Option-Implied Risk Premia with Intertemporal Hedging

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ABSTRACT

We develop new measures of the equity and variance risk premia that account for intertemporal hedging motives. Standard asset pricing models impose restrictive assumptions on the role of hedging demand that are difficult to reconcile with empirical evidence on investor prudence and higher order risk pricing. Our framework is flexible enough to relax these restrictions. It delivers risk premia that embed information on the term structure of market risk and can be computed from option prices. Empirically, intertemporal hedging contributes to large fractions of the equity and variance risk premia and delivers significant improvements in out-of-sample predictability of market returns.

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The equity risk premium—the expected return on the equity market over the risk-free rate—is a crucial input for corporate valuation and portfolio allocation. Unfortunately, it is also notoriously difficult to estimate ex ante. Martin (2017) shows how the risk-neutral market variance discounted at the risk-free rate provides a lower bound for the equity risk premium, in a one-period economy that ignores the higher-order moments of market returns. A major benefit of this approach is that the risk-neutral variance can be computed in real-time from observed option prices. Chabi-Yo and Loudis (2020) and Tetlock et al. (2024) extend the approach of Martin (2017) and provide estimates for the equity risk premium that account for higher-order risks, still in a one-period model.

Restricting the economy to a one-period economy simplifies the analysis, but at the expense of strong assumptions. In particular, the risk of future shifts in the economic environment, e.g., changes in the expected returns or return volatility, is ignored. Consider a forecast horizon $T_1 > t$. A one-period model assumes that investors choose their portfolio allocation at time t ignoring risks beyond time T_1. However, these risks impact future consumption. Merton (1969, 1971, 1973), Samuelson (1969) and later Cox and Huang (1989); Campbell and Viceira (1999, 2002) and Campbell et al. (2003), among others, show that investors optimally seek to hedge these risks by tilting their portfolio allocation towards assets that deliver higher returns when consumption is negatively affected. They show that these shifting investment opportunities can have important effects on optimal portfolios for investors with long horizons. Intertemporal hedging of risk materializing after time T_1 therefore affects demand as well as equilibrium prices and risk premia at horizon T_1.

We derive novel estimates for the equity and variance risk premia that incorporate both higher-order risks and intertemporal hedging. The model is set in a multi-period economy in which the representative investor chooses at date t their optimal allocation to the market portfolio, and is allowed to rebalance their portfolio at dates T_j between T_1 and T_N, to maximize their expected utility of terminal wealth at the investment horizon $T_N \geq T_1$. We show that this environment is equivalent to one in which the investor can trade the market index, the risk-free asset as well as the full term structure of options with maturities between T_1 and T_N, liquidating all contracts maturing after T_1 at T_1. In this setting, the resulting premia depend not only on the conditional moments of market returns over $[t, T_1]$ but also on the investor's time-t expectations of the conditional moments of returns over each interval between trading dates from T_1 through T_N. These time-t expectations of future moments have a direct economic meaning: they price the compensation required to hold short-horizon market exposure that covaries with shocks to post-T_1 investment opportunities. Thus, both short-term risks and anticipated future return moments shape risk premia at horizon T_1. Our model nests the benchmark models of investors endowed with logarithmic utility, both without access to options and with access to them. It also generalizes the model of Chabi-Yo and Loudis (2020), which arises as the special case when $T_N = T_1$.

Under minimal assumptions on the distribution of returns, we show that the impact of intertemporal hedging on the equity risk premium depends on the representative agent's tolerance for skewness ρ_t. This parameter captures the agent's prudence and has been shown to measure the strength of precautionary savings, see Kimball (1990); Gollier and Pratt (1996); Gollier (2001). We highlight a tension between the parameter restrictions typically imposed in asset pricing models and the implications of economic theory on co-skewness pricing and prudence. In particular, asset pricing frameworks featuring CRRA-type preferences—including models with Epstein-Zin recursive utility and habit formation—commonly adopt parameterizations that constrain ρ_t to values below one. This arises because, under these preferences, ρ_t is fully determined by the coefficient of relative risk aversion γ: when $\gamma > 1$, it follows that $\rho_t < 1$. In this case, intertemporal hedging decreases the equity risk premium. However, this restriction implies that the stochastic discount factor is decreasing in market volatility and that co-skewness risk is positively priced, contrary to the evidence reported by Harvey and Siddique (1999), Christoffersen et al. (2013) and Langlois (2020). Moreover, it implies that prudence is less than twice the risk aversion, contradicting the results of Kimball (1990) on the strength of precautionary motives.

Our approach does not restrict the analysis to any particular functional form of utility. The agent's tolerance for skewness is a free parameter, decoupled from the risk aversion parameter. Choosing $\rho_t > 1$, in line with the economic theory on the price of co-skewness risk and on prudence, leads to intertemporal hedging increasing the equity risk premium. We further show that under this constraint, our equity risk premium is a lower bound for expected future returns.

Our equity and variance risk premium measures are functions of risk-neutral moments and expected future moments of S&P 500 returns. To estimate the latter, we project the future risk-neutral variance between T_j and T_k—for T_j and T_k between T_1 and T_N—onto returns at T_j, leveraging the spanning results of Carr and Madan (2001) and Bakshi and Madan (2000). This approach allows the resulting risk premia to be computed in real time from the term structure of observed option prices with maturities from T_1 to T_N. Adopting preference parameters consistent with a negative price of co-skewness risk and strong precautionary motives, we compute these premia for the S&P 500 over the 1996–2023 period, across forecast horizons T_1 ranging from 10 days to 12 months, and investment horizons T_N up to two years. We show that the increase in the equity risk premium due to intertemporal hedging is particularly large during times of market calm. Intertemporal hedging accounts for up to 80% of the total equity risk premium during these periods, and up to 30% during NBER recessions.

Furthermore, our equity risk premium delivers higher out-of-sample R^2 in return prediction than the bounds of Martin (2017), Chabi-Yo and Loudis (2020) and Tetlock et al. (2024). For short forecast horizons T_1 up to one month, the out-of-sample R^2 increases with the investors' horizon T_N, reaching its maximum at an investment horizon T_N of 18 months. For forecast horizons T_1 larger than one month, the maximum R^2 is obtained at $T_N =$ 2 years, i.e., the longest horizon for which we have available option maturities. This result suggests that access to options with maturities longer than two years may help better capture intertemporal hedging effects and improve return forecasts at horizons of two months and longer. Using our measures of expected return and variance, we also construct market-timing strategies and compute realized mean-variance certainty equivalents. These certainty equivalents indicate that our risk premium reaches better forecasts of both the first and second return moments and that the improvements to the forecasts of Chabi-Yo and Loudis (2020) are statistically significant. We compare these results to those obtained under a parameterization consistent with CRRA preferences and $\gamma > 1$ — which implies $\rho_t < 1$ — and find empirical support for $\rho_t > 1$.

Our main results are based on preference parameters that are chosen, not estimated. We extend the analysis and estimate the preference parameters over the whole period 1996-2023, in-sample. We test two specifications: one in which the preference parameters are constant and one in which they vary over time as linear functions of past returns. These estimations further support $\rho_t > 1$. Because estimation over the full sample introduces a look-ahead bias, we next estimate the preference parameters at multiple times t, over windows that do not extend beyond t. We find that the resulting equity risk premium measures are highly sensitive to the length of the estimation window and, overall, do not outperform our baseline estimates in terms of out-of-sample R^2. Our results thus indicate that our risk premia with fixed preference parameters provide the best overall performance: they can be computed in real time from option prices and deliver strong out-of-sample forecasts.

We contribute to several strands of literature.

First, we contribute to the growing literature that constructs bounds on and estimates of physical return moments. Building on Martin (2017), Crescini et al. (2025) use investors' option holdings to recover investor-specific expected returns and risks. Martin and Wagner (2019), Kadan and Tang (2020), and Chabi-Yo et al. (2021) build bounds for the expected return on individual stocks and Kremens and Martin (2019) provide a bound for currency expected exchange rate appreciation using Quanto index options. See Back et al. (2022) for a discussion and empirical tests of bounds for individual stocks and the stock market. We contribute to this literature by quantifying the role of intertemporal hedging in a multi-period framework.

Our intertemporal hedging term includes a risk-neutral leverage effect that is closely related to the asymmetric volatility implied correlation studied by Jackwerth and Vilkov (2019). They use short- and long-term options on the S&P 500 index and options on VIX futures to calibrate the risk-neutral correlation between returns and future volatility. As options on VIX futures are available only starting in 2006, data availability prevents us from using their methodology.

Second, our work is related to the Recovery Theorem of Ross (2015), which shows how to disentangle the physical probability distribution from the pricing kernel and risk-neutral probabilities. This work has been challenged on theoretical and empirical grounds.[1] Instead of making assumptions about the pricing kernel process, Schneider and Trojani (2019) impose sign restrictions on the risk premia of return moments and find predictive power for future returns. Our approach differs in that we express, in a multi-period economy, the equity risk premium and the market's conditional variance under the physical measure as functions of risk-neutral moments of returns at different horizons.

Third, we build on the vast literature on the effect of intertemporal hedging on asset demand. Under Epstein and Zin (1989) preferences, Chacko and Viceira (2005) find that the effect of intertemporal hedging on asset demand is almost negligible, when risk aversion is greater than one and the elasticity of intertemporal substitution exceeds the reciprocal of risk aversion. In contrast, Brandt (1999), Campbell and Viceira (1999, 2002); Campbell et al. (2003), and others show that the estimated time variation in the US risk premia generates large intertemporal hedging demands for investors. This literature generally assumes that risk aversion exceeds one and that intertemporal hedging lowers the equity risk premium. Although our approach does not impose a specific functional form for utility, when one adopts specifica-

[1] Borovička et al. (2016) show that Ross' assumptions rule out realistic models. Bakshi et al. (2018) do not find support for the implications of the Recovery Theorem using U.S. Treasury bond futures. While Audrino et al. (2019) find some forecasting power, Jensen et al. (2019) generalize the assumptions of Ross' (2015) model and find weak predictive power for future realized returns.

tions such as CRRA or Epstein-Zin recursive utility, our theory yields the same implication: intertemporal hedging reduces the equity risk premium for standard parameter values. We further show that under Epstein-Zin preferences, where both the aggregator and the certainty equivalent follow power functional forms, the model enforces a fixed relationship between intertemporal hedging and the equity risk premium. We show that relaxing this power-form assumption for the aggregator and the certainty equivalent eliminates this constraint: the equity risk premium and intertemporal hedging may then move in opposite directions, in contrast to the standard power-form specification.

Our paper proceeds as follows. Section I derives our estimate of the equity risk premium in a multi-period model, based on a second-order approximation. Section II discusses the constraints imposed by standard asset pricing models on the contribution of intertemporal hedging to the equity risk premium. Section III presents estimates for the physical variance and the probability of a crash. Section IV discusses our empirical framework for building risk premium forecasts. Section V presents our main empirical results. In Section VI we show the results when estimating the preference parameters of our model. Finally, Section VII concludes.

I. Equity risk premium in a multi-period economy

In this section, we provide our main theoretical results. We derive an estimate of the equity risk premium in a multi-period economy, accounting for the risks of future intertemporal shifts in the economic environment. All proofs are provided in Section I of the Internet Appendix.

A. Model

We consider a dynamic economy with dates $t = T_0 < T_1 < T_2 < \cdots < T_N$, and a representative agent. Let T_1 be the horizon at which we aim to forecast returns, and T_N be the representative agent's investment horizon. The main innovation of our approach is that the representative agent considers what happens beyond the forecast horizon T_1, up to their investment horizon T_N, when solving the portfolio allocation problem. We assume that this economy is arbitrage-free, which guarantees the existence of a stochastic discount factor (SDF) and of a risk-neutral measure. For simplicity, we assume no interest rate risk.

At time t, the representative agent invests their wealth W_t in an asset delivering the risk-free gross return R_{f,t,T_1}, and in a set of risky assets delivering gross returns $R_{k,t,T_1}, k = 1, ..., K$. Under no-arbitrage conditions, the expected excess return on each risky asset from time t to time T_1 can be expressed as the risk-neutral covariance between the asset return and the inverse of the one-period SDF from t to T_1, m_{t,T_1}:

$$\mathbb{E}_t \left(R_{k,t,T_1} - R_{f,t,T_1} \right) = \mathbb{COV}_t^* \left(R_{k,t,T_1}, \frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}} \right). \tag{1}$$

See Section I.A of the Internet Appendix for the proof of this identity.

We collect the gross returns on risky assets over the interval $[t, T_1]$ in the vector R_{t,T_1}, and those over $[T_1, T_N]$ in the vector R_{T_1,T_N}. At time T_1, the agent rebalances their portfolio. All results extend to the case where the agent rebalances their portfolio at each intermediate time $T_j, 0 < j < N$; however, this generalization involves a more cumbersome notation. Therefore, we present here the case of a single rebalancing date T_1. The general multi-rebalancing setting is developed in Section I.E of the Internet Appendix.

The terminal wealth of the representative agent at their investment horizon T_N is

$$\begin{aligned} W_{T_N} &= W_t \left(R_{f,t,T_1} + \omega_t^\intercal (R_{t,T_1} - R_{f,t,T_1}) \right) \left(R_{f,T_1,T_N} + \omega_{T_1}^\intercal (R_{T_1,T_N} - R_{f,T_1,T_N}) \right) \\ &= W_t \left(\omega_t^\intercal R_{t,T_1} \right) \left(\omega_{T_1}^\intercal R_{T_1,T_N} \right), \end{aligned} \tag{2}$$

where ω_t and ω_{T_1} are the vectors of portfolio weights in risky assets at times t and T_1, respectively. R_{f,T_1,T_N} is the risk-free gross return from time T_1 to time T_N.

The investor chooses the portfolio weights $\{\omega_t, \omega_{T_1}\}$ to maximize their expected utility[2] of terminal wealth over the period $[t, T_N]$. The horizon of this portfolio optimization problem stands in contrast to the frameworks of Martin (2017); Chabi-Yo and Loudis (2020); Tetlock et al. (2024) and Crescini et al. (2025), who feature an economy in which the investor maximizes the expected utility of wealth over $[t, T_1]$.

Provided that no-arbitrage conditions hold in this economy, and assuming that the gross return on the market can be used as a proxy for the return on aggregate wealth, we can express the one-period inverse stochastic discount factor (SDF) from t to T_1, m_{t,T_1} as[3]

$$\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}} = \frac{\upsilon_{T_1}}{\mathbb{E}_t^* (\upsilon_{T_1})} \text{ with } \upsilon_{T_1} = \mathbb{E}_{T_1}^* \left(\frac{u' [W_t R_{f,t,T_N}]}{u' [W_t R_{M,t,T_N}]} \right), \tag{3}$$

where $\mathbb{E}_{T_1}^* (\cdot)$ denotes the expected value at time T_1 under the risk-neutral measure, R_{f,t,T_N} is the risk-free gross return from t to T_N and R_{M,t,T_N} is the gross market return. Thus, the one-period SDF depends on the marginal utility of wealth at the investment horizon of the representative agent T_N.

We do not make any assumptions about the functional form of the marginal utility function. We use a second-order Taylor expansion series of the inverse of the marginal utility to produce a one-period SDF of the form

$$\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}} \approx \frac{(1 + z_{T_1})}{\mathbb{E}_t^* (1 + z_{T_1})}, \tag{4}$$

where

$$z_{T_1} = \frac{a_{1,t}}{R_{f,t,T_1}} (R_{M,t,T_1} - R_{f,t,T_1}) + \frac{a_{2,t}}{R_{f,t,T_1}^2} (R_{M,t,T_1} - R_{f,t,T_1})^2 + \frac{a_{2,t}}{R_{f,T_1,T_N}^2} \mathbb{M}_{T_1,T_N}^{*(2)} \tag{5}$$

[2]We assume that the utility function $u[.]$ is well-defined, its derivatives up to order four exist, and their signs obey the following economic theory restriction: $\text{sign}\left(u^{(i)}[\cdot]\right) = \text{sign}(-1)^{i+1}$ (Eeckhoudt and Schlesinger, 2006; Deck and Schlesinger, 2014).

[3]The derivation of the inverse SDF is given in Section I.B of the Internet Appendix.

and $\mathbb{M}^{*(2)}_{T_1,T_N} = \mathbb{E}^{*}_{T_1}\left(R_{M,T_1,T_N} - R_{f,T_1,T_N}\right)^2$ is the risk-neutral variance at time T_1. The coefficients $a_{1,t}$ and $a_{2,t}$ in the Taylor expansion series are functions of the investor's risk and skewness tolerance parameters τ_t, and ρ_t:[4]

$$a_{1,t} = \tfrac{1}{\tau_t}, \qquad a_{2,t} = \tfrac{(1-\rho_t)}{\tau_t^2}, \tag{6}$$

where

$$\tau_t = -\frac{u^{(1)}\left[W_t R_{f,t,T_N}\right]}{W_t R_{f,t,T_N} u^{(2)}\left[W_t R_{f,t,T_N}\right]}, \tag{7}$$

$$\rho_t = \frac{1}{2!}\frac{u^{(3)}\left[W_t R_{f,t,T_N}\right] u^{(1)}\left[W_t R_{f,t,T_N}\right]}{\left(u^{(2)}\left[W_t R_{f,t,T_N}\right]\right)^2}. \tag{8}$$

The derivation of Equations (4)-(8) is in Section I.C of the Internet Appendix.

Importantly, the utility function in the definition of the preference parameters τ_t and ρ_t in (7)–(8) is evaluated at a specific value of terminal wealth $W_t R_{f,t,T_N}$, and not at the *random* terminal wealth W_{t,T_N}. If this were the case, equations (7) and (8) would define a set of ODEs, which could be used to attempt to recover the functional form of utility $u\left[W_{t,T_N}\right]$ assuming that a constraint is imposed on τ_t and ρ_t.

Equations (4) and (5) show that the inverse of the SDF is a function of three terms: the excess market return, the squared excess market return, and the market risk-neutral variance $\mathbb{M}^{*(2)}_{T_1,T_N}$ at time T_1. The latter term is new and only arises in a multi-period economy.[5] In contrast, the risk and skewness tolerance parameters in equations (7)–(8) differ from those derived

[4]Similarly, a third order expansion produces a one-period SDF of the same form, given in Section VI1 of the Internet Appendix.

[5]We know from Merton's ICAPM that shocks to risk can generate hedging demand and can be priced. But Merton's ICAPM shows that market physical volatility is a key determinant of expected excess return on a stock, not risk neutral market volatility. Strong evidence of time-varying volatility risk premium suggests that the risk neutral market variance and the physical market variance are distinct and carry different sets of information. Our theoretical results are thus distinct from implications of Merton's ICAPM model.

by Chabi-Yo and Loudis (2020) but we expect this difference to be small. They involve risk-free returns over the interval $[t, T_N]$, rather than $[t, T_1]$. However, due to the shape of the yield curve, the risk-free returns from T_1 to T_N tend to be close to 1.

B. *Equity risk premium*

Our main theoretical result combines the expression of the risk premium in Equation (1) with the SDF (4) to provide a closed-form solution to the conditional expected excess market return in terms of risk-neutral moments.

