z} \in [R_F/R, \gamma^{++}(p+\tilde{p}))$ borrow from C at rate $r+p+\tilde{p}$; all other borrowers borrow for F. The social welfare function becomes:

$$
\begin{aligned}
\widetilde{\mathbb{SW}} = \pi \int_{R_F/R}^{1} \Pi_{F,z}(1,\delta) \frac{dz}{1-R_F/R} \\
&+ (1-\pi) \int_{\gamma^{+}(p)}^{1} \int_{\omega^{*}_{F,z}}^{1} \Pi_{F,z}(\omega,\delta) d\omega \frac{dz}{1-R_F/R} + (1-\pi) \int_{\tilde{\gamma}}^{\gamma^{+}(p)} \int_{\omega^{*}_{C,z}}^{1} \Pi_{C,z}(\omega,\delta) d\omega \frac{dz}{1-R_F/R} \\
&+ (1-\pi) \int_{\gamma^{++}(p+\tilde{p})}^{\tilde{\gamma}} \int_{\omega^{*}_{F,z}}^{1} \Pi_{F,z}(\omega,\delta) d\omega \frac{dz}{1-R_F/R} + (1-\pi) \int_{R_F/R}^{\gamma^{++}(p+\tilde{p})} \int_{\omega^{*}_{C,z}}^{1} \Pi_{C,z}(\omega,\delta) d\omega \frac{dz}{1-R_F/R} \\
&+ (1-\pi) \int_{\tilde{\gamma}}^{\gamma^{+}(p)} \left[\int_{\omega^{*}_{C,z}}^{1} (\delta-\ell_z) p \, d\omega + \int_{0}^{\omega^{*}_{C,z}} \omega R (1-\ell_z) \frac{p}{r+p} d\omega - \underline{c} \right] \frac{dz}{1-R_F/R} \\
&+ (1-\pi) \int_{R_F/R}^{\gamma^{++}(p+\tilde{p})} \left[\int_{\omega^{*}_{C,z}}^{1} (\delta-\ell_z)(p+\tilde{p}) d\omega + \int_{0}^{\omega^{*}_{C,z}} \omega R (1-\ell_z) \frac{p+\tilde{p}}{r+p+\tilde{p}} d\omega - \bar{c} \right] \frac{dz}{1-R_F/R},
\end{aligned}
\tag{27}
$$

The first-order condition with respect to $\tilde{p}$ is given by:

[30] Some central banks around the world, such as Bank of Canada, Bank of England, and Riksbank, operate more than one liquidity facilities akin to the discount window: some facilities only accepts collateral of higher quality and offers a lower lending rate, while others also accepts riskier collateral but at a higher lending rate. The Federal Reserve's discount window offers the same rate irrespective of the type of collateral. See Arseneauetal2025 for details.

$$
-\int_{R_F/R}^{\gamma^{++}(p+\tilde{p})} \frac{d\omega^*_{C,z}}{d\tilde{p}} \left[\Pi_{C,z}(\omega^*_{C,z},\delta)+(\delta-\ell_z)(p+\tilde{p})\right] dz + \int_{R_F/R}^{\gamma^{++}(p+\tilde{p})} \frac{d\omega^*_{C,z}}{d\tilde{p}} R_C^{-1}\omega^*_{C,z}R(1-\ell_z)(p+\tilde{p})dz
$$

$$
+\frac{d\gamma^{++}}{d\tilde{p}}\left[\int_{\omega^*_{C,z^{++}}}^{1}(\delta-\ell_{z^{++}})(p+\tilde{p})d\omega+\int_0^{\omega^*_{C,z^{++}}} R_C^{-1}\omega R(1-\ell_{z^{++}})(p+\tilde{p})d\omega\right]
$$

$$
+\int_{R_F/R}^{\gamma^{++}(p+\tilde{p})}\left[\int_{\omega^*_{C,z}}^{1}(\delta-\ell_z)d\omega+\int_0^{\omega^*_{C,z}}\omega R(1-\ell_z)\frac{r}{(r+p+\tilde{p})^2}d\omega\right]dz-\frac{d\gamma^{++}}{d\tilde{p}}\bar{c}+\tilde{\phi}\frac{1-R_F/R}{1-\pi}=0, \quad (28)
$$

Note that we cannot use the same argument in the proof of Proposition 6 to show that $\tilde{p}>0$, because the term in the second line in (28) cannot be taken to zero for $\tilde{p}$ and is actually negative for $p>0$, which we show that continues to be true is equilibrium. To derive a condition under which $\tilde{p}>0$ is optimal, suppose that $\tilde{\gamma}\to R_F/R$, i.e., the mass of banks with high social cost $\bar{c}$ is approaching zero. Then, the first-order condition with respect to $\tilde{p}$, evaluated at $\tilde{p}=0$, becomes:

$$
-\frac{d\gamma^{++}}{d\tilde{p}}\left[\bar{c}-(1-\omega^*_{C,z^{++}})(\delta-\ell_z^{++})p-\frac{(\omega^*_{C,z^{++}})^2}{2}\frac{p}{r+p}R(1-\ell_{z^{++}})\right], \quad (29)
$$

where z^{++} is the bank with $\gamma_{F,z}=\gamma^{++}(p)$. Since $d\gamma^{++}/d\tilde{p}<0$, (29) is positive—and hence $\tilde{p}=0$ is not optimal—if the sum of the bracketed terms is positive. The following Proposition derives the sufficient condition for this to be true.

Proposition 7. *Suppose $\pi\in(\tilde{\pi},1$. For $\bar{c}>(R-r)\max(\delta,0.5)$, it is optimal to set $\tilde{p}>0$ for sufficiently low $\tilde{\gamma}$, while $p>0$ is always optimal.*

The proposition says that it is optimal to price discriminate by setting a higher penalty rate, $p+\tilde{p}$, for collateral of low pledgeability. However, this result does not always hold and it requires both a high enough cost $\bar{c}$ and a small enough number of banks with collateral that imposes higher costs.