PROPOSITION 1: *Up to a second-order expansion-series, under no-arbitrage conditions, the equity risk premium is a function of risk neutral return moments:*

$$RP_{t,T_1,T_N} \equiv \mathbb{E}_t\left(R_{M,t,T_1} - R_{f,t,T_1}\right) = \frac{\frac{a_{1,t}}{R_{f,t,T_1}}\mathbb{M}^{*(2)}_{t,T_1} + \frac{a_{2,t}}{R^2_{f,t,T_1}}\mathbb{M}^{*(3)}_{t,T_1} + \frac{a_{2,t}}{R^2_{f,T_1,T_N}}\mathbb{LEV}^*_{t,T_1,T_N}}{1 + \frac{a_{2,t}}{R^2_{f,t,T_1}}\mathbb{M}^{*(2)}_{t,T_1} + \frac{a_{2,t}}{R^2_{f,T_1,T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1,T_N}}, \tag{9}$$

where

$$\mathbb{LEV}^*_{t,T_1,T_N} = \mathbb{COV}^*_t\left(R_{M,t,T_1} - R_{f,t,T_1}, \mathbb{M}^{*(2)}_{T_1,T_N}\right) \quad \textit{and} \tag{10}$$

$$\mathbb{M}^{*(2)}_{T_1,T_N} = \mathbb{E}^*_1\left(R_{M,T_1,T_N} - R_{f,T_1,T_N}\right)^2. \tag{11}$$

Proof. See Section I.D of the Internet Appendix. □

Two new terms contribute to the equity risk premium in a two-period economy, compared to a one-period economy: the risk-neutral leverage effect $\mathbb{LEV}^*_{t,T_1,T_N}$ and the expected future variance $\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1,T_N}$.[6] These two terms

[6] There is a vast literature on leverage under the physical measure. For example, Hu et al. (2022) show that it is positively related to the variance risk premium. But to our knowledge, this paper is the first to show the relevance of the risk-neutral leverage for computing the equity risk premium.

capture the effect of intertemporal hedging on the equity risk premium.[7] Our conjecture—which will be verified in the data—is that the risk-neutral leverage effect, $\mathbb{LEV}^*_{t,T_1,T_N}$, is negative. The effect of intertemporal hedging on equity risk premium thus depends on the sign of $a_{2,t}$, that is, on whether ρ_t is greater than 1 or not. If $a_{2,t}$ is negative, then intertemporal hedging increases the numerator and decreases the denominator, increasing the resulting equity risk premium. Vice versa, if $a_{2,t}$ is positive, intertemporal hedging decreases the equity risk premium.

In the general setting with multiple rebalancing dates T_j between T_1 and T_N, the result in Proposition 1 naturally extends to:[8]

$$RP_{t,T_1,T_N} = \frac{\frac{a_{1,t}}{R_{f,t,T_1}}\mathbb{M}^{*(2)}_{t,T_1} + \frac{a_{2,t}}{R^2_{f,t,T_1}}\mathbb{M}^{*(3)}_{t,T_1} + a_{2,t}\sum_{j>1}^{N}\frac{1}{R^2_{f,T_{j-1},T_j}}\mathbb{LEV}^*_{t,T_{j-1},T_j}}{1 + \frac{a_{2,t}}{R^2_{f,t,T_1}}\mathbb{M}^{*(2)}_{t,T_1} + a_{2,t}\sum_{j>1}^{N}\frac{1}{R^2_{f,T_{j-1},T_j}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_{j-1},T_j}} \tag{12}$$

where

$$\mathbb{LEV}^*_{t,T_{j-1},T_j} = \mathbb{COV}^*_t\left(R_{M,t,T_1} - R_{f,t,T_1}, \mathbb{M}^{*(2)}_{T_{j-1},T_j}\right) \text{ and} \tag{13}$$

$$\mathbb{M}^{*(2)}_{T_{j-1},T_j} = \mathbb{E}^*_{T_{j-1}}\left(R_{M,T_{j-1},T_j} - R_{f,T_{j-1},T_j}\right)^2, \text{ with } 1 < j \leq N. \tag{14}$$

As in the single rebalancing setting, intertemporal hedging generates two new components: (i) a leverage component that is the sum of leverage effects between returns up to T_1 and return variances over each interval $[T_{j-1}, T_j]$, and (ii) the sum of expected future variances over each $[T_{j-1}, T_j]$. Our equity risk premium thus captures the entire term structure of market risk up to the investment horizon T_N.

[7] In the Internet Appendix VI, we extend our analysis to a third-order approximation. This extension highlights the role of higher-order leverage terms in the contribution of intertemporal hedging to the equity risk premium.

[8] Our measure of equity risk premium in a model with multiple rebalancing dates is derived in Section I.E of the Internet Appendix.

C. *Discussion on model assumptions*

Our model relies on the key assumption that the representative agent allocates all of their wealth between the aggregate market portfolio and the risk-free asset. This assumption follows from Martin (2017) and Chabi-Yo and Loudis (2020). In contrast, Tetlock et al. (2024) and Crescini et al. (2025) allow investors to trade derivative securities that mature at T_1 in addition to the market portfolio, making higher-order moments of market returns directly tradable.

In Section I.H of the Internet Appendix, we show that our model, with a single rebalancing date at T_1, is equivalent to a setting in which the representative agent can trade options maturing at both T_1 and T_N, and liquidates the longer-dated contracts at T_1.

To clarify this intuition, consider first an agent who does not have access to options maturing at T_N. In this setting, a static framework suffices and collapses to the environment of Chabi-Yo and Loudis (2020). In Section I.F of the Internet Appendix, we show that their model nests the framework of Tetlock et al. (2024)[9] and Crescini et al. (2025)[10]. Specifically, the inverse SDF derived in Chabi-Yo and Loudis (2020) can be expressed as the excess return on a portfolio that combines the index with options maturing at T_1, mirroring the construction in these derivative-based models.[11] The relation between the loadings $a_{k,t}$ of Chabi-Yo and Loudis (2020)—and the coefficients in our equation (5)—and the optimal portfolio weights on the market and options maturing at T_1 is uniquely determined by preferences and the

[9]Under logarithmic utility, the inverse SDF is linear in the excess returns on derivative securities, with coefficients proportional to the weights of the growth-optimal portfolio. Tetlock et al. (2024) allow a subset of investors to deviate from log utility; these agents do not alter the functional form of the SDF but affect the number of higher-order moments that enter it.

[10]Crescini et al. (2025) extend the representative-agent analysis by conducting a granular, investor-level study of SDF heterogeneity based on portfolio holdings.

[11]See Section I.G of the Internet Appendix for the derivation of this portfolio.

moments of market returns.

Allowing the agent to trade additional options that mature at T_N, and are sold at T_1, requires a dynamic framework that captures the evolution of state variables between T_1 and T_N. In Section I.H of the Internet Appendix, we show that our inverse SDF can be written as a weighted sum of (i) the excess market return, (ii) the payoffs on call and put options expiring at T_1, and (iii) the payoffs from trading strategies that purchase long-dated options expiring at T_N and liquidate them at T_1. The third component of the inverse SDF captures the investor's hedging demand: the investor wants exposure between t and T_1 to shocks in investment opportunities after T_1, so they take positions whose mark-to-market at T_1 loads on those post-T_1 state variables.

Similarly, our model with multiple rebalancing dates at $T_1 \leq T_j \leq T_N$ is equivalent to a setting in which the representative agent invests in the market, the risk-free asset and the full term structure of options maturing at all rebalancing dates, liquidating all option contracts maturing at $T_j > T_1$ at time T_1. These strategies arise endogenously as intertemporal hedging demands: the agent uses long-dated options to hedge anticipated future shifts in economic conditions. The new "future variance" and "risk-neutral leverage" terms that arise in our equity risk premium estimate therefore have a direct economic meaning: they price the compensation required to hold short-horizon market exposure that covaries with shocks to post-T_1 investment opportunities. As in the static case, the loadings on these dynamic hedging strategies—captured by the coefficients $a_{k,t}$ in our equation (5)—are jointly determined by the agent's preferences and by the moments of the market return distribution.

II. Restrictions on preference parameter values

A strength of our framework is that it does not rely on a given specification of the representative agent's utility function. Instead, our equity risk premium depends on the preference parameters of the representative agent

τ_t and ρ_t. In this section, we discuss the range of values taken by these parameters when a specific utility function is imposed. We discuss several specifications and show that the effect of intertemporal hedging on the equity risk premium crucially depends on the sign of $\rho_t - 1$. We discuss the link between this sign and theories on investors' attitude towards higher-order risks.

A. Example 1. Logarithmic utility

The equity risk premium bound introduced by Martin (2017) corresponds to the case when the representative agent is endowed with a myopic logarithmic utility, i.e., has preference parameters equal to $\tau_t = 1$ ($a_{1,t} = 1$) and $\rho_t = 1$ ($a_{2,t} = 0$). Under this specification, higher-order moments and the risk-neutral leverage term in equation (9) become irrelevant, and (9) reduces to Martin's lower bound.

Tetlock et al. (2024) and Crescini et al. (2025) extend the framework of Martin (2017), and let investors trade derivative securities. The inverse SDF can thus be written as the excess returns on a portfolio of index and options with maturity T_1.[12] Higher order moments become relevant, but not intertemporal hedging as logarithmic utility delivers no intertemporal hedging demand, see Giovannini and Weil (1989). The main novelty of our paper is that our SDF reflects intertemporal hedging; it nests the SDF used in these papers as special case, when $T_1 = T_N$.

B. Example 2. HARA utility

Let the representative agent be endowed with HARA utility:

$$u[x] = \frac{\gamma}{1-\gamma}\left(\frac{ax}{\gamma} + b\right)^{1-\gamma}, \text{ with } \ a > 0 \ \text{ and } \ \frac{ax}{\gamma} + b > 0. \qquad (15)$$

The risk aversion and skewness tolerance parameters can be written as

[12] See Sections I.F and I.G of the Internet Appendix for exact representations of the SDF.

$$\frac{1}{\tau_t} = \frac{aW_t R_{f,t,T_1}}{\frac{aW_t R_{f,t,T_1}}{\gamma} + b} \quad \text{and} \quad \rho_t = \frac{1}{2}\frac{(1+\gamma)}{\gamma}. \tag{16}$$

The parameter γ governs the risk aversion by controlling how rapidly marginal utility declines with aggregate wealth. The parameter a is a scale factor capturing the rate at which absolute risk aversion decreases with wealth, while b shifts the risk-tolerance schedule, introducing an offset wealth effect.

Under the additional assumption of expected returns following a mean-reverting process, Kim and Omberg (1996) underline the role of γ in determining investors' hedging behaviour: if $\gamma > 1$, they speculate on changes in the investment opportunity set, if $0 < \gamma < 1$, they hedge against such changes. Equation (16) shows that γ also fully determines the skewness tolerance ρ_t. If $0 < \gamma < 1$, then $\rho_t > 1$, in which case intertemporal hedging increases the equity risk premium. In turn, if $\gamma > 1$, then $\rho_t < 1$ and intertemporal hedging decreases the equity risk premium.

The HARA utility nests the CARA and CRRA utility as special cases. CRRA utility is obtained by setting $b\gamma = 0$ in (16), leading to $\frac{1}{\tau_t} = \gamma$ where γ is now the relative risk aversion. In the Internet Appendix of his paper, Martin (2017) derives the equity risk premium when the representative agent is endowed with CRRA utility and maximizes expected wealth up to time T_1. Crescini et al. (2025) also investigate these preferences as an extension of their baseline setup and derive the risk aversion level that would be needed to match the equity risk premium of survey data. However, the models in these papers do not feature intertemporal hedging.

Because of the equity premium puzzle, models with CRRA preferences often imply high levels of risk aversion to reconcile aggregate consumption patterns with asset market data. Using values of $\gamma > 1$, Brandt (1999), Campbell and Viceira (1999), Balduzzi and Lynch (1999) and Campbell et al. (2004) show that the estimated time variation in U.S. risk premia generates large positive intertemporal hedging demand, which accounts for above a third of the optimal demand for equity. Campbell and Viceira (1999) and Campbell et al. (2004) also test a specification with $\gamma < 1$ and show that it leads to large negative hedging demand.

CARA utility with absolute risk aversion $\widetilde{\alpha} = \frac{a}{b}$ is obtained when $|b(1-\gamma)| \to \infty$. The risk aversion and skewness tolerance parameters simplify to

$$\frac{1}{\tau_t} = \widetilde{\alpha} W_t R_{f,t,T_N} \quad \rho_t = \frac{1}{2}. \tag{17}$$

Under CARA utility, intertemporal hedging therefore always decreases the equity risk premium.

C. *Example 3. Epstein-Zin utility*

When using the recursive utility of Epstein and Zin (1989), the effect of intertemporal hedging on the equity risk premium is less straightforward. Applying a Taylor-series expansion to the marginal utility helps better understand this effect,[13] by making explicit the contribution of the risk-neutral variance to the SDF. In particular, the resulting SDF includes both the variance of the returns between t and T_1 and the T_1-conditional variance of the returns between T_1 and T_N. The second effect arises due to intertemporal hedging, and shows that the effect of intertemporal hedging on the SDF depends on the parameter $\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$, where γ is the relative risk aversion and ψ is the elasticity of intertemporal substitution. If θ is between 0 and 1, then the inverse SDF decreases with $\mathbb{VAR}^*_{T_1}(R_{M,T_1,T_N})$. As a result, assuming a negative leverage effect, intertemporal hedging increases the equity risk premium. But if θ is outside $[0,1]$, intertemporal hedging decreases the equity risk premium.

The literature using Epstein-Zin preferences has commonly assumed $\theta(\theta-1) > 0$.[14] Under this assumption, intertemporal hedging reduces the equity risk premium. Chacko and Viceira (2005) find that the effect of intertemporal

[13]Derivations are given in Section II.A of the Internet Appendix.

[14]Under the additional assumption that expected returns are mean-reverting, Campbell and Viceira (1999) and Campbell et al. (2004) show that the sign of intertemporal hedging demand depends solely on γ and not anymore on ψ. It is negative for $\gamma < 1$ and positive otherwise.

hedging on asset demand is small, when $\gamma > 1$ and $\psi < 1$. In contrast, Brandt (1999), Campbell and Viceira (1999, 2002); Campbell et al. (2003, 2004), and others show that the estimated time variation in U.S. risk premia generates a large intertemporal hedging demand, which accounts for up to half of the optimal portfolio rule.

In the habit-formation model of Campbell and Cochrane (1999), which additionally assumes that consumption growth can be decomposed into the change in the surplus-consumption ratio and the aggregate wealth return, the impact of intertemporal hedging on the equity risk premium solely depends on the coefficient of relative risk aversion γ, and no longer on the elasticity of intertemporal substitution ψ.[15] Similarly to the case of standard CRRA preferences, intertemporal hedging reduces the equity risk premium when $\gamma > 1$, and increases it when $\gamma < 1$.

Imposing a specific functional form on the utility function inevitably constrains how intertemporal hedging contributes to the equity risk premium. Under CRRA and Epstein-Zin preferences, and in related habit formation models, standard parameterizations imply that intertemporal hedging reduces the equity premium.

Let us consider the general recursive preference framework of Epstein-Zin in which the aggregator is written as

$$\vartheta\left[U_t\right] = (1-\beta)\, g\left[C_t\right] + \beta\vartheta\left[\mu_t\right] \tag{18}$$

where the certainty equivalent is given by

$$\mu_t = \upsilon^{-1}\left[\mathbb{E}_t \upsilon\left[U_{t+1}\right]\right]. \tag{19}$$

Epstein and Zin chose the following functional forms:

$$\vartheta\left[x\right] = g\left[x\right] = x^{1-\frac{1}{\psi}}, \text{ and } \upsilon\left[x\right] = x^{1-\gamma}. \tag{20}$$

Applying a second order Taylor expansion to $\frac{U_{T_N}}{\mu_t}$ reveals that the power

[15] The proof of this claim is in Section II.B of the Internet Appendix.

functionals imposed on the aggregator and on the certainty equivalent restrict the effect of intertemporal hedging to solely depend on the two parameters γ and ψ.[16] Epstein and Zin chose these power specifications for tractability, but a consequence of this choice is that with common values of γ and ψ, intertemporal hedging reduces the equity risk premium. In the general recursive case, we show in the Appendix that the influence of intertemporal hedging depends on four preference parameters that capture the curvature of both the aggregator and the certainty equivalent. These four parameters provide more flexibility to the relation between intertemporal hedging and equity risk premium, which is no longer constrained to be negative.

An alternative way to illustrate the restrictions imposed by the Epstein–Zin framework is to examine first-order departures from the power functional forms of the aggregator and the certainty equivalent. Such first-order deviations introduce two additional parameters.[17] Depending on their values, the effect of intertemporal hedging on the equity risk premium can be either positive or negative.

An advantage of our approach is that it avoids restricting the analysis to any particular functional form. We do not take a stand on the investor's utility function. However, our results still depend on the underlying preference parameters τ_t and ρ_t. The central question therefore is how these values should be chosen in a meaningful way and whether ρ_t should be below or above 1. The answer to this question determines whether intertemporal hedging decreases or increases the equity risk premium.

D. *Sign of $\rho_t - 1$: Theory*

If one believes in the HARA-type of preferences, then common parameter choices imply that $\rho_t < 1$, i.e., intertemporal hedging decreases the equity risk premium. We now give economic arguments in favor of $\rho_t > 1$, which leads to intertemporal hedging increasing the equity risk premium.

[16] See derivations in Section II.C of the Internet Appendix.

[17] See derivations in Section II.D of the Internet Appendix.

First, when $\rho_t > 1$, it follows that both $\frac{a_{2,t}}{R^2_{f,t,T_1}} < 0$ and $\frac{a_{2,t}}{R^2_{f,T_1,T_N}} < 0$, implying that the inverse SDF in equation (4) is a *decreasing* function of market volatility. Consequently, as market volatility increases, so does the SDF. This, in turn, implies that the price of co-skewness risk is negative. Investors then require a positive premium for holding stocks with negative co-skewness, in line with Rubinstein (1973), Kraus and Litzenberger (1976), Harvey and Siddique (2000) and Langlois (2020).

Similarly, as periods of higher volatility are associated with lower realized market returns, this relation implies that the SDF tends to increase when realized returns decline. In other words, the marginal utility increases when the return on aggregate wealth decreases[18], which is consistent with the empirical evidence reported by Christoffersen et al. (2013), and with the large literature on the variance risk premium.[19]

Second, the skewness tolerance parameter can be written as a function of prudence: $\rho_t = \frac{1}{2}$ (prudence/risk aversion), see, e.g., Eeckhoudt and Schlesinger (2006); Deck and Schlesinger (2014); Noussair et al. (2014). Thus, $\rho_t > 1$ implies that prudence is greater than twice the risk aversion. Gollier (2001) and earlier papers (for example, Gollier and Pratt (1996)) explicitly highlight that "reasonable" utility functions that satisfy decreasing absolute risk aversion (DARA) typically imply this inequality. It is often interpreted as a condition for precautionary saving to be quantitatively important, since small risks have a first-order effect on optimal saving decisions (Kimball (1990)'s result). Households save not only because of risk aversion but also because prudence magnifies the response to future uncertainty.

Finally, condition $\rho_t > 1$ is empirically supported. In a different context, Chabi-Yo (2012) applies Judd's small-noise expansion approach without imposing any restriction on the functional form of the utility. Assuming that

[18] We assume that the market return is a suitable proxy for the return on aggregate wealth.

[19] See, among others, Bollerslev et al. (2009); Carr and Wu (2009); Bollerslev and Todorov (2011); Filipović et al. (2016); Dew-Becker et al. (2017); Bardgett et al. (2019); Ait-Sahalia et al. (2020).

ρ_t is constant over time, he finds estimated values of ρ_t that are significantly greater than one.

E. Equity risk premium bounds

We now explore a set of assumptions under which our equity risk premium in (9) is a lower bound for expected future market returns. One of these assumptions requires $\rho_t > 1$.

ASSUMPTION 1: *For $k \geq 1$ and $0 \leq m \leq k$,*

$$\mathbb{E}_t^*\left(\left(R_{M,t,T_1} - R_{f,t,T_1}\right)^{m+1} \mathbb{E}_{T_1}^*\left(R_{M,T_1,T_N} - R_{f,T_1,T_N}\right)^{k-m}\right) > 0 \quad (\textit{resp.} < 0)$$
$$\Rightarrow \quad \mathbb{E}_t^*\left(\left(R_{M,t,T_1} - R_{f,t,T_1}\right)^{m} \mathbb{E}_{T_1}^*\left(R_{M,T_1,T_N} - R_{f,T_1,T_N}\right)^{k-m}\right) < 0 \quad (\textit{resp.} > 0). \tag{21}$$

The assumption 1 generalizes the usual assumption obtained for $k = m$ that odd risk-neutral moments are negative, while even moments are positive, e.g., Chabi-Yo and Loudis (2020). There is wide empirical support for this assumption at low orders.

To formalize the second assumption, let us consider the priced risk factors

$$\left(R_{M,t,T_1} - R_{f,t,T_1}\right)^{m} \mathbb{E}_{T_1}^*\left(R_{M,T_1,T_N} - R_{f,T_1,T_N}\right)^{k-m}. \tag{22}$$

ASSUMPTION 2: *The risk premia associated with the risk factors in (22) are positive.*

Assumption 2 implies, for $k = m = 1$, that $\theta_{1,1} = \frac{a_{1,t}}{R_{f,t,T_1}} \geq 0$, as the risk-neutral variance is positive. This condition leads to $1/\tau_t > 0$, in line with the representative agent being risk averse. For $k = m = 2$, Assumption 2 implies that $\theta_{2,2} = \frac{a_{2,t}}{R^2_{f,t,T_1}} \leq 0$ when the risk-neutral skewness is negative. This condition leads to $\rho_t \geq 1$. Similarly, for $m = 0$ and $k = 2$, Assumption 2 implies that $\theta_{2,0} = \frac{a_{2,t}}{R^2_{f,T_1,T_N}} \leq 0$ when the risk-neutral leverage $\mathbb{LEV}^*_{t,T_1,T_N}$ is negative. This condition also implies that $\rho_t \geq 1$. Assumption 2 is therefore inconsistent with HARA-types of specifications under $\gamma > 1$. It is in line with economic theories on the pricing of higher order risks, as detailed in

Section II.D.

Under Assumptions 1-2, our equity risk premium measure (9) is a lower bound.[20] Furthermore, intertemporal hedging contributes positively to the equity risk premium. This increase captures the compensation required by investors for exposure to the future risk-neutral variance. Importantly, under minimal realistic assumptions, our bound remains a lower bound for the equity risk premium when adding consumption to the representative agent problem.[21]

Finally, under two additional assumptions, we derive a restricted bound that does not rely on preference parameters.

ASSUMPTION 3: *Odd market risk neutral moments are negative.*

This assumption is consistent with the empirical evidence that the risk-neutral skewness is negative.

ASSUMPTION 4: *The preference parameters satisfy* $1/\tau_t \geq 1$ *and* $\rho_t \geq 2$.

The assumption 4 is more restrictive than the assumption 2, and rules out cases $0 < \tau_t < 1$ and $1 \leq \rho_t < 2$. While the first condition on risk aversion is widely supported in the literature, the second condition has been less studied.

Under Assumptions 3 and 4, we can derive the following lower bound for the equity risk premium:[22]

$$RP_{t,T_1,T_N} \geq \frac{\frac{1}{R_{f,t,T_1}}\mathbb{M}^{*(2)}_{t,T_1} - \frac{1}{R^2_{f,t,T_1}}\mathbb{M}^{*(3)}_{t,T_1} - \frac{1}{R^2_{f,T_1,T_N}}\mathbb{LEV}^*_{t,T_1,T_N}}{1 - \frac{1}{R^2_{f,t,T_1}}\mathbb{M}^{*(2)}_{t,T_1} - \frac{1}{R^2_{f,T_1,T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1,T_N}}. \tag{23}$$

The main appeal of this new bound is that it does not involve any preference parameter and is thus straightforward to compute.

[20]See proof in Section II.E of the Internet Appendix.

[21]We prove this result in Section II.F of the Internet Appendix.

[22]See the proof in Section II.E of the Internet Appendix.

F. *Quantifying the effect of intertemporal hedging*

To quantify the effect of intertemporal hedging on the equity risk premium, we define the intertemporal hedging premium as the difference between the equity risk premium from t to T_1 in our multi-period economy and the premium in a one-period (two-date) economy.

DEFINITION 1: *Let us define the intertemporal hedging premium as*

$$IHP_{t,T_1,T_N} = RP_{t,T_1,T_N} - RP_{t,T_1}. \tag{24}$$

Under Assumptions 1–2, the intertemporal hedging premium IHP_{t,T_1,T_N} is positive, i.e., our risk premium, RP_{t,T_1,T_N}, is higher than RP_{t,T_1}. If Assumption 2 does not hold and $\rho_t < 1$, then the intertemporal hedging premium IHP_{t,T_1,T_N} is negative.

III. Physical variance and crash probability

Our framework can be used to write other physical moments of market returns as functions of their risk-neutral moments. In this Section, we derive the physical variance and the probability of a market crash in a multi-period economy.

A. *Physical variance*

The conditional variance of market returns under the physical measure is given by the following proposition, in the model with a single rebalancing date T_1.

PROPOSITION 2: *Up to a second-order expansion-series, under no-arbitrage conditions, the conditional variance of returns under the physical measure is a function of risk neutral return moments:*

$$\begin{aligned}\mathbb{V}ar_t &\equiv \mathbb{E}_t\left(R_{M,t,T_1} - \mathbb{E}_t R_{M,t,T_1}\right)^2 \\ &= \mathbb{E}_t\left(R_{M,t,T_1} - R_{f,t,T_1}\right)^2 - \left(\mathbb{E}_t\left(R_{M,t,T_1} - R_{f,t,T_1}\right)\right)^2 \end{aligned} \tag{25}$$

where $\mathbb{E}_t\left(R_{M,t,T_1} - R_{f,t,T_1}\right)$ *is given by Equation* (9) *and*

$$\mathbb{E}_t\left(R_{M,t,T_1} - R_{f,t,T_1}\right)^2 = \frac{\left\{\begin{array}{c} \mathbb{M}^{*(2)}_{t,T_1} + \frac{a_{1,t}}{R_{f,t,T_1}}\mathbb{M}^{*(3)}_{t,T_1} + \frac{a_{2,t}}{R^2_{f,t,T_1}}\mathbb{M}^{*(4)}_{t,T_1} \\ +\frac{a_{2,t}}{R^2_{f,T_1,T_N}}\mathbb{E}^*_t\left((R_{M,t,T_1} - R_{f,t,T_1})^2\mathbb{M}^{*(2)}_{T_1,T_N}\right) \end{array}\right\}}{1 + \frac{a_{2,t}}{R^2_{f,t,T_1}}\mathbb{M}^{*(2)}_{t,T_1} + \frac{a_{2,t}}{R^2_{f,T_1,T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1,T_N}}. \tag{26}$$

Proof. The result follows from applying the second-order approximation of Equation (4) and substituting it into

$$\mathbb{E}_t\left(R_{M,t,T_1} - R_{f,t,T_1}\right)^2 = \mathbb{E}^*_t\left(\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}}\left(R_{M,t,T_1} - R_{f,t,T_1}\right)^2\right). \tag{27}$$

□

Analogous to the equity risk premium, the conditional variance can be expressed as a function of risk-neutral moments from time t to the forecast horizon T_1. Additionally, it incorporates intertemporal hedging terms, which capture expected changes in state variables between T_1 and T_N.

This estimate of the physical variance presents two key advantages. First, it can be readily computed using available options, and as a result, so can the variance risk premium. This feature provides a comparable advantage over computations that rely on high-frequency data, as in Tetlock et al. (2024) for example. Second, it is model-free and requires minimal assumptions, akin to our measure of the equity risk premium.

Allowing for multiple rebalancing times between T_1 and T_N yields the following measure of physical variance:

$$\mathbb{E}_t\left(R_{M,t,T_1} - R_{f,t,T_1}\right)^2 = \frac{\left\{\begin{array}{c} \mathbb{M}^{*(2)}_{t,T_1} + \frac{a_{1,t}}{R_{f,t,T_1}}\mathbb{M}^{*(3)}_{t,T_1} + \frac{a_{2,t}}{R^2_{f,t,T_1}}\mathbb{M}^{*(4)}_{t,T_1} \\ +a_{2,t}\sum_{j>1}^{N}\frac{1}{R^2_{f,T_1,T_N}}\mathbb{E}^*_t\left((R_{M,t,T_1} - R_{f,t,T_1})^2\mathbb{M}^{*(2)}_{T_{j-1},T_j}\right) \end{array}\right\}}{1 + \frac{a_{2,t}}{R^2_{f,t,T_1}}\mathbb{M}^{*(2)}_{t,T_1} + a_{2,t}\sum_{j>1}^{N}\frac{1}{R^2_{f,T_{j-1},T_j}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_{j-1},T_j}}. \tag{28}$$

The physical variance estimate is now a function of spot return moments as well as expected future return moments up to T_N.

B. Probability of a market crash

We finally use our methodology to obtain the probability of a market crash under the physical measure, in a multi-period economy. We define the probability of a crash as $\mathbb{P}_t\left(R_{M,t,T_1} < \alpha\right)$ where α is given. For example, $\alpha = 0.8$ for a 20% market crash.

Proposition 3 gives our estimate of the probability of a crash in the case of a single rebalancing time.

PROPOSITION 3: *Up to a second-order expansion-series of the inverse marginal utilities, the conditional crash probability defined as* $\Pi_{t,T_1,T_N}[\alpha] \equiv P_t\left(R_{M,t,T_1} < \alpha\right)$ *can be expressed in terms of risk neutral quantities*

$$\Pi_{t,T_1,T_N}[\alpha] = \frac{\left\{\begin{array}{c} \mathbb{M}^{*(0)}_{t,T_1}[\alpha] + \frac{a_{1,t}}{R_{f,t,T_1}}\mathbb{M}^{*(1)}_{t,T_1}[\alpha] + \frac{a_{2,t}}{R^2_{f,t,T_1}}\mathbb{M}^{*(2)}_{t,T_1}[\alpha] \\ + \frac{a_{2,t}}{R^2_{f,T_1,T_N}}\mathbb{E}^*_t\left(\mathbb{M}^{*(2)}_{T_1,T_N}\mathbb{1}_{R_{M,t,T_1}<\alpha}\right) \end{array}\right\}}{1 + \frac{a_{2,t}}{R^2_{f,t,T_1}}\mathbb{M}^{*(2)}_{t,T_1} + \frac{a_{2,t}}{R^2_{f,T_1,T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1,T_N}}, \tag{29}$$

where $\mathbb{M}^{*(n)}_{t,T_1}[\alpha] = \mathbb{E}^*_t\left(\left(R_{M,t,T_1} - R_{f,t,T_1}\right)^n \mathbb{1}_{R_{M,t,T_1}<\alpha}\right)$*, for* $n \geq 0$.

Proof. The result follows from applying the second-order approximation of Equation (4) and substituting it into

$$\mathbb{P}_t\left(R_{M,t,T_1} < \alpha\right) = \mathbb{E}^*_t\left(\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}}\mathbb{1}_{R_{M,t,T_1}<\alpha}\right). \tag{30}$$

□

The probability of a market crash can, therefore, be estimated as a function of truncated market moments. Intertemporal hedging motives are captured by the expected truncated future variance over $[T_1, T_N]$, $\mathbb{E}^*_t\left(\mathbb{M}^{*(2)}_{T_1,T_N}\mathbb{1}_{R_{M,t,T_1}<\alpha}\right)$.

In the general case with multiple rebalancing times, Equation (29) extends to

$$\Pi_{t,T_1,T_N}[\alpha] = \frac{\left\{\begin{array}{c} \mathbb{M}^{*(0)}_{t,T_1}[\alpha] + \frac{a_{1,t}}{R_{f,t,T_1}}\mathbb{M}^{*(1)}_{t,T_1}[\alpha] + \frac{a_{2,t}}{R^2_{f,t,T_1}}\mathbb{M}^{*(2)}_{t,T_1}[\alpha] \\ +a_{2,t}\sum_{j>1}^{N}\frac{1}{R^2_{f,T_1,T_N}}\mathbb{E}^*_t\left(\mathbb{M}^{*(2)}_{T_{j-1},T_j}\mathbb{1}_{R_{M,t,T_1}<\alpha}\right) \end{array}\right\}}{1 + \frac{a_{2,t}}{R^2_{f,t,T_1}}\mathbb{M}^{*(2)}_{t,T_1} + a_{2,t}\sum_{j>1}^{N}\frac{1}{R^2_{f,T_{j-1},T_j}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_{j-1},T_j}}. \tag{31}$$

The probability of a crash depends on the complete set of truncated return moments that are available up to the horizon T_N. As with the equity risk premium and physical variance, a multi-period setting implies that the entire term structure of risk up to the investor's horizon T_N influences the probability of a return crash over the interval $[t, T_1]$.

IV. Empirical framework

In this section, we show how the theoretical expressions derived in Sections I and III for the equity risk premium, physical variance and probability of a crash can be brought to the data.

A. *Computation of moments and expected future moments*

Closed-form expressions of risk-neutral moments $\mathbb{M}^{*(k)}_{t,T_1}$, for a given horizon T_1, are readily available from option prices using the spanning formula of Carr and Madan (2001) and Bakshi and Madan (2000).[23] But similar closed-form expressions are not directly available for expected future moments $\mathbb{E}^*_t\left((R_{M,t,T_1} - R_{f,t,T_1})^k\mathbb{M}^{*(2)}_{T_1,T_N}\right)$. We propose a method to compute these moments from available option prices.

Our method is based on the intuition that $\mathbb{M}^{*(2)}_{T_1,T_N}$ should be a function of S_{T_1}. Using the spanning formula of Carr and Madan (2001) and Bakshi and Madan (2000), the future risk-neutral variance between T_1 and T_N indeed admits a static replication in terms of option prices at time T_1. Each option price depends on the underlying level S_{T_1}, and on the implied variance at the

[23]See Appendix III.A.

corresponding log-moneyness. Under an arbitrage-free parameterization of the implied volatility surface—such as, for example, the raw SVI specification of Gatheral and Jacquier (2014)—the entire surface is summarized by a finite set of parameters, so that option prices, and hence the replicated risk-neutral variances, can be written as functions of S_{T_1} (conditional on the volatility surface parameters).[24] For tractability, we use the excess return $R_{M,t,T_1} - R_{f,t,T_1}$ in place of S_{T_1}, over $[t, T_1]$, and write $\mathbb{M}^{*(2)}_{T_1,T_N}$ as a non-linear function f of $R_{M,t,T_1} - R_{f,t,T_1}$:

$$\mathbb{M}^{*(2)}_{T_1,T_N} = \theta_{T_1,T_N} f[R_{M,t,T_1} - R_{f,t,T_1}] + \epsilon_{T_1}, \tag{32}$$

with $\mathbb{E}^*_t\left(\epsilon_{T_1}|R_{M,t,T_1}\right) = \mathbb{E}^*_t\left(\epsilon_{T_1}\right) = 0$.

Multiplying both sides of Equation (32) by R^2_{M,t,T_1} and taking the time-t risk-neutral expectation, we obtain

$$\theta_{T_1,T_N} = \frac{\mathbb{M}^{*(2)}_{t,T_N} - R^2_{f,T_1,T_N}\mathbb{M}^{*(2)}_{t,T_1}}{\mathbb{E}^*_t\left(R^2_{M,t,T_1} f[R_{M,t,T_1} - R_{f,t,T_1}]\right)}. \tag{33}$$

The final step consists in choosing the function $f[\cdot]$ used in the projection (32). We use $(R_{M,t,T_1} - R_{f,t,T_1})^2$ for two reasons.[25] First, the numerator of θ_{T_1,T_N} is always positive in the data. Therefore, our choice of function $f[\cdot]$ ensures that the expected future variance is a positive number. Second, as $(R_{M,t,T_1} - R_{f,t,T_1})^2$ is a proxy for the first period conditional variance, this function captures the well-documented fact that conditional variances are highly positively correlated over time.

The expected future moments $\mathbb{E}^*_t\left((R_{M,t,T_1} - R_{f,t,T_1})^k \mathbb{M}^{*(2)}_{T_1,T_N}\right)$ are then obtained by combining Equations (32) and (33). For all $k \geq 0$,

$$\mathbb{E}^*_t\left((R_{M,t,T_1} - R_{f,t,T_1})^k \mathbb{M}^{*(2)}_{T_1,T_N}\right) = \theta_{T_1,T_N}\mathbb{M}^{*(k+2)}_{t,T_1} \tag{34}$$

[24]We show in Section III.B of the Internet Appendix that this representation holds under minor assumptions.

[25]We show in Section III.C of the Internet Appendix that the risk-neutral volatility dynamics implied by (32), with this choice of f, is distinct from that of a GARCH process.

where θ_{T_1,T_N} is a function of risk-neutral moments with maturity equal to the investment horizon, $\mathbb{M}^{*(k)}_{t,T_N}$:

$$\theta_{T_1,T_N} = \frac{\mathbb{M}^{*(2)}_{t,T_N} - R^2_{f,T_1,T_N}\mathbb{M}^{*(2)}_{t,T_1}}{\mathbb{M}^{*(4)}_{t,T_1} + 2R_{f,t,T_1}\mathbb{M}^{*(3)}_{t,T_1} + R^2_{f,t,T_1}\mathbb{M}^{*(2)}_{t,T_1}}. \tag{35}$$

For $k = 0$, we recover the expected future risk-neutral variance $\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1,T_N}$, for $k = 1$ the risk-neutral leverage $\mathbb{LEV}^*_{t,T_1,T_N}$, and for $k = 2$ the intertemporal component of the physical variance, $\mathbb{E}^*_t\left((R_{M,t,T_1} - R_{f,t,T_1})^2\mathbb{M}^{*(2)}_{T_1,T_N}\right)$:

$$\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1,T_N} = \theta_{T_1,T_N}\mathbb{M}^{*(2)}_{t,T_1}, \tag{36}$$

$$\mathbb{LEV}^*_{t,T_1,T_N} = \theta_{T_1,T_N}\mathbb{M}^{*(3)}_{t,T_1} \quad \text{and} \tag{37}$$

$$\mathbb{E}^*_t\left((R_{M,t,T_1} - R_{f,t,T_1})^2\mathbb{M}^{*(2)}_{T_1,T_N}\right) = \theta_{T_1,T_N}\mathbb{M}^{*(4)}_{t,T_1}. \tag{38}$$

The computation of expected future risk-neutral moments hence relies on information from options of maturities T_1 and T_N. Substituting (36), (37) and (38) in Equations (9) and (26) highlights that our expressions for the equity risk premium and the physical variance are non-linear functions of T_1-return moments and the T_N-return variance.