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Appendix

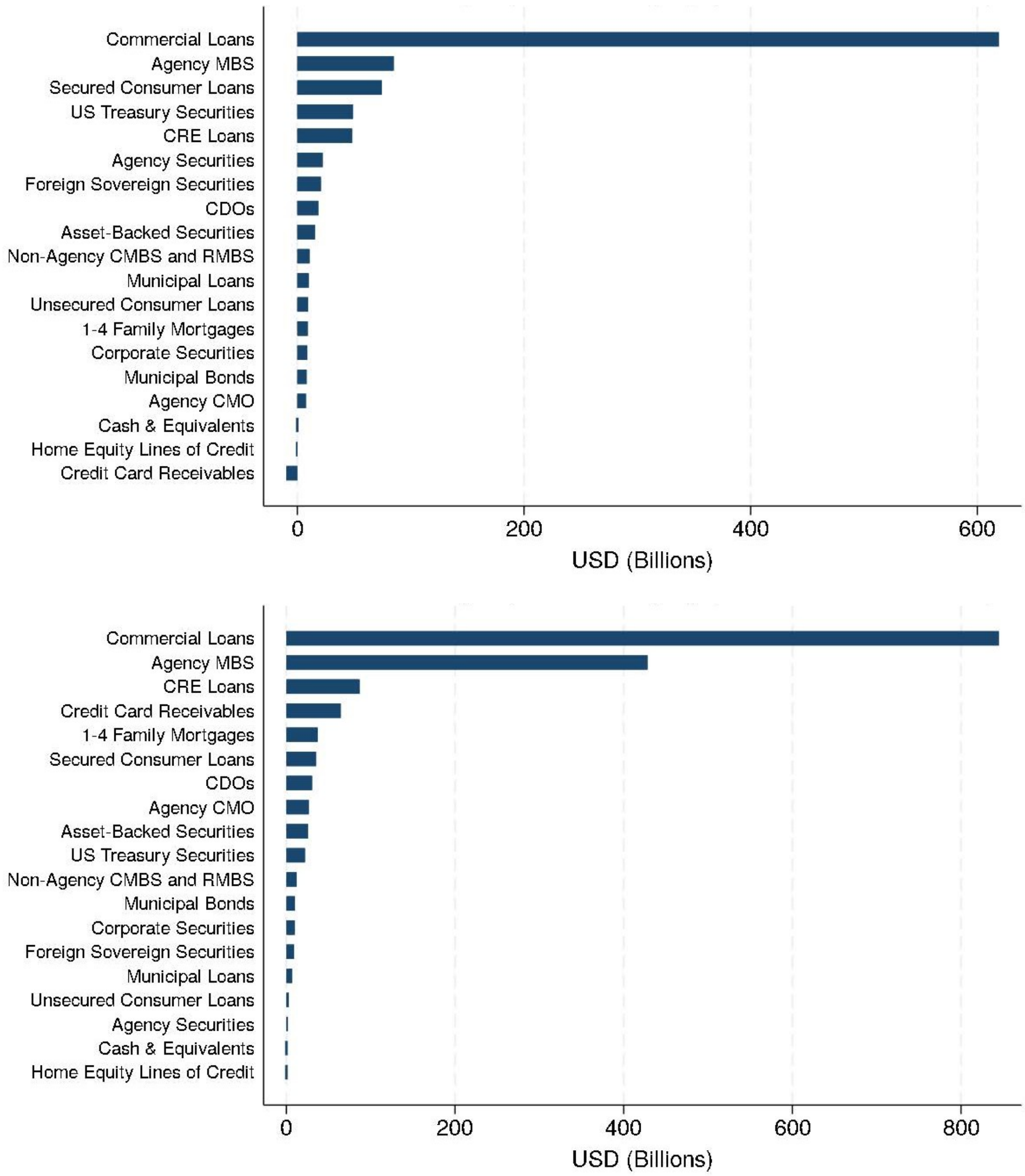
FHLB advances

Daily Federal Home Loan Bank (FHLB) advances outstanding are constructed by combining payment-level information from Fedwire with balance sheet data reported in bank Call Reports. Using Fedwire payments data, all transactions between panel banks and FHLBs from January 1, 2019 through December 31, 2024 are identified. Federal funds transactions are excluded, and net daily flows in FHLB advances are calculated as payments received from FHLBs minus payments sent to FHLBs. Payments sent to FHLBs are adjusted to exclude interest components, with interest rates generally inferred from the payment information. To ensure consistency with regulatory reporting, balance sheet data on FHLB advances are obtained from Call Reports for 2018Q4–2024Q4, defining the panel as banks present in the 2018Q4 filing. For each bank-quarter, a scaling factor is computed as the ratio of quarterly FHLB advance flows implied by Call Reports to the corresponding quarterly flows observed in Fedwire. Daily Fedwire flows are then multiplied by the bank-quarter–specific scaling factor so that the quarterly totals align with Call Report data. The resulting series of daily FHLB advances outstanding is initialized using the stock of advances reported in the 2018Q4 Call Reports and is subsequently updated using the adjusted daily flows. In cases where Call Reports indicate no quarterly change in advances but Fedwire records daily transactions, these payments are retained only when the implied quarterly Fedwire flow is zero. Figure 10 [to be added after clearance] overlays out compured daily FHLB advances, on aggregate, using the above methodology over the quarterly FHLB advances from call reports.

Figure 10: Computed daily aggregate FHLB advances and quarterly FHLB advances from call reports.

Figures and Tables

Figure 11: Collateral Change by Asset Category: 2020 and 2023



Notes: The charts show the collateral change by asset category in 2020 (February 29 to June 30) and 2023 (February 28 to June 30).

Table 5: Haircut Comparison DW-FHLB

DW Asset	FHLB Asset	DW Haircut	FHLB Haircut	Relative Advantage Flag
Agency Backed Mortgages - CMOs	Agency MBS/CMOs	2.3	7.5	1
Agency Backed Mortgages - Pass Throughs	Agency MBS/CMOs	2.5	7.5	1
Commercial Mtge Backed Securities	CMBS	7.9	16.3	1
Commercial Real Estate Loans	CRE	31.1	32.9	1
1-4 Family Mortgages (2nd lien, HE)	HELOCS and SF 2nd lien	51.7	43.3	0
Raw Land Loans	Land ORERC	66.2	55.4	0
Government Sponsored Enterprises - Bills/Notes/Bonds	Non-MBS agency securities	2.2	6.2	1
Government Sponsored Enterprises - Zero Coupon	Non-MBS agency securities	3.1	6.2	1
Agricultural Loans	Other ORERC	55.8	30.3	0
Commercial Loans & Leases	Other ORERC	28.6	30.3	1
Construction Loans	Other ORERC	63.2	30.3	0
Consumer Loans & Leases (Auto, Boat, etc.)	Other ORERC	49.0	30.3	0
Consumer Loans - Credit Card Receivables	NA	26.2	100	1
Consumer Loans - Unsecured	Other ORERC	38.5	30.3	0
Student Loans	Other ORERC	22.4	30.3	1
US Agency Guaranteed Loans	Other ORERC	19.6	30.3	1
US Agency Non-Guaranteed Loans	Other ORERC	51.8	30.3	0
Asset Backed Securities	Other securities	2.1	21.9	1
Auction Rate Securities	Other securities	2.0	21.9	1
Bank Loans to State/Local	Other securities	67.2	21.9	0
Cash/TDF	Other securities	0.0	21.9	1
Certificates of Deposit	Other securities	2.0	21.9	1
Collateralized Debt Obligations / Collateralized Loan Obligations	Other securities	11.2	21.9	1
Corporate Bonds / Covered Bonds	Other securities	5.7	21.9	1
Corporate Bonds / Covered Bonds, Foreign denominated	Other securities	10.0	21.9	1
Foreign Government Agencies	Other securities	2.3	21.9	1
Foreign Government Agencies, Foreign Denominated	Other securities	6.6	21.9	1
Foreign Government, Foreign Gov't Guaranteed, and Brady Bonds - FX Denominated	Other securities	6.7	21.9	1
Foreign Government, Foreign Government Guaranteed, and Brady Bonds	Other securities	2.3	21.9	1
German Jumbo Pfandbriefe, Foreign denominated	Other securities	6.1	21.9	1
Municipal Bonds	Other securities	3.3	21.9	1
Municipal Bonds, Foreign denominated	Other securities	9.4	21.9	1
Supranationals - Bills/Notes/Bonds	Other securities	3.1	21.9	1
Supranationals - Bills/Notes/Bonds, Foreign denominated	Other securities	6.3	21.9	1
Trust Preferred Securities	Other securities	11.0	21.9	1
U.S. Treasuries & Fully Guaranteed Agencies - Bill/Notes/Bonds/Inflation Indexed	Other securities	1.4	21.9	1
U.S. Treasuries & Fully Guaranteed Agencies - Zero Coupons, STRIPs	Other securities	4.0	21.9	1
Non-Agency RMBS	PLMBS	20.9	40.3	1
1-4 Family Mortgages (first lien)	SF 1st lien	20.3	17.8	0

Table 6: Summary Statistics

Variable	Mean	Median	Std. Deviation
	2020 COVID Crisis		
On-BS Advantaged Collateral (2019q4)/Assets	0.285	0.274	0.153
DW Pledged Advantaged Collateral/Assets	0.016	0.000	0.053
DW Borrowing/(DW Borrowing + FHLB Borrowing)	0.004	0.000	0.052
	2023 Banking Turmoil		
On-BS Advantaged Collateral (2022q4)/Assets	0.302	0.287	0.162
DW Pledged Advantaged Collateral/Assets	0.024	0.000	0.066
DW Borrowing/(DW Borrowing + FHLB Borrowing)	0.014	0.000	0.102

Proofs

Proof of Lemma 1

Proof. Because (2) is decreasing in ω_j^* and increasing in r_D for $\mathbb{I}_{r_D<1}=1$, it suffices that it does not hold for $\omega_j^* \to 0$ while taking $r_D \to 1^-$, yielding $P > \underline{P} \equiv [(1-\delta)(R_D - r) + V]/\delta$. $\square$

Proof of Lemma 2

Proof. Consider that the condition in Lemma 1 holds so only $r_D \geq 1$ is possible. The right-hand side of the (2) is decreasing in ω_j^* and increasing in r_D if $\pi + (1-\pi)(1-\omega_j^*) > P(1-\pi)\int_0^{\omega_j^*} L(\omega)/r_D^2 d\omega$. Because $L(\omega) < r_D$ for all ω, $r_D \geq 1$, and $\omega_j^* \leq 1$, it suffices that $\pi > \underline{\pi}' \equiv P/(1+P)$ for the latter condition to hold. Then, for these $\pi's$, take $\omega_j^* \to 1$ and $r_D \to 1^+$ in (2), yielding:

$$
\begin{aligned}
&\pi((1-\delta)R_D + \delta + V) - \delta P(1-\pi)\int_0^1 (1 - L(\omega))d\omega - ((1-\delta)r + \delta)[1 - (1-\pi)\int_0^1 L(\omega)d\omega] \\
>&\pi((1-\delta)R_D + \delta + V) - \delta P(1-\pi) - ((1-\delta)r + \delta),
\end{aligned}
$$

which is a lower bound for the right-hand side of (2). Next, taking the limit $\pi \to 1$, the above condition becomes $(1-\delta)(R_D + V - r) + \delta V > 0$, from the incentive compatibility constraint (1). Because (2) is increasing in π, from continuity there exists a $\underline{\pi}''$ such that (2) holds for $\pi \geq \underline{\pi}''$ when $r_D \geq 1$. Thus, a sufficient condition for feasibility of $r_D \geq 1$ is that $\pi > \underline{\pi} = \max\{\underline{\pi}', \underline{\pi}''\}$. $\square$

Proof of Lemma 3

Proof. From Lemmas 1 and 2, we can focus on $r_D \geq 1$ and $\pi \geq \underline{\pi}$. Then, (2) is decreasing in ω_j^*, and increasing in π, r_D and R_D. Thus, to derive a condition for the maximum R_D such that $\max R_D < R_j$, consider first that $r_D \to 1$ and $\omega_j^* \to 1$. Because the bank will offer an R_D such that the participation constraint binds, we then have

$$
\max R_D = \frac{((1-\delta)r + \delta)[1 - (1-\pi)\int_0^1 L(\omega)d\omega] + \delta P(1-\pi)\int_0^1 (1 - L(\omega))d\omega}{(1-\delta)\pi} - \frac{\delta + V}{1-\delta}. \tag{30}
$$