It is straightforward to extend this method to the calculation of $\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_{j-1},T_j}$, $\mathbb{LEV}^*_{t,T_{j-1},T_j}$ and $\mathbb{E}^*_t\left((R_{M,t,T_1} - R_{f,t,T_1})^2\mathbb{M}^{*(2)}_{T_{j-1},T_j}\right)$, which capture the effect of intertemporal motives on our equity risk premium and physical variance in the general model with multiple rebalancing dates. Using our projection method, these moments can be written as functions of the entire term structure of spot return moments, up to T_N.[26]

Similarly, the probability of a crash (29) depends on truncated risk-neutral moments and expected future moments. The former can be readily computed from options expiring at T_1 and the latter from options expiring between T_1 and T_N.[27]

[26] Derivations are provided in Section III.D of the Internet Appendix.

[27] Derivations are provided in Sections III.A and III.E of the Internet Appendix.

B. Data

We use the S&P 500 index as the market portfolio. We obtain volatility surfaces, index levels, and forward term structures for the S&P 500 Index and the zero-coupon rate term structures from Ivy DB OptionMetrics. The data cover the period from January 1996 to February 2023. When computing the excess returns on the S&P 500 index before January 1996, we use its levels and the Fama term structures on U.S. Treasuries from the Center for Research in Security Prices (CRSP).

Implementing our risk premia requires the evaluation of different functions of risk-neutral moments. We estimate these moments at the end of each week and for each maturity provided in OptionMetrics' Volatility Surface File (10, 30, 60, 91, 122, 152, 182, 273, 365, 547, and 730 days). We refer to these maturities as one week, one month, two months, one quarter, four, five, six, and nine months, one year, 18 months, and two years.

We import annualized continuously-compounded zero-coupon yields from Jing Cynthia Wu's website, Liu and Wu (2021). We interpolate the term structure of zero-coupon rates using the model of Nelson and Siegel (1987) to find the risk-free rate for each maturity.

Following Chabi-Yo et al. (2021), we define a moneyness grid of 1,000 equally spaced points from 1/3 to 3. We use a piecewise cubic Hermite polynomial to interpolate the implied volatility surface on the moneyness grid. We extrapolate the implied volatility using the closest value for moneyness points outside the implied volatility surface. Finally, we use the Black-Scholes formula to convert implied volatilities into call and put prices for each moneyness level.

C. Risk-neutral moments

We report in Figure 1 the risk-neutral moments of excess returns over time for horizons of one to six months. To compare values across horizons, we report the annualized volatility in the top graph $\left(\sqrt{(365/T_1)\,\mathbb{M}^{*(2)}_{t,T_1}}\right)$, skewness in the middle graph $\left(\mathbb{M}^{*(3)}_{t,T_1}/\left(\mathbb{M}^{*(2)}_{t,T_1}\right)^{\frac{3}{2}}\right)$, and kurtosis in the bottom graph $\left(\mathbb{M}_{t,T_1}^{*(4)} / \left(\mathbb{M}_{t,T_1}^{*(2)}\right)^2\right)$.

[FIGURE 1 ABOUT HERE]

In Figure 2, we report the expected future volatility and the leverage effect that arise in the equity risk premium due to intertemporal hedging motives, calculated from Equations (36) and (37).

[FIGURE 2 ABOUT HERE]

Risk-neutral volatilities and expected future volatilities vary over time, reaching a peak during the financial crisis of 2008. Furthermore, they tend to increase with T_1 in times of market calm, and to decrease with T_1 in times of market turmoil. The risk-neutral skewness is almost always negative and tends to decrease over the sample period. The risk-neutral kurtosis, in turn, is positive and trends upward. The risk-neutral leverage effect is negative and tends to decrease with T_1. These time series are consistent with our Assumption 1.

V. Results

In this section, we present our estimates of the equity and variance risk premia obtained within the general framework that allows for multiple rebalancing dates. We treat all available option maturities as rebalancing dates.

The expressions for the one-period equity risk premium and crash probabilities derived in Section I depend on the investor's preference parameters τ_t and ρ_t. In our baseline specification, we assume that these parameters are constant over time. This assumption allows us to compute the equity and variance risk premia in a straightforward manner using readily available option prices.

As discussed in Section II, the parameter ρ_t plays a central role, as it determines whether intertemporal hedging reduces the equity risk premium—as implied by HARA-type utility—or increases it, in line with theories emphasizing the pricing of higher-order risks. In view of this distinction, we examine two alternative parameter configurations.

Our baseline results are obtained by setting the preference parameters to $\tau = 1$ and $\rho = 2$,[28] which implies $a_{1,t} = 1$ and $a_{2,t} = -1$. Under this specification, we show that RP_{t,T_1,T_N}, as defined in Equation (12), delivers stronger return predictability than the existing risk premium measures across all investment horizons T_N.

Alternatively, we consider a parameterization consistent with CRRA utility and a coefficient of relative risk aversion $\gamma = 3$, which implies $\tau = \frac{1}{3}$ and $\rho = \frac{2}{3}$. We show that the baseline specification delivers superior out-of-sample performance relative to this alternative calibration.

A. *Estimated equity risk premium*

Figure 3 reports the time series of equity risk premia for forecast horizons T_1 of one and six months and investment horizons T_N of one and two years.

[FIGURE 3 ABOUT HERE]

Consistent with the model's predictions, RP_{t,T_1,T_N} exceeds RP_{t,T_1} throughout the sample and increases with the investment horizon T_N. The summary statistics in Table I further indicate that the intertemporal hedging premium—defined in (1) as the difference between RP_{t,T_1,T_N} and RP_{t,T_1} —is larger at shorter forecast horizons T_1. This pattern admits two interpretations. First, the intertemporal hedging premium may naturally decline as the forecast horizon lengthens. Alternatively, our estimates may capture only part of the premium, since the maximum investment horizon considered is limited to two years. For some forecast horizons, options with longer maturities may be required to fully account for intertemporal hedging motives.

[TABLE I ABOUT HERE]

[28]For notational simplicity, we omit the time subscript.

We further compare our equity risk premium estimates to the Implied Equity Risk Premium (IERP) of Tetlock et al. (2024), in Figure 4.[29] Using an investment horizon T_N of two years, the two risk premium estimates are close during quiet times. During NBER recessions, the IERP is larger than our premium. The summary statistics in Table IAI of the Internet Appendix confirm and complement these results.

[FIGURE 4 ABOUT HERE]

Figure 5, Panel A, displays the intertemporal hedging premium for a forecast horizon of one month, as a percentage of the total premium RP_{t,T_1,T_N}. The contribution of intertemporal hedging to the equity risk premium exhibits an upward trend over time. Furthermore, it is volatile and depends on the investment horizon T_N. For an investment horizon of one year, intertemporal hedging accounts for between 30 and 60% of the total equity risk premium, while for a two-year horizon, it explains up to 80% of the equity risk premium. These shares are larger during quiet periods. In contrast, during NBER recessions, intertemporal hedging represents roughly 20% of the total premium.

[FIGURE 5 ABOUT HERE]

Panel B indicates that at longer forecast horizons (six months), intertemporal hedging represents a smaller fraction of the total equity risk premium. For an investment horizon T_N of one year, its contribution ranges between 5 and 30%, while for a two-year investment horizon it is nearly twice as large.

Overall, our results indicate that intertemporal hedging accounts for a sizeable share of the equity risk premium, especially during tranquil market conditions.

[29] We thank Paul Tetlock for providing us with the growth optimal weights needed to calculate the IERP, from January 1997 to December 2021. Based on these weights, we computed the IERP for all forecast horizons over this time period. For comparability, all graphs with the IERP are from January 1997 to December 2021.

B. *Conditional variance and variance risk premium*

Figure 6 compares the conditional physical variance obtained in a static model without intertemporal hedging to its counterpart in our dynamic framework, as given by Equation (28). Panel A presents the results for a one-month forecast horizon, and Panel B for a six-month horizon. Across both horizons, the estimated physical variance is systematically lower when intertemporal hedging is incorporated.

[FIGURE 6 ABOUT HERE]

Since the risk-neutral variance is directly inferred from option prices, it does not depend on the investment horizon T_N. As a result, the decline in physical variance induced by intertemporal hedging translates mechanically into a variance risk premium that is larger in absolute value than the premium obtained in the absence of intertemporal hedging.

C. *Out-of-sample performance*

We assess whether incorporating intertemporal hedging improves the out-of-sample forecasting performance of the equity risk premium. Performance is evaluated using two different metrics.

First, we follow Welch and Goyal (2008) and Campbell and Thompson (2008) and compute the out-of-sample R^2 measure as

$$R^2_{OOS} = 1 - \frac{\sum_t \left(r_{M,t,T_1} - \tilde{r}_{M,t,T_1}\right)^2}{\sum_t \left(r_{M,t,T_1} - \bar{r}_{M,t,T_1}\right)^2}, \tag{39}$$

where $r_{M,t,T_1} = R_{M,t,T_1} - R_{f,t,T_1}$ captures the ex-post excess returns realized between t and T_1, $\bar{r}_{M,t,T_1}$ is the sample average of excess returns at horizon T_1 prior to week t and $\tilde{r}_{M,t,T_1}$ is a risk premium forecast at time t. A positive R^2_{OOS} indicates that the prediction $\tilde{r}_{M,t,T_1}$ is more accurate than the past average realized returns, while a negative R^2_{OOS} would favor the past average realized returns.

Panel A of Table II reports the out-of-sample R^2_{OOS} (in percent) obtained using the two bounds $\tilde{r}_{M,t,T_1}$ of Martin (2017) and Chabi-Yo and Loudis

(2020), namely $RP^{Log}_{t,T_1}$and RP_{t,T_1}, as well as our bound RP_{t,T_1,T_N} computed using Equation (12), over the full sample period. Forecast horizons T_1 range from one to twelve months, and we consider all investment horizons T_N up to two years. Comparing columns 2 and 3 (RP^{Log}_{t,T_1} and RP_{t,T_1}) shows that RP_{t,T_1} outperforms RP^{Log}_{t,T_1} for all T_1, indicating that higher order moments are important for out-of-sample performance. Furthermore, RP_{t,T_1,T_N} (columns 4 to 13) outperforms RP_{t,T_1} (column 3) for all investment horizons T_N. For example, for the 10-day forecast horizon, both RP_{t,T_1} and RP^{Log}_{t,T_1} deliver negative R^2_{OOS}, performing worse than a naive forecast based on the historical average return. By contrast, RP_{t,T_1,T_N} yields positive R^2_{OOS} for T_N above one month.

[TABLE II ABOUT HERE]

We test whether the performance differences between RP_{t,T_1} and RP_{t,T_1,T_N} are statistically significant, using the Diebold and Mariano (1995) test. The outperformance of RP_{t,T_1,T_N} is significant for forecast horizons T_1 between two and six months, and for most investment horizons T_N. Overall, these results indicate that incorporating intertemporal hedging into the equity risk premium—under preference parameters $\tau = 1$ and $\rho = 2$—yields a substantial and statistically significant improvement in out-of-sample predictive performance.

Furthermore, the out-of-sample R^2_{OOS} delivered by RP_{t,T_1,T_N} tends to increase with T_N. For short forecast horizons T_1 of 10 days and 1 month, the maximum is attained for $T_N = 18$ months and the R^2_{OOS} slightly declines for T_N at 24 months. But for all other forecast horizons, the R^2_{OOS} reaches its maximum for a T_N of 24 months.

Taken together, these findings suggest that intertemporal hedging motives operating over horizons of up to two years—and potentially longer—play a role in the determination of asset prices and risk premia.

The best overall predictions are obtained using the 24-month investment horizon. The R^2 is about 1.5 that of Chabi-Yo and Loudis (2020). Using a prediction based on the average prediction across investment horizons T_N (last column) also allows to outperform $RP_{t,T_1}$for all forecast horizons T_1.

In the Internet Appendix IV, we replicate the analysis over the 1997–2021 sample, for which the IERP measure of Tetlock et al. (2024) is available. Table IAII reports the corresponding out-of-sample results. For all forecast horizons T_1 except the 10-day horizon, the IERP outperforms RP_{t,T_1} , although the differences are not statistically significant. This relative outperformance may reflect differences in parameter treatment: in our specification, the preference parameters are fixed ex ante, whereas in the construction of the IERP they are implicitly estimated through the high-frequency estimation of the growth-optimal portfolio. Incorporating intertemporal hedging and extending the investment horizon T_N substantially improves the predictability of the equity risk premium. For forecast horizons up to six months, RP_{t,T_1,T_N} outperforms the IERP for T_N sufficiently large—most notably at T_N at two years—. At the longest forecast horizons (six months and one year), however, the IERP delivers an R^2 approximately twice as large as that of $RP_{t,T_1}$and slightly higher than that of RP_{t,T_1,T_N}.

Taken together, these findings point to the importance of options with maturities beyond two years to fully capture the intertemporal hedging premium at longer forecast horizons.

Second, we evaluate market-timing strategies and compute their realized mean-variance certainty equivalents. While the R^2_{OOS} values reported in Panel A of Table II assess how well our methodology captures the expected excess market return, the results in Panel B incorporate both return and variance forecasts, thereby reflecting performance across the first and second moments. For each forecasting method, we compute the weight of the market portfolio in the optimal portfolio at time t as,

$$\omega_{t,T_1} = \frac{\tilde{r}_{M,t,T_1}}{\gamma \tilde{\sigma}^2_{t,T_1}} \tag{40}$$

where γ is a risk aversion parameter and $\tilde{\sigma}^2_{t,T_1}$ is the physical variance of returns computed for each method, as described in Section V.B. We cap these portfolio weights at 2 to prevent excessive leverage, which can arise when the estimated physical variance approaches zero. Then, we compute the realized mean-variance certainty equivalent as,

$$CE = E(r_{p,t,T_1}) - \frac{\gamma}{2}\text{Var}(r_{p,t,T_1}), \tag{41}$$

where $r_{p,t} = r_{f,t,T_1} + \omega_{t,T_1} r_{M,t,T_1}$ are portfolio returns. The certainty equivalent is estimated using the sample return average and variance using non-overlapping returns over horizon T_1.

We report realized certainty equivalents, annualized and expressed in percentage terms, for a coefficient of relative risk aversion $\gamma = 3$. Across most forecast and investment horizons, RP_{t,T_1,T_N} delivers higher certainty equivalents than RP_{t,T_1} and RP^{Log}_{t,T_1}. In line with the results reported in Panel A of Table II, the certainty equivalents generally increase with T_N. We block-bootstrap the time-series of realized portfolio returns to compute the significance of the certainty equivalent differences for each strategy, compared to the one based on RP_{t,T_1} (see Politis and Romano, 1994).[30] We find that almost all differences between RP_{t,T_1,T_N}-based and RP_{t,T_1}-based strategies are statistically significant at the 5% level, except for short forecast horizons up to one month.

Both out-of-sample performance metrics—the out-of-sample R^2 and realized certainty equivalents—indicate that incorporating intertemporal hedging into the construction of the equity risk premium improves the forecasting of both the first and second moments of returns. Most of these differences are statistically significant.

Under Assumptions 1 and 2, these equity risk premium measures are lower bounds for the equity risk premium. As a final exercise, we follow the methodology of Back et al. (2022) to assess both the validity and the tightness of these bounds. The results, reported in Section V of the Internet Appendix, indicate that for all measures and forecast horizons T_1, we do not reject the hypothesis that the bounds are valid lower bounds across all investment horizons T_N. However, for most specifications, we reject the hypothesis that the bounds are tight. Consistent with the model's implica-

[30] We use 10,000 bootstrap samples and a mean block length equivalent to three years.

tions, the magnitude of the approximation error associated with our bound is smaller than that of RP_{t,T_1} and RP^{Log}_{t,T_1}.

D. Changing the preference parameters

We next compare the out-of-sample performance obtained under our baseline parameterization ($\tau = 1, \rho = 2$) to that generated by a set of parameters consistent with CRRA utility. Following the literature, we set $\gamma = 3$. Using (16), this implies $\tau = \frac{1}{3}$ and $\rho = \frac{2}{3}$. Under this alternative calibration, the estimated equity risk premium declines as the investment horizon T_N increases.

Table III reports the out-of-sample performance across investment horizons T_N. In contrast to the baseline specification, predictive performance now deteriorates as T_N increases. The highest out-of-sample R^2 values are obtained for T_N close to the forecast horizon T_1. Although these R^2 values remain higher than those of RP^{Log}_{t,T_1} and RP_{t,T_1}, they are lower than those achieved under our baseline parameterization ($\tau = 1, \rho = 2$).

[TABLE III ABOUT HERE]

In light of these findings, we proceed with the baseline set of parameters in the remainder of the analysis.

E. Crash probabilities

Figure 7 reports the conditional probability of a 10% market crash over forecast horizons of one month (Panel A) and six months (Panel B). We compare crash probabilities computed without intertemporal hedging ($\Pi_{t,T_1}[\alpha]$ with $\alpha = 90\%$) to those obtained under our methodology ($\Pi_{t,T_1,T_N}[\alpha]$), using investment horizons T_N of one and two years. Crash probabilities derived from our framework are uniformly lower than those obtained without intertemporal hedging. Moreover, the estimated crash probability declines as the investment horizon T_N increases.

[FIGURE 7 ABOUT HERE]

To determine whether these lower probabilities are more accurate, we assess in Table IV their out-of-sample prediction performance. For each horizon, we compute the loss function of our prediction as the negative of the log-likelihood function of a Bernouilli outcome:

$$l_{t,T_1,T_N} = -\left(\mathbb{1}_{r_{M,t,T_1}<\alpha} \log\left(\Pi_{t,T_1,T_N}[\alpha]\right) + (1 - \mathbb{1}_{r_{M,t,T_1}<\alpha}) \log(1 - \Pi_{t,T_1,T_N}[\alpha])\right). \tag{42}$$

Similarly, we compute the loss function for $\Pi_{t,T_1}[\alpha]$ and $\Pi^{Log}_{t,T_1}[\alpha]$, which we respectively denote l_{t,T_1} and l^{Log}_{t,T_1}. Next, we test the significance of the average difference in loss functions using the Diebold and Mariano (1995) test. We find that our probabilities for a 10% crash, reported in the third column, lead to significantly lower losses (i.e., higher realized log-likelihoods) than other benchmark probabilities for most horizons. Finally, we similarly find significantly superior predictions for a crash size of 20% for all horizons except one week.

[TABLE IV ABOUT HERE]

VI. Estimating preference parameters

In this section, we try to improve upon the results reported in Section V, which are based on fixed preference parameters $\tau = 1$ and $\rho = 2$, by estimating these parameters directly from the data.

A. *Methodology*

We estimate the preference parameters ρ_t and τ_t using a two-stage non-linear least squares approach, similar to Chabi-Yo and Loudis (2020). Specifically, we estimate the coefficients τ_t, ρ_t, $\beta_0^{(1)}$, and $\beta_0^{(2)}$ by minimizing the weighted sum of squared errors $w_1 \epsilon^{(1)\intercal}_{t,T_1} \epsilon^{(1)}_{t,T_1} + w_2 \epsilon^{(2)\intercal}_{t,T_1} \epsilon^{(2)}_{t,T_1}$ in the following equations,

$$\begin{aligned} R_{M,t,T_1} - R_{f,t,T_1} &= \beta_0^{(1)} + RP_{t,T_1,T_N} + \epsilon_{t,T_1}^{(1)}, \\ \left(R_{M,t,T_1} - R_{f,t,T_1}\right)^2 &= \beta_0^{(2)} + \mathbb{E}_t\left(R_{M,t,T_1} - R_{f,t,T_1}\right)^2 + \epsilon_{t,T_1}^{(2)}. \end{aligned} \tag{43}$$

In the first stage, we set $w_1 = w_2 = 1$. In the second stage, we weigh each sum of squared errors by the inverse of the standard deviations of first-stage errors. Note that parameters τ_t and ρ_t enter the above equations through RP_{t,T_1,T_N} and $\mathbb{E}_t\left(R_{M,t,T_1} - R_{f,t,T_1}\right)^2$. We estimate parameters separately for each horizon T_1 and T_N. We restrict the parameter space so that the resulting risk premia remain positive.

B. *Performance with in-sample estimation*

We begin by estimating the preference parameters over the full sample from 1996 to 2023.[31] We remain agnostic as to whether ρ lies below or above one. The estimated values of τ fall within a narrow range—between 0.86 and 0.88—across all forecast horizons T_1 and investment horizons T_N, indicating limited variation when estimated over the full sample. By contrast, the estimates of ρ show greater dispersion but consistently exceed one. For the RP_{t,T_1} bound, the estimated ρ declines markedly with the forecast horizon T_1, falling from 5.06 to 1.20. The estimate of ρ also decreases with the investment horizon T_N; for $T_N = 2$ years, it remains relatively stable between 1.20 and 1.60 across all T_1. These findings provide strong empirical support for $\rho > 1$ in the data.

Table V compares the out-of-sample R^2 obtained under the fixed-parameter specification ($\tau = 1, \rho = 2$)—as in Section V—with the R^2 obtained when τ

[31] As in Chabi-Yo and Loudis (2020), this procedure introduces a look-ahead bias in the evaluation of out-of-sample performance. Our objective here is not to propose an estimation strategy for the preference parameters, but rather to assess whether the results obtained in Section V continue to hold when the parameters are optimally chosen. We address this source of bias in Section VI.C.

and ρ are estimated from the data. Column (4) reports the results under the estimated preference parameters. The out-of-sample R^2 achieved with estimated parameters is comparable to that obtained under the fixed-parameter specification ($\tau = 1,\ \rho = 2$).

[TABLE V ABOUT HERE]

As an alternative specification, we allow τ_t and ρ_t to vary over time by modeling each parameter as a linear function of past three-month returns, and estimating the associated loadings over the full sample. Figure 8 reports the resulting time series for τ_t and ρ_t for a forecast horizon $T_1 = 1$ month and an investment horizon $T_N = 2$ years. The estimated τ_t ranges between 0.2 and 0.9. It approaches one during recessions and declines toward zero in their immediate aftermath. The estimated ρ_t varies between 1 and 2 and remains consistently above one throughout the sample.

[FIGURE 8 ABOUT HERE]

Column (5) of Table V reports the out-of-sample performance obtained using the time-varying estimates of τ_t and ρ_t. The additional flexibility delivers only negligible improvements in predictive accuracy, suggesting potential overfitting. Moreover, the estimation of the physical variance deteriorates under this specification, resulting in realized certainty equivalents that are lower than those obtained with fixed preference parameters.

Taken together, these results indicate that more finely parameterized preference specifications provide limited and mixed gains in out-of-sample performance.

C. Expanding and rolling window estimations

To eliminate the look-ahead bias inherent in full-sample estimation, we re-estimate the preference parameters using data windows that exclude all observations beyond the forecast date. We consider two alternative window specifications: (i) an expanding window that begins in 1996 and grows over time, and (ii) a rolling window that uses the most recent five years of data.

Out-of-sample performance is evaluated starting in 2006, once a full decade of observations is available for parameter estimation.

Table VI reports the out-of-sample performance obtained when τ and ρ are estimated using expanding and rolling windows. Entries marked with "-" correspond to R^2 values below -1. The expanding-window specification performs poorly, delivering R^2 values almost always below those obtained under the fixed-parameter benchmark ($\tau = 1, \rho = 2$). The rolling-window specification performs even worse, with forecasts that, in most cases, underperform the historical mean return benchmark. Table VII presents the same performance measures using an out-of-sample window starting in 2009, so that the estimated preference parameters incorporate information from the financial crisis. Under this specification, the expanding-window approach yields R^2 values and realized certainty equivalents that are slightly higher than those obtained with fixed parameters, although the differences are not statistically significant. The rolling-window approach performs comparably to RP_{t,T_1} and remains inferior to RP_{t,T_1,T_N} for most horizons.

[TABLES VI AND VII ABOUT HERE]

Overall, these results highlight the sensitivity of out-of-sample performance to the choice of estimation window when preference parameters are estimated. More broadly, they underscore the difficulty of achieving robust out-of-sample gains when preference parameters must be inferred from the data rather than imposed ex ante.

VII. Conclusion

We develop estimates of the equity risk premium, physical variance, and crash probability of market returns that incorporate intertemporal hedging motives. Intertemporal hedging has a large effect on all three moments. Incorporating it delivers sizable improvements in out-of-sample predictability for expected returns, return variance, and crash risk.

A central insight of the paper is that the performance of the equity risk premium measure depends on the specification of the representative agent's skewness parameter. In standard asset pricing models, skewness preferences are mechanically tied to risk aversion. Our results show that this restriction is too tight: strong out-of-sample performance requires disentangling skewness from risk aversion. This additional flexibility is essential to capture higher-order risk pricing in the data. Characterizing utility specifications—or, in the Epstein-Zin framework, aggregators and certainty equivalents—that are consistent with $\gamma \geq 1$ and $\rho > 1$ remains an important open question.

Our work opens the door to a general framework for bounding or characterizing expected excess returns across asset classes, including individual stocks, currencies, commodities, interest rates, and other assets for which options are traded.

More broadly, our framework provides a tractable and option-based approach to measuring and bounding expected excess returns. It can be extended beyond the aggregate market to individual equities, currencies, commodities, fixed income securities, and other assets with traded options.

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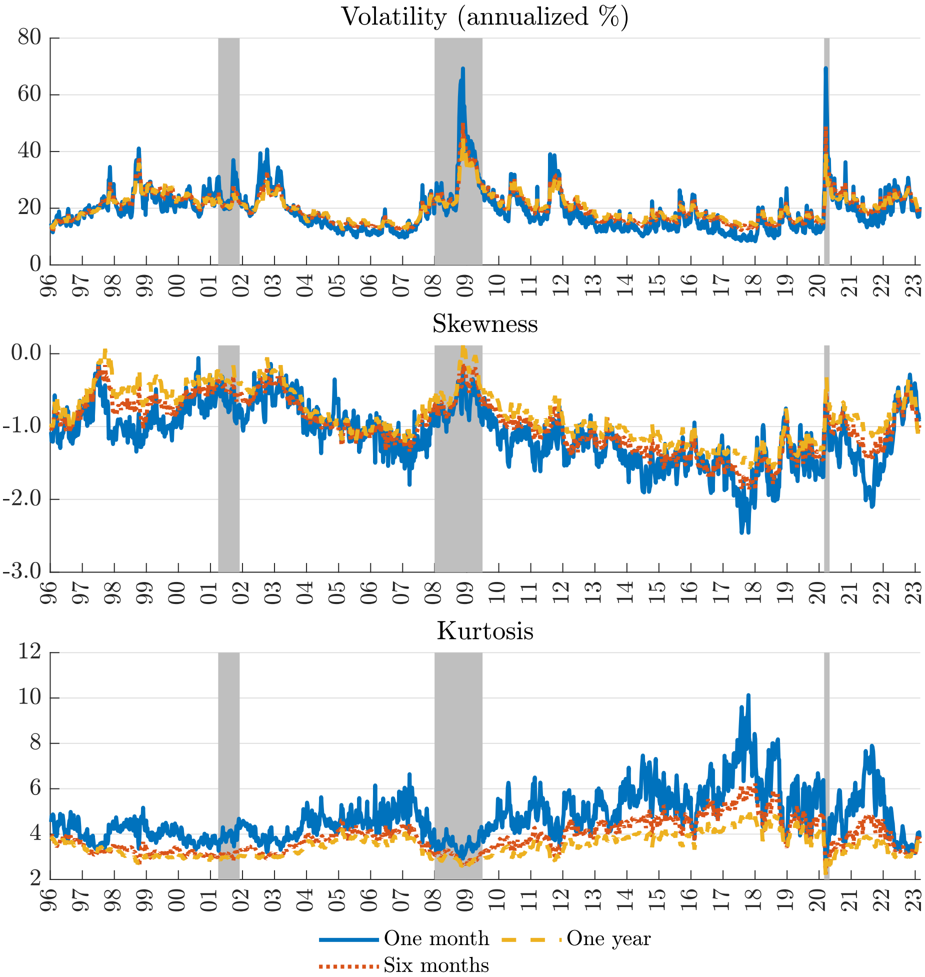

Figure 1. Risk-neutral moments.
We report option-implied risk-neutral volatility, skewness, and kurtosis for the S&P 500 index at a horizon of one month, six months and one year. Data are weekly from January 1996 to February 2023. Gray areas are NBER recessions.

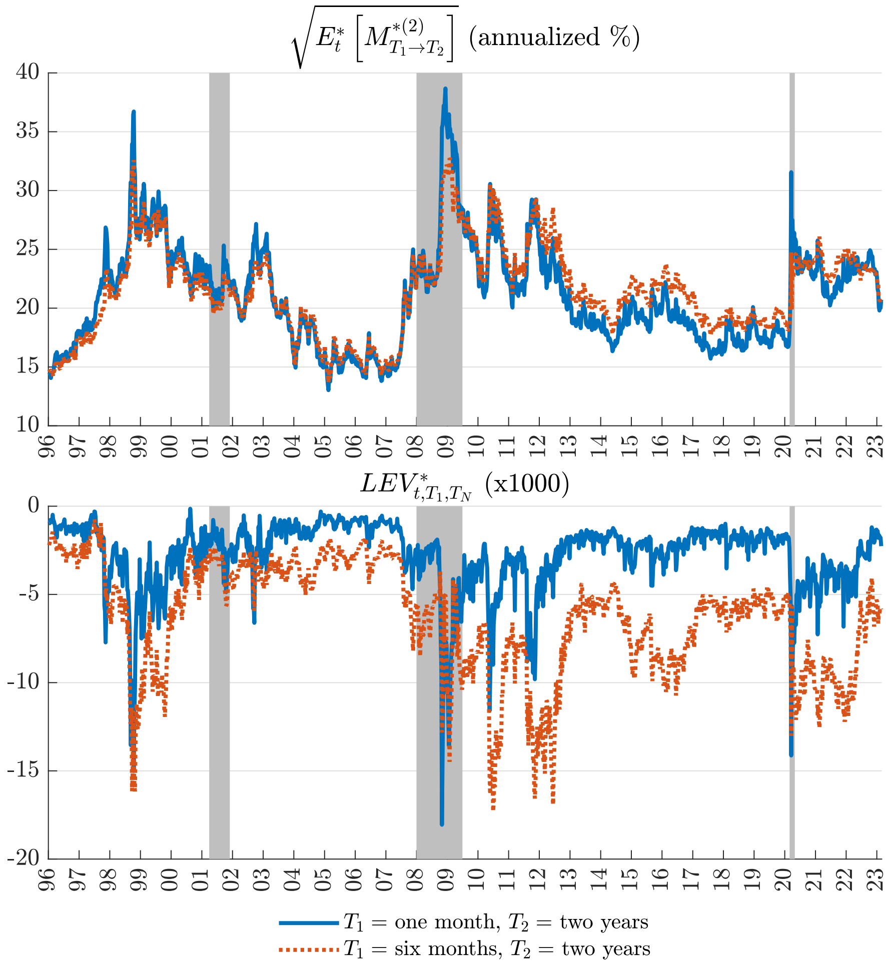

Figure 2. Risk-neutral expected future volatility and leverage. This figure displays the two components that arise in the equity risk premium estimate (9) due to intertemporal hedging motives. We report in the top graph the risk-neutral expected future volatility of the S&P 500 index, and in the bottom graph the leverage effect, as defined in Equation (10). We use forecast horizons T_1 of one and six months, and an investment horizon T_N of two years. We annualize both the variance and the leverage measures by multiplying them by $\frac{365}{T_N - T_1}$. Data are weekly from January 1996 to February 2023. Gray areas are NBER recessions.

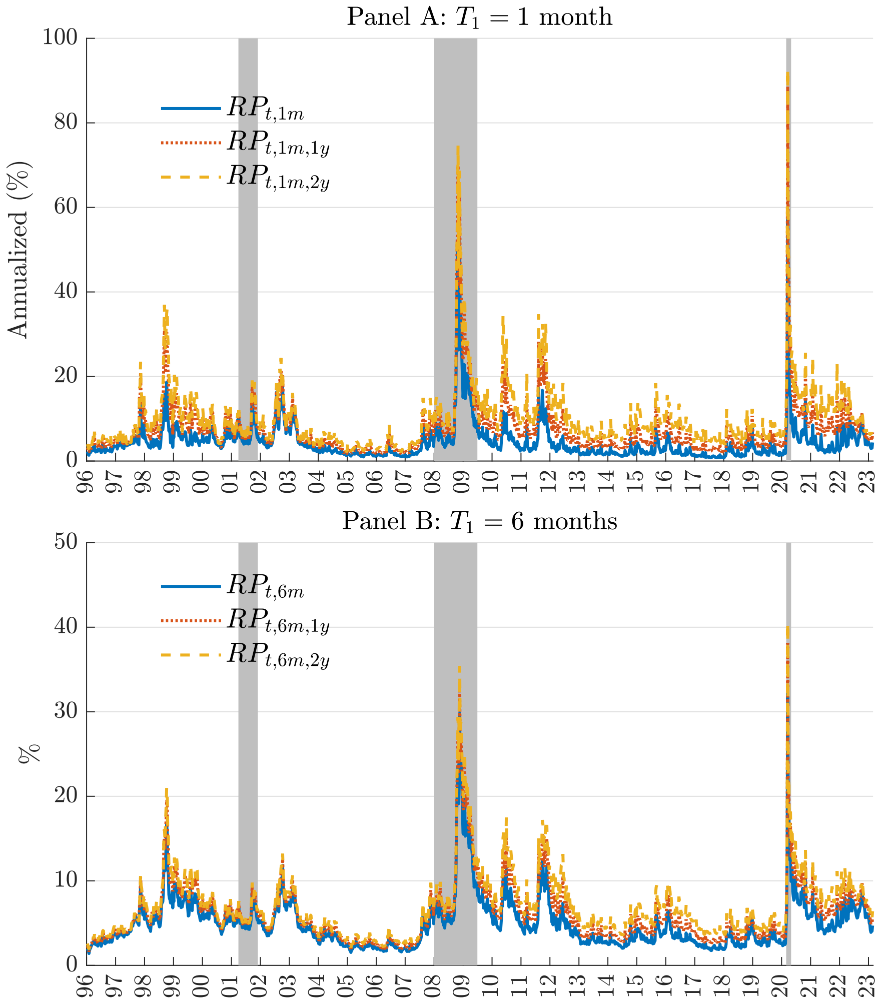

Figure 3. Equity risk premium.
This graph represents the different equity risk premium estimates, for a forecast horizon of 1 month (Panel A) and 6 months (Panel B), using $\tau = 1$ and $\rho = 2$. The following estimates are compared: the bound of Chabi-Yo and Loudis (2020), RP_{t,T_1}, and our bound RP_{t,T_1,T_N} in Equation (12), for $T_N =$ 1 year and 2 years.

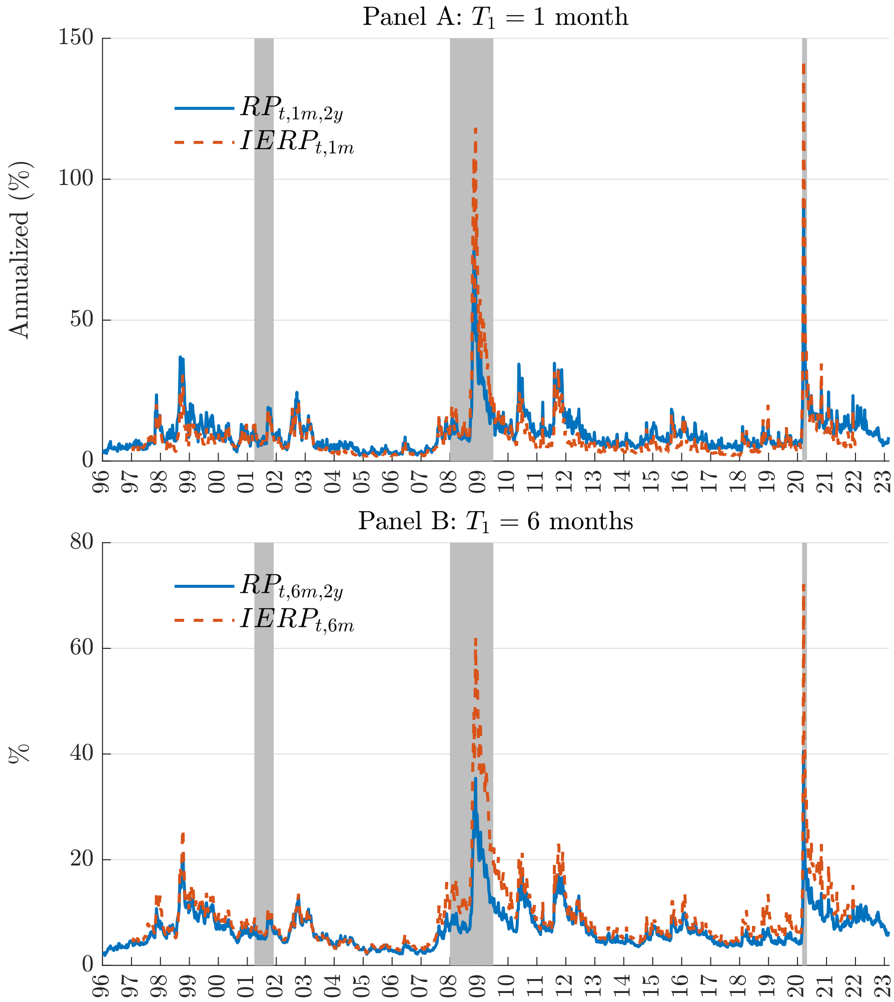

Figure 4. Comparison with $IERP_{t,T_1}$
This graph compares our estimate of the equity risk premium, RP_{t,T_1,T_N}, to the Implied Equity Risk Premium of Tetlock et al. (2024), $IERP_{t,T_1}$, for a forecast horizon of 1 month (Panel A) and 6 months (Panel B). The preference parameters used are $\tau_t = 1$ and $\rho_t = 2$. The investment horizon T_N is taken equal to 2 years.