Now take $\pi \to 1$. We have that $\lim_{\pi \to 1} \max R_D < R_j \Rightarrow V > (r - R_j)(1-\delta)$, which is always true for positive V because $R_j \geq r$. Because $\max R_D$ is decreasing in π and V, we can maintain this relationship as π decreases from one if at the same time we keep increasing V. A remaining concern is that increasing V also increases $\underline{P}$. Although the initial P may be higher than $\underline{P}$, the $P \geq \underline{P}$ condition of Lemma 1 may start binding as V increases, reversing the positive effect of a higher V. Suppose that $P \geq \underline{P}$ starts binding. Then, the change in (2) from increasing V is $\pi + (1-\pi)(1-\omega_j^*) - d\underline{P}/dV \cdot \delta(1-\pi)\int_0^{\omega_j^*}(1 - L(\omega)/r_D)d\omega$. Because $d\underline{P}/dV = 1/\delta$, $L(\omega) < r_D$, $r_D \geq 1$, and $\omega_j^* \leq 1$, the latter condition is bigger than $\pi + (1-\pi)(1-\omega_j^*) - (1-\pi)\omega_j^*$, which is strictly positive if $\pi > 1/2$. Hence, the sufficient condition for the Lemma is $\pi > \max\{\underline{\pi}, 0.5\}$

$\square$

Proof of Lemma 4

Proof. From Lemmas 1 and 2, the bank offers a deposit contract that satisfies $r_D \geq 1$. Denote by ψ and ϕ the Lagrange multipliers on the participation constraint (2) and on $r_D \geq 1$, respectively. Then, the first order-conditions with respect to r_D and R_D are:

$$
\begin{aligned}
&-\left[\pi+(1-\pi)(1-\omega_j^*)\right]\delta R_j-\Pi(\omega_j^*,\delta)\frac{d\omega_j^*}{dr_D}+\psi\left[\pi+(1-\pi)(1-\omega_j^*)\right]\delta\\
&-\psi(1-\pi)\left[(1-\delta)R_D+\delta r_D+V-L(\omega_j^*)((1-\delta)r+\delta)+\delta P(1-L(\omega_j^*)/r_D)\right]\frac{d\omega_j^*}{dr_D}\\
&-\psi(1-\pi)\delta P\int_0^{\omega_j^*}L(\omega)/r_D^2 d\omega+\phi=0, \qquad (31)
\end{aligned}
$$

and

$$
\begin{aligned}
&-\left[\pi+(1-\pi)(1-\omega_j^*)\right](1-\delta)-\Pi(\omega_j^*,\delta)\frac{d\omega_j^*}{dR_D}+\psi\left[\pi+(1-\pi)(1-\omega_j^*)\right](1-\delta)\\
&-\psi(1-\pi)\left[(1-\delta)R_D+\delta r_D+V-L(\omega_j^*)((1-\delta)r+\delta)+\delta P(1-L(\omega_j^*)/r_D)\right]\frac{d\omega_j^*}{dR_D}=0, \qquad (32)
\end{aligned}
$$

Dividing (31) and (32) by δ and $1-\delta$, respectively, and combining the two we get that:

$$
\begin{aligned}
\phi=\psi(1-\pi)\delta P\int_0^{\omega_j^*}L(\omega)/r_D^2 d\omega+\left(\frac{d\omega_j^*}{dr_D}\frac{1}{\delta}-\frac{d\omega_j^*}{dR_D}\frac{1}{1-\delta}\right)\big\{\Pi(\omega_j^*,\delta)\\
+\psi(1-\pi)\left[(1-\delta)R_D+\delta r_D+V-L(\omega_j^*)((1-\delta)r+\delta)+\delta P(1-L(\omega_j^*)/r_D)\right]\big\}. \qquad (33)
\end{aligned}
$$

Using (1), we have that $(1-\delta)R_D+\delta r_D+V>(1-\delta)r_D r+\delta r_D+\delta V>r_D\geq L(\omega_j^*)$, because $r\geq 1$ and the bank does not have enough resources when a run materializes to cover early withdrawals by all depositors, amounting to r_D. Hence, $\phi>0$, i.e., $r_D=1$, as long as $\frac{d\omega_j^*}{dr_D}\frac{1}{\delta}-\frac{d\omega_j^*}{dR_D}\frac{1}{1-\delta}>0$. □

Proof of Lemma 5

Proof. Showing that $\overline{\omega}_j<1$ given Assumption 1 is straightforward as is to derive the relationship between ω_F^* and ω_C^* for different levels of γ_F. □

Proof of Lemma 6

Proof. Because $\delta>\ell$ in the equilibria we focus, the lower dominance region always exists. Because $\gamma_C=1$, $\underline{\omega}_C=\underline{\omega}'_C$, while $\underline{\omega}_F=\underline{\omega}'_F$ for $\gamma_F\geq\underline{\gamma}_F$ and $\underline{\omega}_F=\underline{\omega}''_F$ for $\gamma_F<\underline{\gamma}_F$.

Then, for $\gamma_F\geq\underline{\gamma}_F$, it follows that $\underline{\omega}_F<\underline{\omega}_C$ because $R_F<R_C$. Alternatively, for $\gamma_F<\underline{\gamma}_F$, $\underline{\omega}_F=\underline{\omega}''_F>\underline{\omega}_C=\underline{\omega}'_C$ requires that $\gamma_F<\underline{\gamma}_C R_F/R_C<\bar{\gamma}_F$. Thus, $\underline{\omega}_F>\underline{\omega}_C$ for $\gamma_F<\underline{\gamma}_C R_F/R_C$ and $\underline{\omega}_F<\underline{\omega}_C$ for $\gamma_F\in[\underline{\gamma}_C R_F/R_C,\underline{\gamma}_F)$. These results can be re-organized into the cases reported in the Lemma. □

Proof of Lemma 7

Proof. Consider first lender C. Using the definitions of the lower and upper thresholds, $\underline{\omega}_c<\overline{\omega}_C$ requires that

$$\frac{(1-\delta)R_D+(\delta-\ell)R_C}{R(1-\ell)}<\frac{R_C}{R}\Rightarrow R_D<R_C.$$

Next, consider lender F. For $\gamma_F < \underline{\gamma}_F$, it is straightforward that $\underline{\omega}_F = \underline{\omega}''_F < \overline{\omega}''_F = \overline{\omega}_F$. For $\gamma_F \geq \underline{\gamma}_F$, $\underline{\omega}_F = \underline{\omega}'_F$ and $\underline{\omega}_F < \overline{\omega}_F$ requires that

$$\frac{(1-\delta)R_D + (\delta-\ell)R_F}{R(1-\ell)} < \frac{R_F}{\gamma_F R} \Rightarrow \gamma_F < \frac{(1-\ell)R_F}{(1-\delta)R_D + (\delta-\ell)R_F}.$$

The right-hand side is higher than $\overline{\gamma}_F$, thus there is a region of γ_F's that coordinated runs are possible. If $R_F > R_D$, the right-hand side is higher than one and thus the condition is satisfied for any γ_F. If $R_D > R_F$, then coordinated runs are not possible for γ_F's greater that the right-hand side. □

Proof of Lemma 8

Proof. $\hat{\lambda}_j > \overline{\lambda}_j$ requires γ_j to satisfy (16). For $\gamma_j = 1$, this relationship boils down to $\omega > R_j/R = \overline{\omega}_j$, which is never true. Hence, $\hat{\lambda}_j < \overline{\lambda}_j$, for $\gamma_j = 1$, i.e., $\hat{\lambda}_C < \overline{\lambda}_C$.

For $\gamma_j = 0$, (16) is always true since the right-hand side is strictly positive in the intermediate region of ω's. Hence, $\exists \gamma_j^P(\omega) \equiv \dfrac{\omega R - R_D}{R_j - R_D}\dfrac{R_j}{\omega R} \in (0,1)$ such that $\hat{\lambda}_j < \overline{\lambda}_j$ for $\gamma_j > \gamma_j^P(\omega)$, and $\hat{\lambda}_j > \overline{\lambda}_j$ otherwise. This proves the last part of the Lemma.