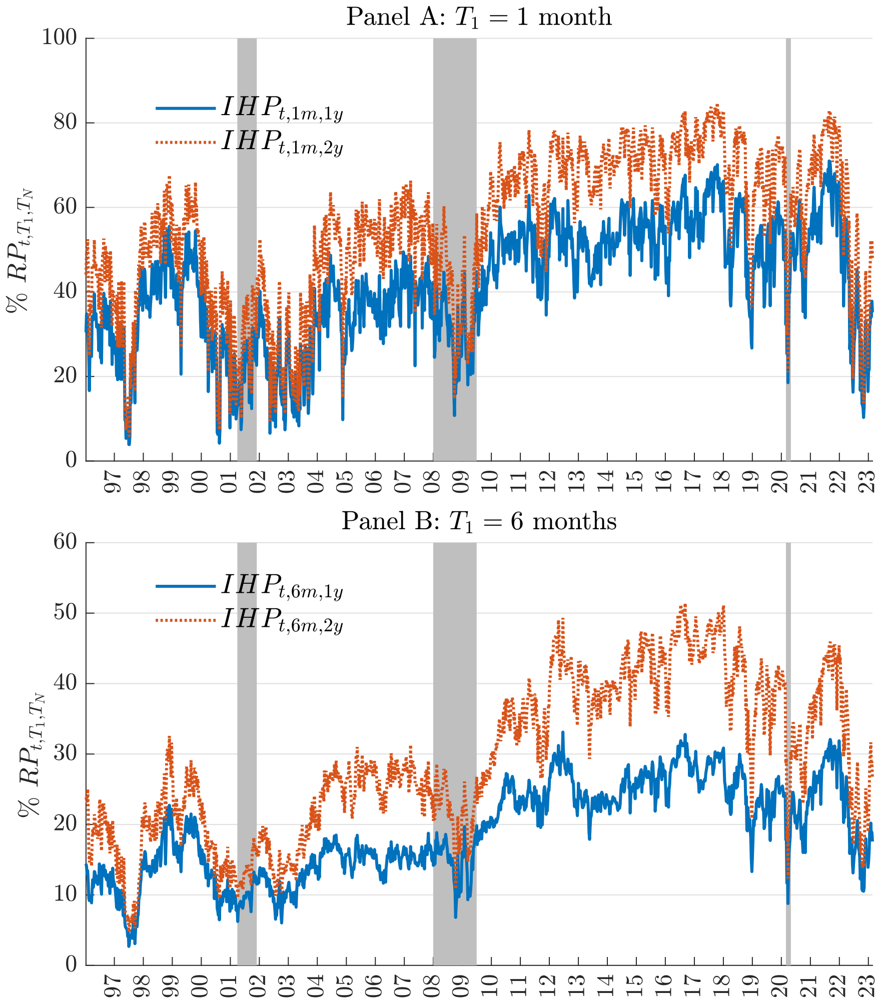

Figure 5. Intertemporal hedging premium IHP_{t,T_1,T_N}
This graph represents the intertemporal hedging premium, IHP_{t,T_1,T_N}, defined in Equation (1), as a fraction of RP_{t,T_1,T_N} for different equity risk premium estimates RP_{t,T_1,T_N}. The forecast horizon is 1 month (Panel A) and 6 months (Panel B).

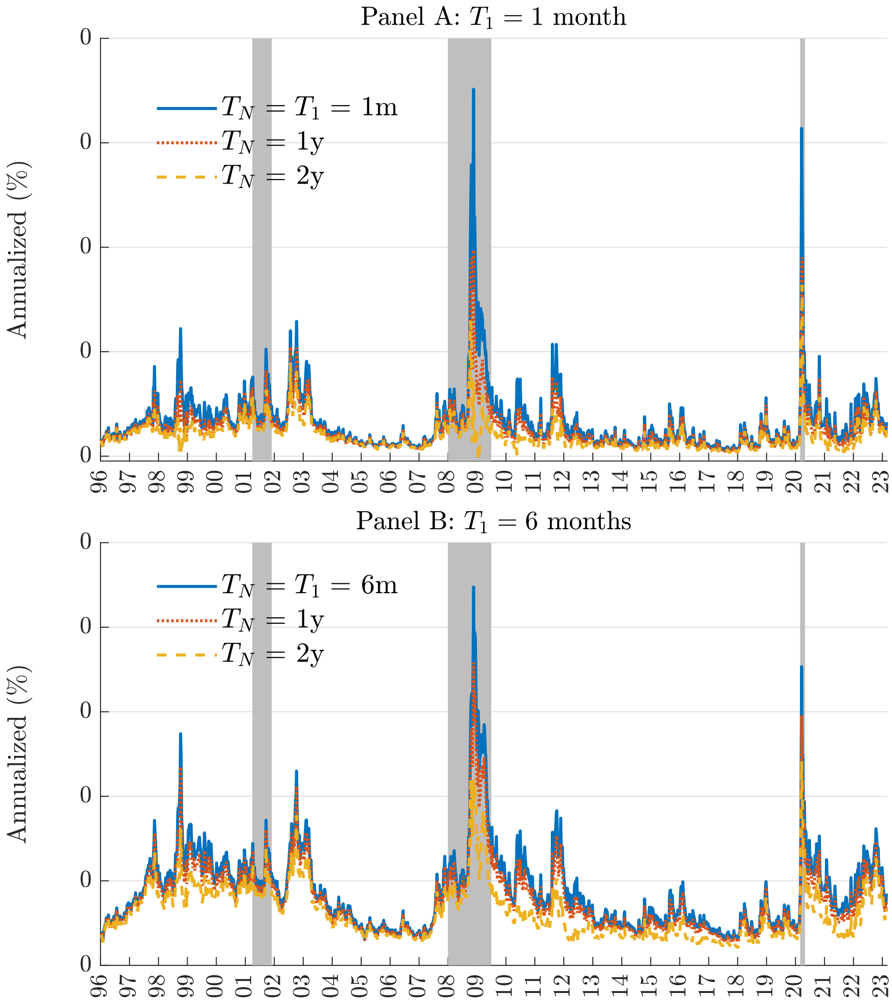

Figure 6. Conditional variance under the physical measure.
This graph represents the conditional physical variance as defined in Equation (28), for $T_1 = 1$ month (Panel A) and $T_1 = 6$ months (Panel B). The conditional variance without intertemporal hedging ($T_N = T_1$) is compared to the variance with intertemporal hedging, using $T_N = 1$ year and $T_N = 2$ years.

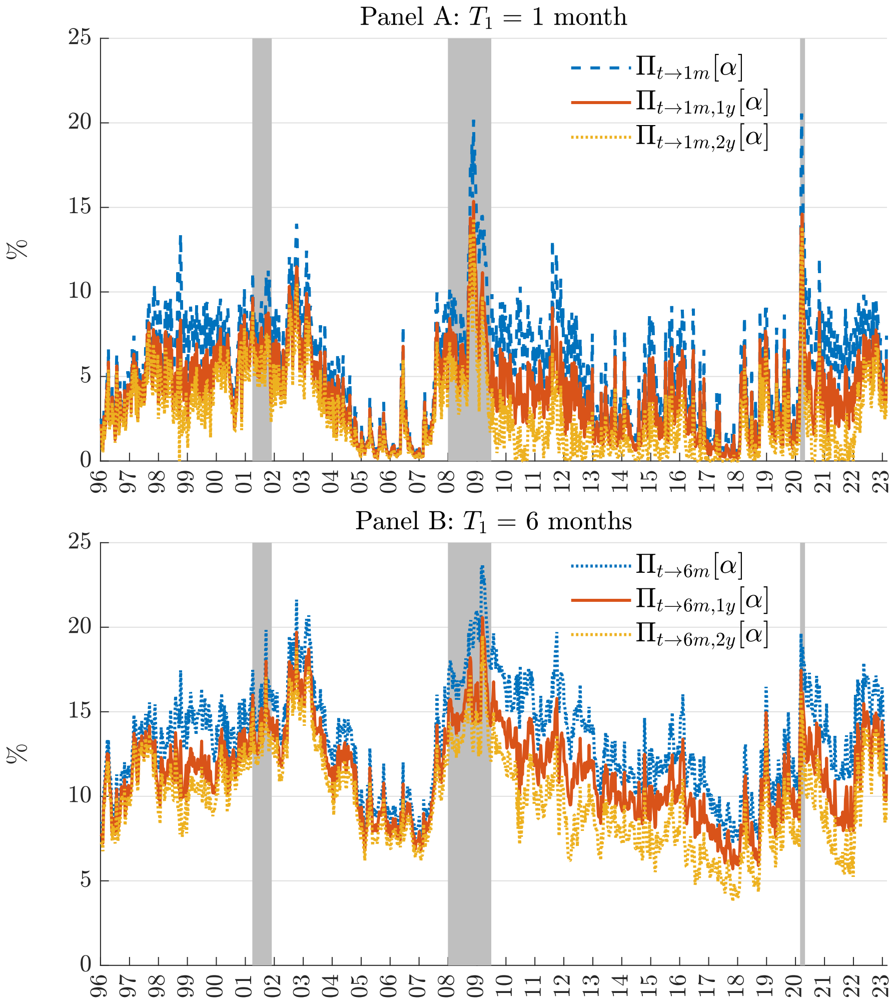

Figure 7. Probability of a 10% market crash
We report the time-varying probability of a 10% stock market crash from Equation 31, for a forecast horizon T_1 of 1 month (Panel A) and 6 months (Panel B). The crash probabilities without intertemporal hedging $\Pi_{t,T_1}[\alpha]$ are compared to those with intertemporal hedging $\Pi_{t,T_1,T_N}[\alpha]$ with investment horizons T_N at 1 and 2 years. Gray areas are NBER recessions.

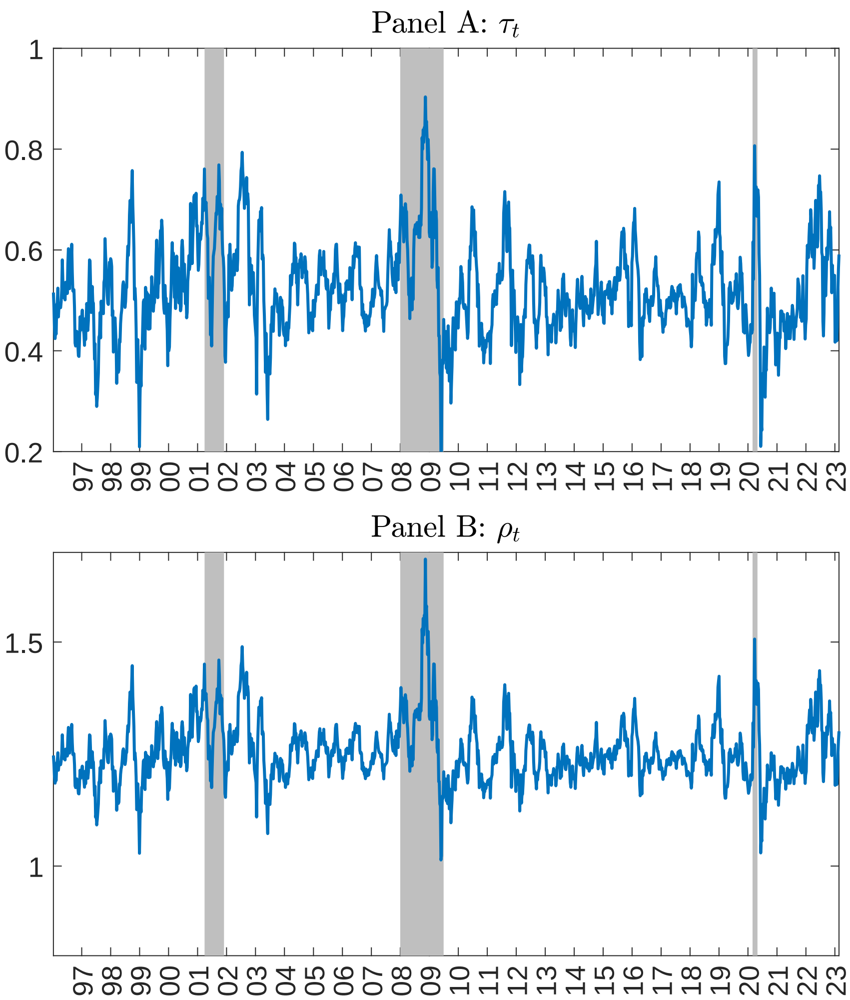

Figure 8. Estimated preference parameters τ_t and ρ_t over the period 1996-2023.
This graph represents the estimated time series of risk aversion parameter τ_t and skewness tolerance parameter ρ_t, for $T_1 = 1$ month $T_N = 2$ years. Estimates are obtained by letting the preference parameters be linear functions of past 3-month returns, and applying the estimation methodology described in Section VI.A on the whole dataset.

Table I
Summary statistics for risk premia

We report summary statistics for times series of risk premia predictions. We use weekly time series of overlapping values for horizons longer than 10 days. Each panel reports the results for a forecast horizon T_1 between one week and six months. All values are annualized and in percent. RP^{Log}_{t,T_1} is the lower bound of Martin (2017). RP_{t,T_1} is the second-order lower bound of Chabi-Yo and Loudis (2020). RP_{t,T_1,T_N} is the risk premium measure in Equation (12), with $T_N = 1$ and 2 years. Data are weekly from January 1996 to February 2023.

Prediction	Mean	Standard deviation	Skew.	Kurt.	10%	25%	50%	75%	90%
Panel A: One week									
RP^{Log}_{t,T_1}	3.58	4.71	6.25	60.48	1.03	1.41	2.27	3.96	6.71
RP_{t,T_1}	3.74	5.12	6.70	69.05	1.06	1.45	2.33	4.03	7.00
$RP_{t,T_1,1y}$	9.36	9.63	7.13	87.99	3.78	4.74	6.89	10.72	15.69
$RP_{t,T_1,2y}$	14.59	11.07	5.65	62.83	6.91	8.71	11.46	17.09	23.76
Panel B: One month									
RP^{Log}_{t,T_1}	4.27	4.24	4.66	36.24	1.34	1.89	3.25	5.11	7.72
RP_{t,T_1}	4.57	4.73	4.94	40.19	1.41	1.98	3.45	5.41	8.28
$RP_{t,T_1,1y}$	7.59	6.66	4.42	35.65	2.78	3.84	5.90	9.07	13.22
$RP_{t,T_1,2y}$	9.93	7.37	3.68	27.03	4.14	5.50	8.09	12.04	16.93
Panel C: One quarter									
RP^{Log}_{t,T_1}	4.32	3.29	3.43	22.09	1.70	2.21	3.53	5.34	7.34
RP_{t,T_1}	4.84	3.86	3.76	26.43	1.87	2.43	3.86	5.93	8.25
$RP_{t,T_1,1y}$	6.36	4.63	3.57	25.55	2.57	3.44	5.24	7.69	10.66
$RP_{t,T_1,2y}$	7.55	4.92	3.36	23.53	3.50	4.51	6.49	9.07	12.36
Panel D: Six months									
RP^{Log}_{t,T_1}	4.30	2.62	2.60	14.30	1.95	2.45	3.71	5.35	6.94
RP_{t,T_1}	4.99	3.16	2.80	16.34	2.21	2.82	4.29	6.12	8.22
$RP_{t,T_1,1y}$	6.13	3.70	2.54	14.16	2.81	3.60	5.30	7.46	10.05
$RP_{t,T_1,2y}$	6.98	3.94	2.44	13.45	3.33	4.36	6.11	8.49	11.23

Table II
Out-of-sample prediction and allocation performance with $\tau_t = 1$ and $\rho_t = 2$

We report the out-of-sample performance of different risk premium prediction methods. RP^{Log}_{t,T_1} is the lower bound of Martin (2017). RP_{t,T_1} is the second-order lower bound of Chabi-Yo and Loudis (2020). RP_{t,T_1,T_N} is our risk premium estimate, given in Equation (12). We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (39)). The results in the last column are based on predicted returns obtained by averaging RP_{t,T_1,T_N} across T_N. For each prediction method, we test for the significance of the R^2_{OOS} difference relative to RP_{t,T_1} using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (41)). The physical variances are computed using option prices, using Equation (28). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to RP_{t,T_1} using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively. Data are from January 1996 to February 2023.

			RP_{t,T_1,T_N} with $T_N =$ (in months)										
Horizon T_1 (in months)	RP^{Log}_{t,T_1}	RP_{t,T_1}	1	2	3	4	5	6	9	12	18	24	Average across T_N
Panel A: Out-of-sample R^2													
10d	−0.10	−0.08	−0.04	0.02	0.08	0.14	0.19	0.26	0.36	0.42	0.46	0.33	0.37
1	1.10	1.24	-	1.38	1.59	1.66	1.73	1.78	1.88	1.95	2.00	1.94	1.87
2	1.65	2.04	-	-	2.26	2.63	2.77*	2.89*	3.16*	3.37*	3.67	3.81	3.17*
3	1.65	2.33	-	-	-	2.60*	3.09*	3.27*	3.68*	3.98*	4.45*	4.74*	3.78*
4	2.35	3.36	-	-	-	-	3.64**	4.15**	4.66**	5.04**	5.60**	5.96*	4.93**
5	3.17	4.56	-	-	-	-	-	4.86**	5.85**	6.29**	6.93**	7.36**	6.34**
6	3.50	5.19	-	-	-	-	-	-	6.00**	7.27**	7.96**	8.44**	7.51**
12	2.84	5.63	-	-	-	-	-	-	-	-	6.85	8.79	7.90
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma = 3$													
10d	2.80	2.85	2.94	3.17	3.31	3.46	3.59	3.71	3.49	2.97	1.66	1.77	3.80
1	4.47	4.63	-	4.80	5.15	5.31	5.48	5.65	6.03	6.27	6.32	7.02	6.13
2	4.79	5.15	-	-	5.40**	5.93**	6.16**	6.39**	7.01**	7.64**	8.76**	9.03*	7.14**
3	5.29	5.83	-	-	-	6.08**	6.59**	6.81**	7.45**	8.10**	9.51**	10.45**	7.81**
4	5.38	6.03	-	-	-	-	6.26**	6.73**	7.34**	7.93**	9.17**	10.07**	7.85**
5	5.18	5.85	-	-	-	-	-	6.02**	6.78**	7.26**	8.29**	9.25**	7.47**
6	5.43	6.35	-	-	-	-	-	-	6.99***	8.54***	9.78**	10.23**	9.08**
12	5.07	5.92	-	-	-	-	-	-	-	-	6.53	6.25	6.90

Table III

Out-of-sample prediction and allocation performance with $\tau_t = \frac{1}{3}$ and $\rho_t = \frac{2}{3}$

We report the out-of-sample performance of different risk premium prediction methods. RP^{Log}_{t,T_1} is the lower bound of Martin (2017). RP_{t,T_1} is the second-order lower bound of Chabi-Yo and Loudis (2020). RP_{t,T_1,T_N} is our risk premium estimate, given in Equation (12). We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (39)). The results in the last column are based on predicted returns obtained by averaging RP_{t,T_1,T_N} across T_N. For each prediction method, we test for the significance of the R^2_{OOS} difference relative to RP_{t,T_1} using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (41)). The physical variances are computed using option prices, using Equation (28). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to RP_{t,T_1} using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively. Data are from January 1996 to February 2023.

Horizon T_1			RP_{t,T_1,T_N} with $T_N =$ (in months)										
(in months)	RP^{Log}_{t,T_1}	RP_{t,T_1}	1	2	3	4	5	6	9	12	18	24	Average across T_N
Panel A: Out-of-sample R^2													
10d	−0.10	−0.20	0.08	0.15	−0.04	−0.27	−0.53	−0.84	−1.79	−2.89	−5.29	−7.65	−1.28
1	1.10	0.52	-	0.87	1.00	0.94	0.81	0.67	0.20	−0.40	−1.78	−3.16	0.41
2	1.65	0.25	-	-	0.61	0.67	0.57	0.42	−0.06	−0.65	−2.06	−3.45	0.04
3	1.65	0.02	-	-	-	0.32	0.24	0.13	−0.29	−0.84	−2.22	−3.56	−0.35
4	2.35	0.95	-	-	-	-	1.19	1.14	0.92	0.44	−0.83	−2.08	0.68
5	3.17	2.66	-	-	-	-	-	2.83	2.65	2.19	0.96	−0.26	2.19
6	3.50	3.61	-	-	-	-	-	-	3.89	2.48	0.97	−0.40	2.20
12	2.84	3.32	-	-	-	-	-	-	-	-	3.49	−0.32	1.92
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma = 3$													
10d	2.80	3.28	3.67	3.71	3.34	2.79	2.08	1.31	-	-	-	-	-
1	4.47	5.73	-	5.83	5.35	5.02	4.64	4.25	3.22	2.24	0.59	-	3.37
2	4.79	6.76	-	-	6.34	5.42	5.05	4.72	3.90	3.18	2.03	1.18	3.93
3	5.29	8.41	-	-	-	7.86	6.75	6.34	5.35	4.55	3.39	2.64	5.08
4	5.38	7.89	-	-	-	-	7.51	6.73	5.91	5.22	4.21	3.53	5.45
5	5.18	7.26	-	-	-	-	-	7.02	5.86	5.23	4.31	3.70	5.13
6	5.43	7.31	-	-	-	-	-	-	6.42	4.79	3.96	3.46	4.54
12	5.07	4.87	-	-	-	-	-	-	-	-	4.78	3.95	4.36

Table IV
Out-of-sample crash prediction

We report the out-of-sample performance of different crash prediction methods. Each month, we use the crash probability from Martin (2017) ($\Pi^{Log}_{t,T_1}[\alpha]$), the one from Chabi-Yo and Loudis (2020), $\Pi_{t,T_1}[\alpha]$, and the one from our methodology, $\Pi_{t,T_1,T_N}[\alpha]$, defined in Equation (31). T_N is set equal to 2 years. We compute the loss function for $\Pi^{Log}_{t,T_1}[\alpha]$, $\Pi_{t,T_1}[\alpha]$ and $\Pi_{t,T_1,T_N}[\alpha]$, as given in Equation (42). For each method indicated in the rows, we test whether the average loss functions are significantly larger than those of the method indicated in the columns using the Diebold and Mariano (1995) test. A significantly positive test statistic indicates that the column-method outperforms the row-method. We estimate the variance of the difference in loss functions using a Newey-West correction with 12 lags. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively. We report on a 10% ($\alpha = 0.90$), and 20% ($\alpha = 0.80$) crash size. Data are from January 1996 to February 2023.

	10% crash		20% crash	
	$\Pi_{t,T_1}[\alpha]$	$\Pi_{t,T_1,T_N}[\alpha]$	$\Pi_{t,T_1}[\alpha]$	$\Pi_{t,T_1,T_N}[\alpha]$
Panel A: One week				
$\Pi^{Log}_{t\to T_1}[\alpha]$	1.56*	1.67**	1.29*	−0.91
$\Pi_{t\to T_1}[\alpha]$	-	1.68**	-	−0.91
Panel B: One month				
$\Pi^{Log}_{t\to T_1}[\alpha]$	1.76**	−1.28	5.71***	4.11***
$\Pi_{t\to T_1}[\alpha]$	-	−1.29	-	4.01***
Panel C: One quarter				
$\Pi^{Log}_{t\to T_1}[\alpha]$	4.42***	3.58***	2.67***	1.57*
$\Pi_{t\to T_1}[\alpha]$	-	3.43***	-	1.41*
Panel D: Six months				
$\Pi^{Log}_{t\to T_1}[\alpha]$	3.91***	3.02***	3.36***	2.48***
$\Pi_{t\to T_1}[\alpha]$	-	2.79***	-	2.24**
Panel E: Nine months				
$\Pi^{Log}_{t\to T_1}[\alpha]$	2.66***	1.93**	1.48*	1.10
$\Pi_{t\to T_1}[\alpha]$	-	1.63*	-	0.94
Panel F: One year				
$\Pi^{Log}_{t\to T_1}[\alpha]$	2.18**	1.48*	1.25	0.92
$\Pi_{t\to T_1}[\alpha]$	-	1.16	-	0.75

Table V
Out-of-sample prediction and allocation performance with τ_t and ρ_t estimated in-sample

We report the out-of-sample performance of different risk premium prediction methods, from January 1997 to February 2023. RP^{Log}_{t,T_1} is the lower bound of Martin (2017). RP_{t,T_1} is the second-order lower bound of Chabi-Yo and Loudis (2020). RP_{t,T_1,T_N} is the risk premia measure in Equation (12). In columns (2) and (3), results are reported setting the preference parameters to $\tau = 1$ and $\rho = 2$ (benchmark). In column (4), they are kept constant over the time series of data, but the constants are estimated. In column (5), they are modelled as linear functions of past 3-month returns. We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (39)). For each prediction method, we test for the significance of the R^2_{OOS} difference relative to RP_{t,T_1} using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (41)). The physical variances are computed using option prices, using Equation (28). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to RP_{t,T_1} using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively.

T_1		$\tau = 1$ and $\rho = 2$		ρ, τ estimated	
(months)	RP^{Log}_{t,T_1} (1)	RP_{t,T_1} (2)	$RP_{t,T_1,2y}$ (3)	$RP_{t,T_1,2y}$ (4)	$RP_{t,T_1,2y}$ (5)
Panel A: Out-of-sample R^2					
10d	−0.09	−0.07	0.33	0.54	0.83
1	1.11	1.25	1.94	1.91	1.70
2	1.66	2.04	3.82	3.67	4.18
3	1.65	2.33	4.74*	4.68*	5.25
4	2.36	3.36	5.97*	6.13*	6.71
5	3.18	4.57	7.37**	7.78*	8.43
6	3.53	5.22	8.47**	8.72	8.99
12	2.86	5.66	8.82	8.17	9.39
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma = 3$					
10d	5.28	5.40	15.75	17.19**	20.02**
1	4.06	4.21	2.18	2.95	-
2	4.15	4.37	6.80	5.89	4.05
3	4.82	5.28	8.92**	9.50*	4.20
4	4.48	4.86	7.43*	6.06	1.54
5	4.94	5.59	9.29***	9.95*	5.26
6	4.64	5.34	7.31	6.18	3.95
12	4.80	5.64	5.85	6.97*	8.10*

Table VI
Out-of-sample prediction and allocation performance for expanding and rolling window estimations

We report the out-of-sample performance of different risk premium prediction methods, from January 2006 to February 2023. RP^{Log}_{t,T_1} is the lower bound of Martin (2017). RP_{t,T_1} is the second-order lower bound of Chabi-Yo and Loudis (2020). RP_{t,T_1,T_N} is the risk premia measure in Equation (12). In columns (2) and (3), results are reported setting the preference parameters to $\tau = 1$ and $\rho = 2$ (benchmark). In column (4), they are set constant and estimated on an expanding window of time. In column (5), they are set constant and estimated on a rolling window of ten years. We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (39)). Values smaller than -1 are not reported and left blank. For each prediction method, we test for the significance of the R^2_{OOS} difference relative to RP_{t,T_1} using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (41)). The physical variances are computed using option prices, using Equation (28). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to RP_{t,T_1} using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively.

T_1		$\tau = 1$ and $\rho = 2$		expanding window	rolling window
	RP^{Log}_{t,T_1}	RP_{t,T_1}	$RP_{t,T_1,2y}$	$RP_{t,T_1,2y}$	$RP_{t,T_1,2y}$
	(1)	(2)	(3)	(4)	(5)
Panel A: Out-of-sample R^2					
10d	−0.10	−0.08	0.17	0.32	-
1	0.73	0.87	1.57	-	-
2	1.03	1.41	3.29	-	-
3	0.30	0.98	3.81	-	-
4	1.43	2.57	5.78*	3.87*	-
5	2.64	4.35	7.94**	5.85	-
6	2.94	5.13	9.53**	9.18**	-
12	2.30	6.73	11.77	10.79	7.37
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma = 3$					
10d	2.67	2.73	-	-	-
1	3.75	3.94	3.01	0.16	-
2	3.42	3.69	5.92	-	0.43**
3	4.26	4.79	9.19	5.65	-
4	3.94	4.21	2.96	-	0.10
5	4.05	4.67	8.41	-	-
6	4.55	5.29	7.50	5.92	2.78
12	5.01	6.43	7.02	6.48	7.17

Table VII
Out-of-sample prediction and allocation performance for expanding and rolling window estimations from 2009

We report the out-of-sample performance of different risk premium prediction methods, from January 2009 to February 2023. RP^{Log}_{t,T_1} is the lower bound of Martin (2017). RP_{t,T_1} is the second-order lower bound of Chabi-Yo and Loudis (2020). RP_{t,T_1,T_N} is the risk premia measure in Equation (12). In columns (2) and (3), results are reported setting the preference parameters to $\tau = 1$ and $\rho = 2$ (benchmark). In column (4), they are modelled constant and estimated on a telescopic window of time. In column (5), they are modelled constant and estimated on a rolling window of ten years. We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (39)). Values smaller than -1 are not reported and left blank. For each prediction method, we test for the significance of the R^2_{OOS} difference relative to RP_{t,T_1} using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (41)). The physical variances are computed using option prices, using Equation (28). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to RP_{t,T_1} using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively.

T_1		$\tau = 1$ and $\rho = 2$		expanding window	rolling window
	RP^{Log}_{t,T_1}	RP_{t,T_1}	$RP_{t,T_1,2y}$	$RP_{t,T_1,2y}$	$RP_{t,T_1,2y}$
	(1)	(2)	(3)	(4)	(5)
Panel A: Out-of-sample R^2					
10d	0.38	0.45	1.76	0.85	-
1	2.18	2.65	5.37*	5.42	3.55
2	4.52	5.83	11.48***	12.46*	7.24
3	5.09	7.59	15.58***	17.93*	8.20*
4	6.07	9.59	18.16***	23.15**	8.98**
5	6.41	10.72	19.55***	23.50	8.27
6	5.42	10.52	21.12***	23.54**	10.40***
12	-	8.14	19.62	21.46	24.40
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma = 3$					
10d	3.18	3.29	0.88	-	-
1	4.85	5.21	8.00	8.23	4.51***
2	5.57	6.27	14.98***	16.16**	6.32
3	5.24	6.15	14.73***	15.35*	3.47
4	6.25	7.53	16.56**	18.73*	13.10***
5	6.18	7.69	16.33***	18.39***	12.11
6	6.58	8.40	18.40***	18.59***	17.36**
12	6.47	9.22	15.90***	16.57***	12.10***

Internet Appendix for
"Option-Implied Risk Premia with Intertemporal Hedging"

FOUSSENI CHABI-YO, ELISE GOURIER, and HUGUES LANGLOIS

ABSTRACT

This Internet Appendix provides proofs and derivations of results in the main text, extensions to the models presented in Section I, as well as additional empirical empirical tests.

I . Proofs and derivations of Section I in the main text

This section contains the proofs and derivations of the results presented in Section I.

A. *Proof of Equation (1)*

Let R_{k,t,T_1} be the return of risky asset k from time t to time T_1 and m_{t,T_1} be the one-period SDF. We show that the conditional expected return of risky assets can be expressed as the risk-neutral covariance between the asset return and the inverse of the SDF m_{t,T_1}. This result is not new, and was derived in Equation (2) of Chabi-Yo and Loudis (2020).

The conditional expected return of asset k can be expressed using the identity

$$\mathbb{E}_t\left(R_{k,t,T_1}\right) = \mathbb{E}_t\left(R_{k,t,T_1}\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}}\frac{m_{t,T_1}}{\mathbb{E}_t m_{t,T_1}}\right). \tag{I1}$$

The ratio $\frac{m_{t,T_1}}{\mathbb{E}_t m_{t,T_1}}$ defines the risk-neutral distribution. Hence, the Radon-Nykodym theorem allows us to express the conditional expected return of asset k as a function of moments under the risk-neutral measure:

$$\begin{aligned}
\mathbb{E}_t\left(R_{k,t,T_1}\right) &= \mathbb{E}_t^*\left(R_{k,t,T_1}\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}}\right) \\
&= \mathbb{COV}_t^*\left(\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}}, R_{k,t,T_1}\right) + \underbrace{\mathbb{E}_t^*\left(\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}}\right)}_{=1}\mathbb{E}_t^*\left(R_{k,t,T_1}\right) \\
&= \mathbb{COV}_t^*\left(\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}}, R_{k,t,T_1}\right) + R_{f,t,T_1}. \qquad (I2)
\end{aligned}$$

This identity is reminiscent of the well-known asset pricing equation in which the expected excess return is negatively related to the covariance between the return and the SDF under the physical measure.

B. *Proof of Equation* (3)

We show that the inverse of the one-period SDF m_{t,T_1} can be expressed as a function of the marginal utility of wealth and expectations under the risk-neutral measure.

The representative agent's optimization problem can be re-written as

$$\max_{\omega_t} \mathbb{E}_t \left(u\left[W_{T_N}\right]\right) = \max_{\omega_t} \mathbb{E}_t \left(\max_{\omega_{T_1}} \mathbb{E}_{T_1}\left(u\left[W_{T_N}\right]\right)\right). \tag{I3}$$

Solving Problem (I3) backward, the first step is to solve

$$\max_{\omega_{T_1}} \mathbb{E}_{T_1}\left(u\left[W_{T_N}\right]\right). \tag{I4}$$

Equation (I4) produces an optimal weight $\omega^*_{T_1}$, and the terminal wealth achieved with this weight is $W^*_{T_N} = W_{T_1}\left(\omega^{*\intercal}_{T_1} R_{T_1,T_N}\right)$. The corresponding one-period SDF from time T_1 to time T_N, m_{T_1,T_N}, has the form

$$m_{T_1,T_N} = \delta_{T_1} u^{'}\left[W^*_{T_N}\right]. \tag{I5}$$

Given the optimal value, $\omega^*_{T_1}$, the second step solves

$$\max_{\omega_t} \mathbb{E}_t \left(\mathbb{E}_{T_1}\left(u\left[W^*_{T_N}\right]\right)\right). \tag{I6}$$

This produces a one-period SDF from time t to time T_1 of the form

$$m_{t,T_1} = \delta_t \mathbb{E}_{T_1}\left(u'\left[W^*_{T_N}\right]\left(\omega^{*\intercal}_{T_1} R_{T_1,T_N}\right)\right). \tag{I7}$$

Equation (I7) gives us the constant δ_t:

$$\delta_t = m_{t,T_1}\left(\mathbb{E}_{T_1}\left(u'\left[W^*_{T_N}\right]\left(\omega^{*\intercal}_{T_1} R_{T_1,T_N}\right)\right)\right)^{-1}. \tag{I8}$$

Because parameter δ_t is a constant, we have $\delta_t = \mathbb{E}_t\delta_t$. We exploit the no-arbitrage conditions that allow us to move from the physical measure to the risk-neutral measure to obtain,

$$\begin{aligned}
\delta_t &= \mathbb{E}_t\left(m_{t,T_1}\left(\mathbb{E}_{T_1}\left(u'\left[W^*_{T_N}\right]\left(\omega^{*\intercal}_{T_1} R_{T_1,T_N}\right)\right)\right)^{-1}\right) \\
&= \mathbb{E}_t\left(m_{t,T_1}\right)\mathbb{E}_t\left(\frac{m_{t,T_1}}{\mathbb{E}_t\left(m_{t,T_1}\right)}\left(\mathbb{E}_{T_1}\left(u'\left[W^*_{T_N}\right]\left(\omega^{*\intercal}_{T_1} R_{T_1,T_N}\right)\right)\right)^{-1}\right) \\
&= \mathbb{E}_t\left(m_{t,T_1}\right)\mathbb{E}^*_t\left(\left(\mathbb{E}_{T_1}\left(u'\left[W^*_{T_N}\right]\left(\omega^{*\intercal}_{T_1} R_{T_1,T_N}\right)\right)\right)^{-1}\right).
\end{aligned} \tag{I9}$$

Next, we replace δ_t by its expression in (I8) and obtain

$$\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}} = \frac{1/\mathbb{E}_{T_1}\left(u'\left[W^*_{T_N}\right]\left(\omega^{*\intercal}_{T_1} R_{T_1,T_N}\right)\right)}{\mathbb{E}^*_t\left(1/\mathbb{E}_{T_1}\left(u'\left[W^*_{T_N}\right]\left(\omega^{*\intercal}_{T_1} R_{T_1,T_N}\right)\right)\right)}. \tag{I10}$$

Similarly, we can use the SDF (I5) and show that

$$\frac{\mathbb{E}_{T_1} m_{T_1,T_N}}{m_{T_1,T_N}} = \frac{1/u'\left[W^*_{T_N}\right]}{\mathbb{E}^*_{T_1}\left(1/u'\left[W^*_{T_N}\right]\right)}. \tag{I11}$$

Next, we write $\mathbb{E}_{T_1}\left(u'\left[W^*_{T_N}\right]\left(\omega^{*\intercal}_{T_1} R_{T_1,T_N}\right)\right)$ in (I10) as a function of risk-neutral quantities:

$$
\begin{aligned}
\mathbb{E}_{T_1}\left(u^{'}\left[W^*_{T_N}\right]\left(\omega^{*\intercal}_{T_1}R_{T_1,T_N}\right)\right) &= \mathbb{E}_{T_1}\left(\frac{m_{T_1,T_N}}{\mathbb{E}_{T_1}m_{T_1,T_N}}\frac{\mathbb{E}_{T_1}m_{T_1,T_N}}{m_{T_1,T_N}}u^{'}\left[W^*_{T_N}\right]\left(\omega^{*\intercal}_{T_1}R_{T_1,T_N}\right)\right) \\
&= \mathbb{E}^*_{T_1}\left(\frac{\mathbb{E}_{T_1}m_{T_1,T_N}}{m_{T_1,T_N}}u^{'}\left[W^*_{T_N}\right]\left(\omega^{*\intercal}_{T_1}R_{T_1,T_N}\right)\right) \\
&= \frac{\omega^{*\intercal}_{T_1}\mathbb{E}^*_{T_1}R_{T_1,T_N}}{\mathbb{E}^*_{T_1}\left(1/u^{'}\left[W^*_{T_N}\right]\right)} = \frac{R_{f,T_1,T_N}}{\mathbb{E}^*_{T_1}\left(1/u^{'}\left[W^*_{T_N}\right]\right)}, \qquad \text{(I12)}
\end{aligned}
$$

where we have used the no-arbitrage conditions to move from the physical measure to the risk-neutral measure in the second equation, and Equation (I11) to obtain the third equation.