To find the ω for which $\hat{\lambda}_j < \overline{\lambda}_j$, $\gamma_j > \gamma_j^P(\omega)$ can be re-written as

$$\omega < \omega_j^P \equiv \frac{R_D}{R}\frac{1}{1-\gamma_j(1-R_D/R_j)}$$

We need to show that, for $\gamma_j < 1$, there exists $\omega_j^P \in (\underline{\omega}_j, \overline{\omega}_j)$ such that the above condition is satisfied for some ω. First, note that $\omega_j^P < \overline{\omega}_j$ requires $\gamma_j < 1$, which is always true in the intermediate region. Next, we examine whether $\omega_j^P > \underline{\omega}_j$. For $\gamma_j < \underline{\gamma}_j$ we have that $\underline{\omega}_j = \underline{\omega}''$ using 13. Because $\omega_j^P < \underline{\omega}''_j$ for $\gamma_j < \underline{\gamma}_j$, we have that $\hat{\lambda}_j > \overline{\lambda}_j$ for all ω's in the intermediate region. Finally, for $\gamma_j \geq \underline{\gamma}_j$ we have that $\underline{\omega}_j = \underline{\omega}'$. In turn, $\omega_j^P > \underline{\omega}'$ requires that

$$\frac{R_D}{R}\frac{1}{1-\gamma_j(1-R_D/R_j)} > \frac{(1-\delta)R_D + (\delta-\ell)R_j}{R(1-\ell)} \Rightarrow \gamma_j > \underline{\gamma}_j.$$

So there always exists an ω_j^P in the intermediate region for $\gamma_j > \underline{\gamma}_j$ such that $\hat{\lambda}_j < \overline{\lambda}_j$ for $\omega < \omega_j^P$. □

Proof of Lemma 9

Proof. For $x_j^* \in [\underline{\omega}_j + \varepsilon, \overline{\omega}_j - \varepsilon)$, all ω lie in $[\underline{\omega}_j, \overline{\omega}_j)$. Then, $\omega_j^{\hat{\lambda}}(x_j^*)$ is given by $\lambda(\omega_j^{\hat{\lambda}}(x_j^*), x_j^*) = \hat{\lambda}_j(\omega_j^{\hat{\lambda}}(x_j^*))$, which yields

$$\omega_j^{\hat{\lambda}}(x_j^*) = \frac{\varepsilon\left[(1+\delta)(R_j - R_D) + 2R_D - 2\ell R_j\right] + x_j^*(1-\delta)(R_j - R_D)}{2\varepsilon R(1-\ell) + (1-\delta)(R_j - R_D)}. \tag{34}$$

Similarly, $\lambda(\omega_j^{\overline{\lambda}}(x_j^*), x_j^*) = \overline{\lambda}_j(\omega_j^{\overline{\lambda}}(x_j^*))$ yields

$$\omega_j^{\overline{\lambda}}(x_j^*) = \frac{\varepsilon\left[(1+\delta) - 2\ell\right] + x_j^*(1-\delta)}{2\varepsilon\gamma_j R_j^{-1} R(1-\ell) + (1-\delta)}, \tag{35}$$

Both cutoffs lie in $(x_j^* - \varepsilon, x_j^* + \varepsilon)$ because: (i) $\lambda(\omega, x_j^*)$ is continuous and decreasing in ω, while $\hat{\lambda}_j$, $\overline{\lambda}_j$ are continuous and increasing in ω, (ii) $\lambda(x_j^* - \varepsilon, x_j^*) = 1 > \hat{\lambda}_j(x_j^* - \varepsilon), \overline{\lambda}_j(x_j^* - \varepsilon)$ for $x_j^* < \overline{\omega}_j - \varepsilon$, (iii)

$\lambda(x_j^*+\varepsilon, x_j^*) = \delta < \hat{\lambda}_j(x_j^*+\varepsilon), \overline{\lambda}_j(x_j^*+\varepsilon)$ for $x_j^* > \underline{\omega}_j + \varepsilon$. So, the two functions cross once in $(x_j^*-\varepsilon, x_j^*+\varepsilon)$ and there exist unique $\omega_j^{\hat{\lambda}}(x_j^*)$ and $\omega_j^{\overline{\lambda}}(x_j^*)$.

Given the same monotonicity properties the relationship between the two ω cutoff is the inverse of the relationship between the two λ cutoff, derived in Lemma 8. $\square$

Proof of Proposition 2

Proof. We already established in step 6 that a threshold strategy, whereby an individual patient depositor withdraws only if x_i is below a common x_j^*, is an equilibrium. Moreover, we showed in this step that x_j^* satisfying (20) exists. To complement the proof we need to show that x_j^* is unique for limiting noise whereby $x_j^* \to \omega_j^*$ and (20) can be expressed as in (21). We examine the case of $j = C$ and $j = F$ separately.

First, consider $j = C$. From Lemma 8 we have that $\hat{\lambda}_C < \overline{\lambda}_C$, and also $\gamma_C = 1$. Hence, the derivative of (21) with respect to ω_C^* is

$$\frac{d\bar{\Delta}_C^*}{d\omega_C^*} = \frac{d\hat{\lambda}_C}{d\omega_C^*} V \frac{1}{1-\delta} + \int_{\hat{\lambda}_C}^{\overline{\lambda}_C} \frac{R(1-\ell)}{1-\lambda} \frac{d\lambda}{1-\delta} - \int_{\overline{\lambda}_C}^{1} \frac{R_C^{-1} R(1-\ell)}{\lambda} r \frac{d\lambda}{1-\delta}, \tag{36}$$

the sign of which cannot be unambiguously determined. So, instead, of relying on a strictly increasing $\bar{\Delta}_C^*$ to establish uniqueness of ω_C^*, we will really on its strict convexity. The idea is simple and depicted in Figure 12 below. The left chart shows a function that is not monotonic but strictly convex. The right chart shows a function that neither monotonic nor (strictly) convex. The argument goes as follows: Since $\bar{\Delta}_C^*$ in negative for low ω and positive for high ω by continuity it must cross zero from below at least once. Then to cross zero again from above would require that $\bar{\Delta}_C^*$ starts decreasing while positive. In turn this would require that $\bar{\Delta}_C^*$ is not strictly convex.

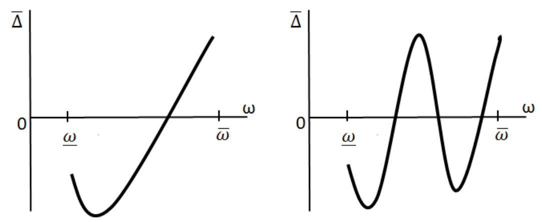


Figure 12: Graphical argument for uniqueness given strict convexity.

Thus, it suffices to show that $\bar{\Delta}_C^*$ is strictly convex. Taking the second derivative we get that

$$\frac{d^2\bar{\Delta}_C^*}{d\omega_C^{*2}} = \frac{R(1-\ell)}{1-\delta}\left[\frac{d\overline{\lambda}_C}{d\omega_C^*}\frac{1}{1-\overline{\lambda}_C} - \frac{d\hat{\lambda}_C}{d\omega_C^*}\frac{1}{1-\hat{\lambda}_C}\right] + \frac{d\overline{\lambda}_C}{d\omega_C^*}\frac{R_C^{-1}}{\overline{\lambda}_C}\frac{R(1-\ell)}{1-\delta} r, \tag{37}$$

since $d^2\hat{\lambda}_C/d\omega_C^{*2} = 0$. Also, note that (after some algebra) the bracketed terms in (37) cancel out because $d\overline{\lambda}_C/d\omega_C^* = R_C^{-1}R(1-\ell)$, $d\hat{\lambda}_C/d\omega_C^* = (R_C - R_D)^{-1}R(1-\ell)$, and $\hat{\lambda}_C = (\overline{\lambda}_C R_C - R_D)/(R_C - R_D)$. Hence, the second derivative is strictly positive, $\bar{\Delta}_C^*$ is strictly convex, and ω_C^* that satisfies (21) for $j = C$ is unique.

Next, consider $j = F$. Given Lemma 8, we will examine separately the cases that $\gamma_F \leq \underline{\gamma}_F$ and $\gamma_F > \underline{\gamma}_F$. For $\gamma_F \leq \underline{\gamma}_F$, $\hat{\lambda}_F \geq \overline{\lambda}_F$ from Lemma 8. Hence,

$$\frac{d\bar{\Delta}_F^*}{d\omega_F^*} = \frac{d\overline{\lambda}_F}{d\omega_F^*}(R_D + V)\frac{1}{1-\delta} - \int_{\overline{\lambda}_F}^{1} \frac{\gamma_F R_F^{-1} R(1-\ell)}{\lambda} r \frac{d\lambda}{1-\delta}, \tag{38}$$

and

$$\frac{d^2\bar{\Delta}_F^*}{d\omega_F^{*2}} = \frac{d\bar{\lambda}_F}{d\omega_F^*}\frac{\gamma_F R_F^{-1} R(1-\ell)}{\bar{\lambda}_F} r \frac{1}{1-\delta} > 0. \tag{39}$$

Hence, $\bar{\Delta}_F^*$ is strictly convex for this region of γ_F, and ω_F^* is unique.

For $\gamma_F > \underline{\gamma}_F$, $\hat{\lambda}_F \lesseqgtr \bar{\lambda}_F$ depending on ω from Lemma 8, so we need to consider the properties of $\bar{\Delta}_F^*$ separately for $\omega_F^* < \omega_F^P$ and $\omega_F^* \geq \omega_F^P$. Showing that $\bar{\Delta}_F^*$ is piece-wise convex in these two segments is not sufficient to establish uniqueness: a piece-wise convex function may cross zero is more that one points. The strict convexity of $\bar{\Delta}_F^*$ for $\omega_F^* < \omega_F^P$ is still useful but we need to augment it with an additional condition that $\bar{\Delta}_F^*$ may cross zero only from below for $\omega_j^* \geq \omega_F^P$, or that the derivative of $\bar{\Delta}_F^*$ at any candidate crossing point is strictly positive (see Kashyap et al. 2024 for how this condition is sufficient to establish uniqueness under perverse state monotonicity).

For $\omega_j^* < \omega_F^P$ we have that $\hat{\lambda}_F < \bar{\lambda}_F$. Hence,

$$\begin{aligned}\frac{d\bar{\Delta}_F^*}{d\omega_F^*} &= \frac{d\hat{\lambda}_F}{d\omega_F^*} V \frac{1}{1-\delta} + \int_{\hat{\lambda}_F}^{\bar{\lambda}_F} \frac{R(1-\ell)}{1-\lambda}\frac{d\lambda}{1-\delta} \\ &\quad - \int_{\bar{\lambda}_F}^{1} \frac{\gamma_F R_F^{-1} R(1-\ell)}{\lambda} r \frac{d\lambda}{1-\delta} + \frac{d\bar{\lambda}_F}{d\omega_F^*}\left[\frac{\omega_F^* R(1-\ell)(1-\gamma_F)}{1-\bar{\lambda}_F}\right]\end{aligned} \tag{40}$$

and

$$\begin{aligned}\frac{d^2\bar{\Delta}_F^*}{d\omega_F^{*2}} &= \frac{R(1-\ell)}{1-\delta}\left[\frac{d\bar{\lambda}_F}{d\omega_F^*}\frac{1}{1-\bar{\lambda}_F} - \frac{d\hat{\lambda}_F}{d\omega_F^*}\frac{1}{1-\hat{\lambda}_F}\right] + \frac{d\bar{\lambda}_F}{d\omega_F^*}\frac{\gamma_F R_F^{-1}}{\bar{\lambda}_F}\frac{R(1-\ell)}{1-\delta} r \\ &\quad + \frac{d\bar{\lambda}_F}{d\omega_F^*}\frac{R(1-\ell)(1-\gamma_F)}{1-\bar{\lambda}_F} + \left(\frac{d\bar{\lambda}_F}{d\omega_F^*}\right)^2\left[\frac{\omega_F^* R(1-\ell)(1-\gamma_F)}{(1-\bar{\lambda}_F)^2}\right],\end{aligned} \tag{41}$$

since $d^2\hat{\lambda}_F/d\omega_F^{*2} = 0$. Also, because $d\bar{\lambda}_F/d\omega_F^* = \gamma_F R_F^{-1} R(1-\ell)$, $d\hat{\lambda}_F/d\omega_F^* = (R_F - R_D)^{-1} R(1-\ell)$, and $\hat{\lambda}_F = ((\bar{\lambda}_F - \ell)R_F/\gamma_F - R_D + \ell R_F)/(R_F - R_D)$, we have that

$$\frac{d\bar{\lambda}_F}{d\omega_F^*}\frac{1}{1-\bar{\lambda}_F} - \frac{d\hat{\lambda}_F}{d\omega_F^*}\frac{1}{1-\hat{\lambda}_F} = \frac{R(1-\ell)}{R_F}\frac{(\bar{\lambda}_F - \ell)(\gamma^{-1}-1)}{[1-\bar{\lambda}_F\gamma_F^{-1} + \ell(\gamma_F^{-1}-1)](1-\bar{\lambda}_F)} > 0.$$

Hence, the second derivative is strictly positive and $\bar{\Delta}_F^*$ is strictly convex for $\omega_F^* < \omega_F^P$.

For $\omega_j^* \geq \omega_F^P$ we have that $\hat{\lambda}_F \geq \bar{\lambda}_F$. Hence,

$$\frac{d\bar{\Delta}_F^*}{d\omega_F^*} = \frac{d\bar{\lambda}_F}{d\omega_F^*}\frac{R_D + V}{1-\delta} - \int_{\bar{\lambda}_F}^{1}\frac{\gamma_F R_F^{-1} R(1-\ell)}{\lambda} r \frac{d\lambda}{1-\delta} \tag{42}$$

To evaluate this derivative at candidate crossing point we substitute $\bar{\Delta}_F^* = 0$ in (42), which yields

$$\left.\frac{d\bar{\Delta}_F^*}{d\omega}\right|_{\omega=\omega^*} = \frac{1}{\omega}\left[(\delta-\ell)(R_D + V - r)\frac{1}{1-\delta} + \int_{\bar{\lambda}_F}^{1}\frac{\ell}{\lambda} r \frac{d\lambda}{1-\delta}\right] > 0$$

Hence $\bar{\Delta}_F^*$ may only cross zero from below for $\omega_j^* \geq \omega_F^P$. In sum, the ω_F^* that satisfies (21) for $j = F$ is also unique. □

Proof of Proposition 3

Proof. Totally differentiating $\bar{\Delta}_C^* = 0$ and $\bar{\Delta}_F^*$ given by (21) we get that:

$$\frac{d\omega_C^*}{dR_C} = -\frac{d\bar{\Delta}_C^*/dR_C}{d\bar{\Delta}_C^*/d\omega_C^*} \quad \& \quad \frac{d\omega_F^*}{dR_F} = -\frac{d\bar{\Delta}_F^*/dR_F}{d\bar{\Delta}_F^*/d\omega_F^*} \quad \& \quad \frac{d\omega_F^*}{d\gamma_F} = -\frac{d\bar{\Delta}_F^*/d\gamma_F}{d\bar{\Delta}_F^*/d\omega_F^*}. \tag{43}$$

We will evaluate these derivative at the equilibrium thresholds ω_C^* and ω_F^* that satisfy $\bar{\Delta}_C^* = 0$ and $\bar{\Delta}_F^* = 0$. We have already shown that $d\bar{\Delta}_C^*/d\omega_C^* > 0$ and $d\bar{\Delta}_F^*/d\omega_F^* > 0$ in Proposition 2. Thus, we just need to sign the derivatives in the denominator.

The derivative of $\bar{\Delta}_C^*$ with respect to R_C is:

$$\begin{aligned}
\frac{d\bar{\Delta}_C^*}{dR_C} &= \frac{d\hat{\lambda}_C}{dR_C} V \frac{1}{1-\delta} - \int_{\hat{\lambda}_C}^{\bar{\lambda}_C} \frac{\lambda-\ell}{1-\lambda}\frac{d\lambda}{1-\delta} + \int_{\bar{\lambda}_C}^{1} \frac{R_C^{-2}\omega_C^* R(1-\ell)}{\lambda} r \frac{d\lambda}{1-\delta} \\
&\overset{\bar{\Delta}_C^*=0}{=} \left[\frac{d\hat{\lambda}_C}{dR_C} R_C + \hat{\lambda}_C - \delta\right] \frac{V}{1-\delta}\frac{1}{R_C} - \int_{\hat{\lambda}_C}^{\bar{\lambda}_C} \frac{\lambda-\ell}{1-\lambda}\frac{d\lambda}{1-\delta} + \frac{1}{R_C}\int_{\delta}^{\hat{\lambda}_C} (R_D - r)\frac{d\lambda}{1-\delta} \\
&\quad + \frac{1}{R_C}\int_{\hat{\lambda}_C}^{\bar{\lambda}_C} \left[\frac{\omega_C^* R(1-\ell) - (\lambda-\ell)R_C}{1-\lambda} - r\right]\frac{d\lambda}{1-\delta} - \frac{1}{R_C}\int_{\bar{\lambda}}^{1} \frac{\ell}{\lambda}\frac{d\lambda}{1-\delta} < 0,
\end{aligned} \tag{44}$$

because the first term can be written as $(d\hat{\lambda}_C/dR_C)R_C + \hat{\lambda}_C - \delta = \ell - \delta < 0$, the second term is negative, $R_D - r < 0$ in the third term, and the integrand in the fourth term is equal $R_D - r < 0$ for $\lambda = \hat{\lambda}_C$ and decreasing in λ. Note that we have substituted $\bar{\Delta}_C^* = 0$ in the second step.

For the case of F, need to consider separately the cases that $\gamma_F > \gamma_F^P$, i.e., $\hat{\lambda}_F < \bar{\lambda}_F$, and $\gamma_F \le \gamma_F^P$, i.e., $\hat{\lambda}_F \ge \bar{\lambda}_F$. For $\gamma_F > \gamma_F^P$, the derivative of $\bar{\Delta}_F^*$ with respect to R_F is:

$$\begin{aligned}
\frac{d\bar{\Delta}_F^*}{dR_F} &= \frac{d\hat{\lambda}_F}{dR_F} V \frac{1}{1-\delta} - \int_{\hat{\lambda}_F}^{\bar{\lambda}_F} \frac{\lambda-\ell}{1-\lambda}\frac{d\lambda}{1-\delta} + \frac{d\bar{\lambda}_F}{dR_F}\frac{\omega_F^* R(1-\ell)(1-\gamma_F)}{1-\bar{\lambda}_F}\frac{1}{1-\delta} + \int_{\bar{\lambda}_F}^{1} \frac{\gamma_F R_F^{-2}\omega_F^* R(1-\ell)}{\lambda} r \frac{d\lambda}{1-\delta} \\
&\overset{\bar{\Delta}_F^*=0}{=} \left[\frac{d\hat{\lambda}_F}{dR_F} R_F + \hat{\lambda}_F - \delta\right] \frac{V}{1-\delta}\frac{1}{R_F} - \int_{\hat{\lambda}_F}^{\bar{\lambda}_F} \frac{\lambda-\ell}{1-\lambda}\frac{d\lambda}{1-\delta} + \frac{d\bar{\lambda}_F}{dR_F}\frac{\omega_F^* R(1-\ell)(1-\gamma_F)}{1-\bar{\lambda}_F}\frac{1}{1-\delta} \\
&\quad + \frac{1}{R_F}\int_{\delta}^{\hat{\lambda}_F} (R_D - r)\frac{d\lambda}{1-\delta} + \frac{1}{R_F}\int_{\hat{\lambda}_F}^{\bar{\lambda}_F} \left[\frac{\omega_F^* R(1-\ell) - (\lambda-\ell)R_F}{1-\lambda} - r\right]\frac{d\lambda}{1-\delta} - \frac{1}{R_F}\int_{\bar{\lambda}}^{1} \frac{\ell}{\lambda}\frac{d\lambda}{1-\delta} < 0,
\end{aligned} \tag{45}$$

because the first term can be written as $d\hat{\lambda}_F/dR_F R_F + \hat{\lambda}_F - \delta = \ell - \delta < 0$, the second term is negative, $d\bar{\lambda}_F/dR_F < 0$ in the third term, $R_D - r < 0$ in the fourth term, and the integrand in the fifth term is equal $R_D - r < 0$ for $\lambda = \hat{\lambda}_F$ and decreasing in λ. Note that again we have substituted $\bar{\Delta}_F^* = 0$ in the second step.

For $\gamma_F \le \gamma_F^P$, the derivative of $\bar{\Delta}_F^*$ with respect to R_F is:

$$\begin{aligned}
\frac{d\bar{\Delta}_F^*}{dR_F} &= \frac{d\bar{\lambda}_F}{dR_F}(R_D + V)\frac{1}{1-\delta} + \int_{\bar{\lambda}_F}^{1} \frac{\gamma_F R_F^{-2}\omega_F^* R(1-\ell)}{\lambda} r \frac{d\lambda}{1-\delta} \\
&\overset{\bar{\Delta}_F^*=0}{=} \left[\frac{d\bar{\lambda}_F}{dR_F} R_F + \bar{\lambda}_F - \delta\right] \frac{R_D+V}{1-\delta}\frac{1}{R_F} - \frac{1}{R_F}\int_{\delta}^{\bar{\lambda}_F} r\frac{d\lambda}{1-\delta} - \frac{1}{R_F}\int_{\bar{\lambda}}^{1} \frac{\ell}{\lambda}\frac{d\lambda}{1-\delta} < 0
\end{aligned} \tag{46}$$

because $d\bar{\lambda}_F/dR_F R_F + \bar{\lambda}_F - \delta = \ell - \delta < 0$.

Following the exact same step, it is easy to show that $d\bar{\Delta}_F^*/d\gamma_F > 0$ for both cases.

$\square$

Proof of Proposition 4

Proof. Consider $p \to 0$ and $\gamma_F \to 1$ such that $R_C \to R_C$, $\omega_C^* \to \omega_F^*$, and $\int_{\omega_C^*}^1 \Pi_C(\omega,\delta)d\omega \to \int_{\omega_F^*}^1 \Pi_F(\omega,\delta)d\omega$. Then, increase p which results in lower profitability under C borrowing because

$$\frac{d}{dp}\int_{\omega_C^*}^1 \Pi_C(\omega,\delta)d\omega = -\int_{\omega_C^*}^1 (\delta-\ell)d\omega - \frac{d\omega_C^*}{dR_C}\frac{dR_C}{dp}\Pi_C(\omega_C^*,\delta) < 0, \tag{47}$$

from Proposition 3 and $\delta > \ell$.

To compensate for the lower profitability under C increase decrease γ_F that reduces the profitability under F borrowing because

$$\frac{d}{d\gamma_F}\int_{\omega_F^*}^1 \Pi_F(\omega,\delta)d\omega = -\frac{d\omega_F^*}{d\gamma_F}\Pi_F(\omega_F^*,\delta) > 0, \tag{48}$$

from Proposition 3. The decrease in pledgeability to borrow from F required to compensate for the higher penalty rate when borrowing from C is equal to

$$\frac{d\int_{\omega_F^*}^1 \Pi_F(\omega,\delta)d\omega}{d\gamma_F}\left(\frac{d\int_{\omega_C^*}^1 \Pi_C(\omega,\delta)d\omega}{dp}\right)^{-1}, \tag{49}$$

yielding the $\gamma^+(p)$ in the Proposition. A higher (lower) γ_F than that makes borrowing from F more (less) desirable.

$\square$

Proof of Proposition 5

Proof. Under unilateral runs, the bank will choose the lender j that maximizes its expected profits conditional on the uncertain state:

$$\max_j \int_{\underline{\omega}_j}^1 \Pi_j(\omega,\delta)d\omega = \max_j \int_{\underline{\omega}_j}^1 [\omega R(1-\ell) - (1-\delta)R_D - (\delta-\ell)R_j]\,d\omega \tag{50}$$

The relationship between $\underline{\omega}_F$ and $\underline{\omega}_C$ depends on the level of γ from Lemma 6. Note that $\underline{\gamma}_F\frac{1-\ell}{\delta-\ell} > \underline{\gamma}_C\frac{R_F}{R_C}$, i.e., the γ_F under which only unilateral runs are possible also implies that $\underline{\omega}_F < \underline{\omega}_C$ in Lemma 6. Hence, the bank's profits are unambiguously higher under borrowing from F.

$\square$

Proof of Proposition 6

Proof. As a preliminary result we sign the elasticity of $\omega^*_{C,z}$ with respect to R_C as $p \to 0$ given by

$$\left.\frac{d\omega^*_{C,z}}{dR_C}\frac{R_C}{\omega^*_{C,z}}\right|_{p=0} = \frac{-\frac{d\hat{\lambda}_C}{dR_C}R_C V + \int_{\hat{\lambda}_C}^{\bar{\lambda}_C}\frac{\lambda-\ell}{1-\lambda}R_C d\lambda - \int_{\bar{\lambda}_C}^{1}\frac{R_C^{-1}\omega^*_{C,z}R(1-\ell_z)}{\lambda}rd\lambda}{\frac{d\hat{\lambda}_C}{d\omega}\omega^*_{C,z}V + \int_{\hat{\lambda}_C}^{\bar{\lambda}_C}\frac{\omega R(1-\ell_z)}{1-\lambda}d\lambda - \int_{\bar{\lambda}_C}^{1}\frac{R_C^{-1}\omega^*_{C,z}R(1-\ell_z)}{\lambda}rd\lambda}. \tag{51}$$

This elasticity is less than one if

$$\left(\frac{d\hat{\lambda}_C}{d\omega}\omega^*_{C,z} + \frac{d\hat{\lambda}_C}{dR_C}R_C\right)V + \int_{\hat{\lambda}_C}^{\bar{\lambda}_C}\frac{\omega R(1-\ell_z)-(\lambda-\ell_z)R_C}{1-\lambda}d\lambda > 0$$

$$\Rightarrow \frac{(1-\ell_z)V}{R_C-R_D}\left(\omega R - \frac{\omega R - R_D}{R_C-R_D}\right) + \int_{\hat{\lambda}_C}^{\bar{\lambda}_C}\frac{\bar{\lambda}_C-\lambda}{1-\lambda}R_C d\lambda > 0, \tag{52}$$

which is always true because $\omega R > \omega R/R_c > (\omega R - R_D)/(R_C - R_D)$.

Turning to the optimality condition with respect to p, (26), and evaluating it at $p = 0$ and for $c = 0$, we have that the sum of all terms apart from the term including ϕ is:

$$-\int_{R_F/R}^{\gamma^+(p)}\left[\frac{d\omega^*_{C,z}}{dp}\Pi_{C,z}(\omega^*_{C,z},\delta) - (1-\omega^*_{C,z})(\delta-\ell_z) - \frac{(\omega^*_{C,z})^2}{2}r^{-1}R(1-\ell_z)\right]dz$$

$$= -\int_{R_F/R}^{\gamma^+(p)}\frac{\omega^*_{C,z}}{r}\left[\left.\frac{d\omega^*_{C,z}}{dR_C}\frac{R_C}{\omega^*_{C,z}}\right|_{p=0}\Pi_{C,z}(\omega^*_{C,z},\delta) - \left(\frac{1}{\omega^*_{C,z}}-1\right)(\delta-\ell_z)r - \frac{\omega^*_{C,z}}{2}R(1-\ell_z)\right]. \tag{53}$$

The last expression is positive, and hence $p = 0$ cannot be an equilibrium, as long as for every bank z, the sum of the bracketed terms is negative, i.e.,

$$\frac{d\omega^*_{C,z}}{dR_C}\frac{R_C}{\omega^*_{C,z}} < \frac{\left(\frac{1}{\omega^*_{C,z}}-1\right)(\delta-\ell_z)r + \frac{\omega^*_{C,z}}{2}R(1-\ell_z)}{\Pi_{C,z}(\omega^*_{C,z},\delta)}. \tag{54}$$

Given that the elasticity of the run threshold to the discount window rate is less one, as shown above, it suffices that the right-hand side of the latter expression is higher than one, which in turn requires that $(\omega^*_{C,z})^2/2 - (\delta-\ell_z)r - \omega^*_{C,z}(1-\delta)R_{D,z} < 0$. Given that this condition is increasing in ω it suffices that it holds for $\omega^*C,z \to \overline{\omega}_C = R_C/R$, yielding

$$R_{D,z} > \frac{r}{2}\frac{1-\ell_z}{1-\delta} - R\frac{\delta-\ell_z}{1-\delta}. \tag{55}$$

The left-hand side of this condition is increasing in ℓ so it suffices that it holds for $\ell_z \to \delta$, yielding $R_{D,z} > r/2$. Now, we can obtain the minimum $R_{D,z}$ that satisfies depositors' participation constraint (2) by setting $\omega = 0$, i.e., $\min R_{D,z} = r - V/(1-\delta)$. Finally, we require that $\min R_{D,z} > r/2$, which yields the sufficient condition $V < (1-\delta)r/2$ such that $p = 0$ is not optimal. From Lemma 3, V needs to be higher than $\underline{V}(\pi)$. Because $\underline{V}(1) < 0$, we know that there exist π's such that this can hold. Hence, our sufficient condition on V holds for $\pi > \underline{V}^{-1}((1-\delta)r/2)$. Then, the lowest $\tilde{\pi}$ such that $p > 0$ is optimal is given by the maximum of $\underline{V}^{-1}((1-\delta)r/2)$ and the lower bound from Lemma 3. □

Proof of Proposition 7

Proof. Suppose $p > 0$ and $\tilde{p} = 0$. We will first show that $\tilde{p} = 0$ is not optimal and verify $p > 0$ at the end. The bracketed terms in (29) are decreasing in $\omega^*_{C,z^{++}}$ because $(\delta-\ell_z^{++})-R_C^{-1}\omega^*_{C,z^{++}}R(1-\ell^{++})=\delta-\overline{\lambda}_{C,z^{++}}<0$. Thus, if it positive for the higher possible value of $\omega^*_{C,z^{++}}$, then it is always positive. Since the maximum value that $\omega^*_{C,z^{++}}$ can take is equal to the upper dominance cutoff R_C/R, it suffices that

$$\bar{c} > \left(1-\frac{r+p}{R}\right)\delta p+\frac{r+p}{2R}p. \tag{56}$$

If $\delta>1/2$, the right-hand side of this condition is smaller than δp, which is increasing in p. Thus, we can derive a sufficient condition by substituting in the maximum feasible $p=R-r$, yielding $\bar{c}>\delta(R-r)$. If $\delta<1/2$, the right-hand side is smaller than $p/2$, also increasing in p. Similarly, the sufficient condition is $\bar{c}>(R-r)/2$. By continuity, under the condition $\bar{c}>(R-r)\max(\delta,1/2)$, C has an incentive to set $\tilde{p}>0$, i.e., differentiate its pricing across collateral types, if the number of banks pledging collateral with high social costs is sufficiently low—$\tilde{\gamma}$ sufficiently close to R_F/R. For higher $\tilde{\gamma}$, C will weigh in the additional factors described in Section 5.1; whether $\tilde{p}>0$ is optimal would depend on the parametrization of the economy.

To conclude the proof we show $p>0$ is optimal. We will consider separately the cases that $\tilde{p}>0$ and $\tilde{p}=0$. Stating with the former, the first-order condition of (27) with respect to p is:

$$\begin{aligned}
&-\int_{\tilde{\gamma}}^{\gamma^+(p)}\frac{d\omega^*_{C,z}}{dp}\left[\Pi_{C,z}(\omega^*_{C,z},\delta)+(\delta-\ell_z)p\right]dz+\int_{\tilde{\gamma}}^{\gamma^+(p)}\frac{d\omega^*_{C,z}}{dp}R_C^{-1}\omega^*_{C,z}R(1-\ell_z)p\,dz\\
&-\int_{R_F/R}^{\gamma^{++}(p+\tilde{p})}\frac{d\omega^*_{C,z}}{dp}\left[\Pi_{C,z}(\omega^*_{C,z},\delta)+(\delta-\ell_z)(p+\tilde{p})\right]dz+\int_{R_F/R}^{\gamma^{++}(p+\tilde{p})}\frac{d\omega^*_{C,z}}{dp}R_C^{-1}\omega^*_{C,z}R(1-\ell_z)(p+\tilde{p})dz\\
&+\frac{d\gamma^+}{dp}\left[\int_{\omega^*_{C,z^+}}^{1}(\delta-\ell_{z^+})p\,d\omega+\int_0^{\omega^*_{C,z^+}}R_C^{-1}\omega R(1-\ell_{z^+})p)d\omega\right]\\
&+\frac{d\gamma^{++}}{dp}\left[\int_{\omega^*_{C,z^{++}}}^{1}(\delta-\ell_{z^{++}})(p+\tilde{p})d\omega+\int_0^{\omega^*_{C,z^{++}}}R_C^{-1}\omega R(1-\ell_{z^{++}})(p+\tilde{p})d\omega\right]\\
&+\int_{\tilde{\gamma}}^{\gamma^+(p)}\left[\int_{\omega^*_{C,z}}^{1}(\delta-\ell_z)d\omega+\int_0^{\omega^*_{C,z}}\omega R(1-\ell_z)\frac{r}{(r+p)^2}d\omega\right]dz-\frac{d\gamma^+}{dp}\underline{c}\\
&+\int_{R_F/R}^{\gamma^{++}(p+\tilde{p})}\left[\int_{\omega^*_{C,z}}^{1}(\delta-\ell_z)d\omega+\int_0^{\omega^*_{C,z}}\omega R(1-\ell_z)\frac{r}{(r+p+\tilde{p})^2}d\omega\right]dz-\frac{d\gamma^{++}}{dp}\bar{c}+\phi\frac{1-R_F/R}{1-\pi}=0.
\end{aligned} \tag{57}$$

Because both penalty rates p and $\tilde{p}$ are linearly additive in R_C, we have that $d\gamma^{++}/dp=d\gamma^{++}/\tilde{p}$ and $d\omega^*_{C,z}/dp=d\omega^*_{C,z}/d\tilde{p}$. Using these relationship and substituting (28) in (57) we get that:

$$\begin{aligned}
&-\int_{\tilde{\gamma}}^{\gamma^+(p)}\frac{d\omega^*_{C,z}}{dp}\left[\Pi_{C,z}(\omega^*_{C,z},\delta)+(\delta-\ell_z)p\right]dz+\int_{\tilde{\gamma}}^{\gamma^+(p)}\frac{d\omega^*_{C,z}}{dp}R_C^{-1}\omega^*_{C,z}R(1-\ell_z)p\,dz\\
&+\frac{d\gamma^+}{dp}\left[\int_{\omega^*_{C,z^+}}^{1}(\delta-\ell_{z^+})p\,d\omega+\int_0^{\omega^*_{C,z^+}}R_C^{-1}\omega R(1-\ell_{z^+})p)d\omega\right]\\
&+\int_{\tilde{\gamma}}^{\gamma^+(p)}\left[\int_{\omega^*_{C,z}}^{1}(\delta-\ell_z)d\omega+\int_0^{\omega^*_{C,z}}\omega R(1-\ell_z)\frac{r}{(r+p)^2}d\omega\right]dz-\frac{d\gamma^+}{dp}\underline{c}+\phi\frac{1-R_F/R}{1-\pi}=0,
\end{aligned} \tag{58}$$

since $\tilde{\phi}=0$. Then, following the exact same steps as in the proof of Proposition 6, we can show that the sum of the first five terms in (59) is positive—even for $\underline{c}=0$—and hence $p=0$ is not optimal.

Next, suppose that $\tilde{p} = 0$. Because C does not price discriminate across collateral pledgeability, it lends to all borrowers with $\gamma_{F,z} \leq \gamma^{+}(p)$ at rate $r + p$. The first-order condition of (27) with respect to p is:

$$\begin{aligned}
&-\int_{R_F/R}^{\gamma^{+}(p)} \frac{d\omega^{*}_{C,z}}{dp} \left[\Pi_{C,z}(\omega^{*}_{C,z}, \delta) + (\delta - \ell_z)p\right] dz + \int_{\tilde{\gamma}}^{\gamma^{+}(p)} \frac{d\omega^{*}_{C,z}}{dp} R_C^{-1} \omega^{*}_{C,z} R(1-\ell_z) p dz \\
&+ \frac{d\gamma^{+}}{dp} \left[\int_{\omega^{*}_{C,z^{+}}}^{1} (\delta - \ell_{z^{+}}) p d\omega + \int_{0}^{\omega^{*}_{C,z^{+}}} R_C^{-1} \omega R(1-\ell_{z^{+}}) p) d\omega\right] \\
&+ \int_{R_F/R}^{\gamma^{+}(p)} \left[\int_{\omega^{*}_{C,z}}^{1} (\delta - \ell_z) d\omega + \int_{0}^{\omega^{*}_{C,z}} \omega R(1-\ell_z) \frac{r}{(r+p)^2} d\omega\right] dz - \frac{d\gamma^{+}}{dp} \underline{c} + \phi \frac{1 - R_F/R}{1-\pi} = 0. \qquad (59)
\end{aligned}$$

As above, using the steps in the proof of Proposition 6, ϕ cannot be positive and, thus, $p = 0$ is not optimal.

□

Designing the Optimal Discount Window

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Online Appendix

A Derivations

A.1 First-order conditions for Private Equilibrium

$$\overbrace{-(1-\delta-\psi)\left[\pi+(1-\pi)(1-\omega_k^*)\right]}^{\text{Combined effect of higher } R_D \text{ on bank profits and depositors' participation}}$$

$$\underbrace{-(1-\pi)\frac{d\omega_k^*}{dR_D}\left\{\Pi_k(\omega_k^*,\delta)+\psi\left[(1-\delta)R_D+\delta r_D+V-L(\omega_k^*)+\delta P(1-L(\omega_k^*)/r_D)\right]\right\}}_{\text{Combined effect of higher } R_D \text{ on run risk affecting bank profits and depositors' participation}}=0 \tag{A.1}$$

and

$$\overbrace{-\left[\pi(R-min_jR_j)+(1-\pi)\left(\int_{\omega_k^*}^{1}\omega R d\omega-(1-\omega_k^*)R_k\right)+\psi(1-\pi)(1+\delta P)\int_0^{\omega_k^*}(\gamma_k R_k^{-1}\omega R-1)d\omega\right]}^{\text{Combined effect of higher } \ell \text{ on bank profits and depositors' participation}}$$

$$\underbrace{-(1-\pi)\frac{d\omega_k^*}{d\ell}\left\{\Pi_k(\omega_k^*,\delta)+\psi\left[(1-\delta)R_D+\delta r_D+V-L(\omega_k^*)+\delta P(1-L(\omega_k^*)/r_D)\right]\right\}}_{\text{Combined effect of higher } \ell \text{ on run risk affecting bank profits and depositors' participation}}=0, \tag{A.2}$$

B Optimal p with commitment

C will choose a penalty rate to maximize $\mathbb{SW}$ in (25). We assume that C can credit commit to a penalty rate p and internalizes how the choice of p affects the choice of $R_{D,z}$ and ℓ_z, which are determined by the participation constraint (2) and the private optimality condition (24); see Definition (1) of the private equilibrium. Doing so yields the following optimality condition:

$$\frac{d\mathbb{SW}}{dp}+\frac{d\mathbb{SW}}{dR_{D,z}}\frac{dR_{D,z}}{dp}+\frac{d\mathbb{SW}}{d\ell_z}\frac{d\ell_z}{dp}+\phi=0, \tag{B.1}$$

where ϕ is the multiplier on the non-negativity constraint $p\geq 0$. We now unpack each term in (B.1).

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The first term captures the direct effect of a higher p on $\mathbb{SW}$ and can be expanded as follows:[1]

$$\frac{d\mathbb{SW}}{dp} = -(1-\pi)\int_{R_F/R}^{\gamma^+(p)} \frac{d\omega^*_{C,z}}{dp}\left[\omega^*_{C,z}R(1-\ell_z)-(1-\delta)R_{D,z}-(\delta-\ell_z)r\right]dz$$
$$+(1-\pi)\frac{d}{dp}\int_{R_F/R}^{\gamma^+(p)}\int_0^{\omega^*_{C,z}} R_C^{-1}\omega R(1-\ell_z)p\,d\omega dz-(1-\pi)\frac{d\gamma^+}{dp}c. \tag{B.2}$$

The first line in (B.2) captures the decrease in expected bank profitability due to an increase in the run probability from a higher p for those banks that choose to borrow from C; recall that $d\omega^*_{C,z}/dp > 0$ from Proposition 3. This term is unambiguously negative pushing for a lower p. The first term in second line captures the change in the central bank profits from a higher p and can be decomposed further in three parts: (i) decrease in the number of banks borrowing from C, $d\gamma^+/dp < 0$, pushing expected central bank profits down; (ii) increase in the run probability for those banks that borrow from C, $d\omega^*_{C,z}/dp > 0$, and, hence, higher profits to the central bank from seizing collateral; and (iii) increase in the profit margin per unit of lending, $d[(r+p)^{-1}\omega R(1-\ell_z)p]/dp > 0$. The combined effects of these three effects is ambiguous. However, for $\rho \to 0$, the former two are zero, while the third is positive, pushing for a non-zero p. The final term captures the benefit from reducing the costs of discount window use by increasing p and is unambiguously positive ($d\gamma^+(p)/dp < 0$).

The second term in (B.1) captures the indirect effect of a higher p operating via the deposit rate for those banks that borrow from C. $dR_{D,z}/dp = -(dPC_z/dp)/(dPC_z/R_{D,z})$ is obtained by implicitly differentiating the binding participation constraint (2) for those banks, yielding:

$$\frac{d\mathbb{SW}}{dR_{D,z}}\frac{dR_{D,z}}{dp} = -\int_{R_F/R}^{\gamma^+(p)}\left\{(1-\pi)\left[\int_{\omega^*_{C,z}}^1(1-\delta)d\omega+\frac{d\omega^*_{C,z}}{dR_{D,z}}[\omega^*_{C,z}R(1-\ell_z)-(1-\delta)R_{D,z}-(\delta-\ell_z)R_C]\right]\right.$$
$$\left.+\pi(1-\delta)+(1-\pi)\frac{d\omega^*_{C,z}}{dR_{D,z}}p\left[\delta-\ell_z-R_C^{-1}\omega^*_{C,z}R(1-\ell_z)\right]\right\}\frac{dR_{D,z}}{dp}dz$$
$$=\int_{R_F/R}^{\gamma^+(p)}\frac{dPC_z}{dp}\left\{\psi_z-(1-\pi)\frac{d\omega^*_{C,z}}{dR_{D,z}}\left(\frac{dPC_z}{dR_{D,z}}\right)^{-1}p\left[\overline{\lambda}_{C,z}-\delta\right]\right\}dz, \tag{B.3}$$

where in the last step we used the private optimality condition (23) and the definition of $\overline{\lambda}_C$. dPC_z/dp is negative as it affects the (2) only through the run probability, which increases for higher p resulting in lower depositors' payoff and a tighter constraint. Should $d\omega^*_{C,z}/dR_{D,z} < 0$ and $dPC_z/dR_{D,z} > 0$, i.e., a higher deposit rate reduces the incentives to run and increases depositors' payoff, the aggregate effect in (**??**) is negative pushing for a lower penalty rate p.[2]

The third term in (B.1) captures the indirect effect of a higher p operating via the liquid holdings choice of those banks that borrow from C. $d\ell_z/dp = -(dPC_z/dp)/(dPC_z/\ell_z)$ is obtained by implicitly differentiating

[1] For the marginal bank z^+ with $\gamma_{F,z^+} = \gamma^+(p)$, it holds that $\int_{\omega^*_{F,z^+}}^1 \Pi_{F,z^+}(\omega,\delta)d\omega = \int_{\omega^*_{C,z^+}}^1 \Pi_{C,z^+}(\omega,\delta)d\omega$. Hence, the terms that include the derivative $d\gamma^+(p)/dp$ of the limit of integration in the second and third lines cancel out.

[2] These properties are intuitive but cannot be unambiguously established analytically.

the private optimality condition (24) for those banks, yielding:

$$\begin{aligned}\frac{d\mathbb{SW}}{d\ell_z}\frac{d\ell_z}{dp} =& -\int_{R_F/R}^{\gamma^+(p)}\Bigg\{(1-\pi)\left[\int_{\omega^*_{C,z}}^{1}(\omega R - R_C)d\omega + \frac{d\omega^*_{C,z}}{d\ell}[\omega^*_{C,z}R(1-\ell_z) - (1-\delta)R_{D,z} - (\delta-\ell_z)R_C]\right]\\ &+\pi(R-R_F) + (1-\pi)\left[\int_{\omega^*_{C,z}}^{1}d\omega + \int_0^{\omega^*_{C,z}}R_C^{-1}\omega R d\omega\right]p\\ &+(1-\pi)\frac{d\omega^*_{C,z}}{d\ell}p\left[\delta-\ell_z - R_C^{-1}\omega^*_{C,z}R(1-\ell_z)\right]\Bigg\}\frac{d\ell_z}{dp}dz\\ =& \int_{R_F/R}^{\gamma^+(p)}\frac{dPC_z}{dp}\left\{\psi_z + (1-\pi)\left(\frac{dPC_z}{d\ell_z}\right)^{-1}\left[\int_{\omega^*_{C,z}}^{1}d\omega + \int_0^{\omega^*_{C,z}}R_C^{-1}\omega R d\omega - \frac{d\omega^*_{C,z}}{d\ell_z}\left(\bar{\lambda}_{C,z}-\delta\right)\right]p\right\}dz\end{aligned} \tag{B.4}$$

Should $d\omega^*_{C,z}/d\ell_z < 0$ and $dPC_z/d\ell_z > 0$, i.e., higher liquid holdings decrease the probability of a run and increase depositors' payoffs, the aggregate effect in (**??**) is negative since $dPC_z/dp < 0$.

Overall, the indirect effects of a higher p, operating via the deposit rate and liquid holdings, push for a lower penalty. The intuition is that a higher p would result in inefficiently high R_D and ℓ. Moreover, the direct effect on bank profits also pushes for a lower p. By contrast, the direct effect on central bank profits and the social costs from a big central bank footprint push for a higher p.