We replace (I12) in (I10) to obtain

$$
\begin{aligned}
\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}} &= \frac{\frac{\mathbb{E}^*_{T_1}\left(\frac{1}{u^{'}\left[W^*_{T_N}\right]}\right)}{R_{f,T_1,T_N}}}{\mathbb{E}^*_t\left(\frac{\mathbb{E}^*_{T_1}\left(\frac{1}{u^{'}\left[W^*_{T_N}\right]}\right)}{R_{f,T_1,T_N}}\right)} \\
&= \frac{\left(\left(1/R_{f,T_1,T_N}\right)/\mathbb{E}_t\left(1/R_{f,T_1,T_N}\right)\right)\mathbb{E}^*_{T_1}\left(\frac{u^{'}\left[W_tR_{f,t,T_N}\right]}{u^{'}\left[W^*_{T_N}\right]}\right)}{\mathbb{E}^*_t\left(\left(\left(1/R_{f,T_1,T_N}\right)/\mathbb{E}_t\left(1/R_{f,T_1,T_N}\right)\right)\mathbb{E}^*_{T_1}\left(\frac{u^{'}\left[W_tR_{f,t,T_N}\right]}{u^{'}\left[W^*_{T_N}\right]}\right)\right)}. \qquad \text{(I13)}
\end{aligned}
$$

Since there is no interest rate risk, $1/R_{f,T_1,T_N} = \mathbb{E}_t\left(1/R_{f,T_1,T_N}\right)$, and (I13) simplifies to

$$
\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}} = \frac{\mathbb{E}^*_{T_1}\left(\frac{u^{'}\left[W_tR_{f,t,T_N}\right]}{u^{'}\left[W^*_{T_N}\right]}\right)}{\mathbb{E}^*_t\left(\mathbb{E}^*_{T_1}\left(\frac{u^{'}\left[W_tR_{f,t,T_N}\right]}{u^{'}\left[W^*_{T_N}\right]}\right)\right)}. \qquad \text{(I14)}
$$

Assume that the gross return on the market can be used as proxy for the return on aggregate wealth:

$$R_{M,t,T_N} = \frac{W^*_{T_N}}{W_t} \quad \text{and} \quad R_{M,T_1,T_N} = \frac{W^*_{T_N}}{W_{T_1}} \tag{I15}$$

Equation (I14) can be rewritten as

$$\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}} = \frac{\mathbb{E}^*_{T_1}\left(\frac{u'\left[W_t R_{f,t,T_N}\right]}{u'\left[W_t R_{M,t,T_N}\right]}\right)}{\mathbb{E}^*_t\left(\mathbb{E}^*_{T_1}\left(\frac{u'\left[W_t R_{f,t,T_N}\right]}{u'\left[W_t R_{M,t,T_N}\right]}\right)\right)}. \tag{I16}$$

This ends the proof.

C. *Proof of Equation (4)*

We detail below the second-order expansion of the inverse of the marginal utility, which we use to derive Proposition 1.

Define the function

$$f\left[x,y\right] = \frac{u'\left[W_t x_0 y_0\right]}{u'\left[W_t x y\right]}, \tag{I17}$$

with $x = R_{M,t,T_1}$, $y = R_{M,T_1,T_N}$, $x_0 = R_{f,t,T_1}$ and $y_0 = R_{f,T_1,T_N}$. Since we assume no interest rate risk, $R_{M,t,T_N} = xy$ and $R_{f,t,T_N} = x_0 y_0$.

We write the inverse of the SDF as

$$\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}} = \frac{\mathbb{E}^*_{T_1}\left(f\left[x,y\right]\right)}{\mathbb{E}^*_t\left(\mathbb{E}^*_{T_1}\left(f\left[x,y\right]\right)\right)}. \tag{I18}$$

We adopt the following short notations. First, we use f_x and f_y to denote the first partial derivatives of the function f, f_{xx} and f_{yy} the second partial derivatives, and f_{xy} the cross-derivative, all evaluated at (x_0, y_0). Second, we denote as $u^{'}$, $u^{''}$, and $u^{'''}$ the first, second, and third derivatives of $u[\cdot]$ evaluated at (x_0, y_0). We perform a second-order Taylor expansion series of $f[x, y]$ around $(x, y) = (x_0, y_0)$:

$$\begin{aligned} f[x,y] \approx\ & 1 + \frac{1}{1!}(x - x_0) f_x + \frac{1}{1!}(y - y_0) f_y + \frac{1}{2!}(x - x_0)^2 f_{xx} \\ & + \frac{1}{2!}(y - y_0)^2 f_{yy} + \frac{2}{2!}(x - x_0)(y - y_0) f_{xy}, \end{aligned} \tag{I19}$$

where

$$f_x = \frac{y_0}{x_0} f_y = \frac{1}{x_0}\left(-\frac{(W_t x_0 y_0)\, u^{''}}{u^{'}}\right), \tag{I20}$$

$$f_{xy} = \frac{1}{x_0 y_0}\left(-\frac{W_t x_0 y_0 u^{''}}{u^{'}}\right) + \frac{1}{x_0 y_0}\frac{\left(W_t x_0 y_0 u^{''}\right)^2}{(u^{'})^2}\left(2 - \frac{u^{'''} u^{'}}{(u^{''})^2}\right) \text{ and} \tag{I21}$$

$$f_{xx} = \frac{y_0^2}{x_0^2} f_{yy} = \frac{1}{(x_0)^2}\frac{\left(W_t x_0 y_0 u^{''}\right)^2}{(u^{'})^2}\left(2 - \frac{u^{'''} u^{'}}{(u^{''})^2}\right). \tag{I22}$$

Implicit in this approach is the assumption that the expansion provides a good local approximation in the neighborhood of $(x, y) = (x_0, y_0)$.[1]

[1] The approximation may lose accuracy in the tails of the distribution. One possible refinement would be to partition the support of x (and similarly y) into three regions: the left tail, the center (which includes the safe return), and the right tail. This would produce nine partitions in total, with a separate expansion in each region (as in Chabi-Yo and Loudis (2024), in a static one-period model). While theoretically feasible, this approach introduces up to nine parameters, which could be reduced to five if continuity and differentiability of the inverse marginal utility are imposed. Although possible, the empirical burden of additional parameters is substantial, so we leave this extension to

Note that $f_{xy} = f_{yx}$. Thus, we obtain,

$$\begin{aligned} f[x,y] &\approx 1 + \frac{1}{x_0}\frac{1}{\tau_t}(x-x_0) + \frac{1}{y_0}\frac{1}{\tau_t}(y-y_0) \\ &+ \frac{1}{(x_0)^2}\frac{(1-\rho_t)}{\tau_t^2}(x-x_0)^2 + \frac{1}{(y_0)^2}\frac{(1-\rho_t)}{\tau_t^2}(y-y_0)^2 \\ &+ \frac{1}{x_0 y_0}\left(\frac{1}{\tau_t} + \frac{2(1-\rho_t)}{\tau_t^2}\right)(x-x_0)(y-y_0), \end{aligned} \tag{I23}$$

where τ_t and ρ_t are defined in Equations (7) and (8). Replacing x, x_0, y, and y_0 by their expressions and using preference parameters $a_{1,t}$ and $a_{2,t}$ defined in (6), we obtain

$$\begin{aligned} \mathbb{E}^*_{T_1}(f[x,y]) &\approx 1 + \frac{a_{1,t}}{R_{f,t,T_1}}(R_{M,t,T_1} - R_{f,t,T_1}) + \frac{a_{1,t}}{R_{f,T_1,T_N}}\mathbb{E}^*_{T_1}(R_{M,T_1,T_N} - R_{f,T_1,T_N}) \\ &+ \frac{a_{2,t}}{(R_{f,t,T_1})^2}(R_{M,t,T_1} - R_{f,t,T_1})^2 + \frac{a_{2,t}}{(R_{f,T_1,T_N})^2}\mathbb{E}^*_{T_1}\left((R_{M,T_1,T_N} - R_{f,T_1,T_N})^2\right) \\ &+ \frac{a_{1,t} + 2a_{2,t}}{R_{f,t,T_N}}(R_{M,t,T_1} - R_{f,t,T_1})\,\mathbb{E}^*_{T_1}(R_{M,T_1,T_N} - R_{f,T_1,T_N}). \end{aligned} \tag{I24}$$

Using the martingale condition $\mathbb{E}^*_{T_1}(R_{M,T_1,T_N} - R_{f,T_1,T_N}) = 0$, (I24) simplifies to

$$\mathbb{E}^*_{T_1} f[x,y] = \mathbb{E}^*_{T_1}\left(\frac{u'[W_t R_{f,t,T_1} R_{f,T_1,T_N}]}{u'[W_t R_{M,t,T_1} R_{M,T_1,T_N}]}\right) \approx 1 + z_{T_1}, \tag{I25}$$

where z_{T_1} is given by Equation (5). We then replace Equation (I25) in (I18) to obtain Equation (4).

---

future research.

D. Proof of Proposition 1

We use the expression for the SDF (4) derived in Section I.C, and plug it in the expected return expression identity (1). We obtain

$$
\begin{aligned}
\mathbb{E}_t\left(R_{M,t,T_1}-R_{f,t,T_1}\right) &\approx \mathbb{COV}_t^*\left(R_{M,t,T_1},\frac{1+z_{T_1}}{1+\mathbb{E}_t^* z_{T_1}}\right)\\
&\approx \mathbb{COV}_t^*\left(R_{M,t,T_1}-R_{f,t,T_1},\frac{1+z_{T_1}}{1+\mathbb{E}_t^* z_{T_1}}\right)\\
&\approx \mathbb{E}_t^*\left((R_{M,t,T_1}-R_{f,t,T_1})\frac{1+z_{T_1}}{1+\mathbb{E}_t^* z_{T_1}}\right)
\end{aligned}
\tag{I26}
$$

We then substitute (5) in (I26) and obtain the estimate for the market risk premium in Equation (9).

E. Model with multiple portfolio rebalancing times

The results presented in the previous subsections rely on the assumption that the representative agent can rebalance their portfolio only at time T_1. We now relax this restriction and allow the agent to rebalance at discrete times T_j, $1 \leq j < N$. Gross market and risk-free returns between t and T_N, with $t = T_0 < T_1$, can be written as

$$
R_{M,t,T_N} = \prod_{j=1}^{N} R_{M,T_{j-1},T_j} \text{ and } R_{f,t,T_N} = \prod_{j=1}^{N} R_{f,T_{j-1},T_j}.
$$

Let us denote $x_j = R_{M,T_{j-1},T_j}$ and $x_{0,j} = R_{f,T_{j-1},T_j}$. A second-order Taylor expansion-series of the inverse marginal utility around $(x_1,...,x_N) =$

$(x_{0,1,}, ..., x_{0,N})$ yields

$$z_{T_1} = \frac{a_{1,t}}{x_{0,1}} (x_1 - x_{0,1}) + \frac{a_{2,t}}{x_{0,1}^2} (x_1 - x_{0,1})^2 + a_{2,t} \sum_{j>1}^{N} \frac{1}{x_{0,j}^2} \mathbb{E}_{T_1}^* (x_j - x_{0,j})^2 . \quad \text{(I27)}$$

We replace expression (I27) in (3) and plug it in (1), to derive the equity risk premium in Equation (12).

F. Comparison of the models of Chabi-Yo and Loudis (2020) and Tetlock, McCoy, and Shah (2024)

In this subsection, we show that the SDF of Chabi-Yo and Loudis (2020) nests as a special case the SDF of Tetlock, McCoy, and Shah (2024). This result was also shown in parallel work by Vidal Nunes, Ruas, and Dias (2025).

In the model of Tetlock, McCoy, and Shah (2024), the representative agent can trade the market as well as $K-1$ derivatives securities, with payoffs equal to the higher-order moments of the market returns. The excess return on the market is $\widetilde{R}_{M,t,T_1} = R_{M,t,T_1} - R_{f,t,T_1}$ and the excess returns on derivatives securities are $\widetilde{R}^k_{M,t,T_1} - c_k$, with $c_k = \mathbb{E}_t^* \left(\widetilde{R}^k_{M,t,T_1} \right)$.

Equation (9) of Tetlock, McCoy, and Shah (2024) gives the inverse SDF as follows:

$$\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}} = 1 + \sum_{k=1}^{K} \frac{1}{R_{f,t,T_1}} \omega_{k,t} \left(\widetilde{R}^k_{M,t,T_1} - c_k \right) \quad \text{(I28)}$$

where $\omega_{k,t}$ are the weights of the growth-optimal portfolio, held by logarithmic investors.

In comparison, the SDF of Chabi-Yo and Loudis (2020) is given by Equations (13) and (21) in their paper, as

$$\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}} = \frac{1 + \sum_{k=1}^{K} \widetilde{a}_{k,t} \widetilde{R}_{M,t,T_1}^k}{1 + \sum_{k=1}^{K} \widetilde{a}_{k,t} c_k} \tag{I29}$$

with

$$\widetilde{a}_{k,t} = \frac{a_{k,t}}{R_{f,t,T_1}^k}$$

The SDF (I29) can be expanded as

$$\begin{aligned} \frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}} &= \frac{1}{1 + \sum_{k=1}^{K} \widetilde{a}_{k,t} c_k} + \sum_{k=1}^{K} \left(\frac{\widetilde{a}_{k,t}}{1 + \sum_{k=1}^{K} \widetilde{a}_{k,t} c_k} \right) \widetilde{R}_{M,t,T_1}^k & \text{(I30)} \\ &= 1 + \sum_{k=1}^{K} \left(\frac{\widetilde{a}_{k,t}}{1 + \sum_{k=1}^{K} \widetilde{a}_{k,t} c_k} \right) \left(\widetilde{R}_{M,t,T_1}^k - c_k \right) & \text{(I31)} \end{aligned}$$

Setting the weights as follows:

$$\omega_{k,t} = R_{f,t,T_1} \frac{\widetilde{a}_{k,t}}{1 + \sum_{k=1}^{K} \widetilde{a}_{k,t} c_k}, \tag{I32}$$

we recover the SDF of Tetlock, McCoy, and Shah (2024).

The model of Tetlock, McCoy, and Shah (2024) allows for investors who are not logarithmic investors. The complexity of these investors' portfolios determines the value of K. Consider $K = 2$, for illustration purposes. If the weights $\omega_{k,t}$ for $k = 1, ..., K$ are known (or computed) as it is the case in Tetlock, McCoy, and Shah (2024), then the coefficients $\widetilde{a}_{k,t}$ can be recovered from these weights and from the moments of market returns c_k. The preference parameters τ_t and ρ_t can be uniquely identified from these coefficients. Conversely, if the preference parameters τ_t and ρ_t are known, $\widetilde{a}_{k,t}$ can be recovered and the weights can be computed. The same reasoning extends to any finite $K > 2$.

G. Link to the model of Crescini, Trojani, and Vedolin (2025)

In this subsection, we show that the SDFs of Chabi-Yo and Loudis (2020) and Tetlock, McCoy, and Shah (2024) can be rewritten in the form of the SDF used by Crescini, Trojani, and Vedolin (2025).

From the spanning formula of Carr and Madan (2001), $\widetilde{R}^k_{M,t,T_1}$, for $k > 1$ can be written as a collection of calls and puts:

$$\begin{aligned}\widetilde{R}^k_{M,t,T_1} &= \left(R_{M,t,T_1} - R_{f,t,T_1}\right)^k \\ &= \frac{k\,(k-1)}{S_t^2}\left\{\begin{array}{c} \int_{S_t R_{f,t,T_1}}^{\infty}\left(\frac{K}{S_t} - R_{f,t,T_1}\right)^{k-2}\left(S_{T_1} - K\right)^+ dK \\ + \int_0^{S_t R_{f,t,T_1}}\left(\frac{K}{S_t} - R_{f,t,T_1}\right)^{k-2}\left(K - S_{T_1}\right)^+ dK \end{array}\right\}\end{aligned} \tag{I33}$$

where S_t is the spot market price at time t.

Substituting (I33) in (I28) and (I31), it follows that $\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}}$ is a weighted sum of the excess market return, and payoffs of calls and puts (both tradable) with maturity T_1 and strike K. This SDF can thus be rewritten in form of the SDF of Crescini, Trojani, and Vedolin (2025):

$$\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}} = 1 + \theta' \mathbf{R}^e \tag{I34}$$

where $\mathbf{R}^e = \mathbf{R} - 1$ contains the excess forward return of the index and options. $\theta' \mathbf{R}^e$ is a weighted sum of the excess market return and payoffs of tradable call and put options.

In the model of Crescini, Trojani, and Vedolin (2025), portfolio weights are allowed to vary across investors: $\theta = \theta_i$, to reflect varying demand of investors for the different assets. Logarithmic investors determine the SDF, which has the same form as the one of Tetlock, McCoy, and Shah (2024) and is a special case of Chabi-Yo and Loudis (2020). There is a mapping between the coefficients $a_{k,t}$ of Chabi-Yo and Loudis (2020) and the investors holdings in the vector θ of Equation (I34). Therefore, the coefficients $a_{k,t}$ embed information on aggregate demand for options with maturity T_1.

H. Our inverse SDF as a portfolio of options

We show below that in our setting, the inverse SDF can be written as a portfolio of the market index and options with maturities T_1 and T_N.

Our inverse SDF can be viewed as an extension of the inverse SDFs of Chabi-Yo and Loudis (2020) in Equation (I29), augmented with the conditional risk-neutral variance of the market over $[T_1, T_N]$, $\mathbb{M}^{*(2)}_{T_1,T_N}$. We showed in Appendices I.F and I.G that the inverse SDF of Chabi-Yo and Loudis (2020) can be expressed as a weighted sum of the excess market return and a collection of option payoffs maturing at T_1. As $\mathbb{M}^{*(k)}_{T_1,T_N} = \mathbb{E}^*_{T_1} \widetilde{R}^k_{M,T_1,T_N}$ for $k > 1$ depends on future option prices, the spanning formula implies

$$\begin{aligned} \widetilde{R}^k_{M,T_1,T_N} &= \left(\frac{S_{T_N}}{S_{T_1}} - R_{f,T_1,T_N} \right)^k \\ &= \frac{k\,(k-1)}{S^2_{T_1}} \left\{ \begin{array}{c} \int_{S_{T_1} R_{f,T_1,T_N}}^{\infty} \left(\frac{K}{S_{T_1}} - R_{f,T_1,T_N} \right)^k (S_{T_N} - K)^+ \, dK \\ + \int_0^{S_{T_1} R_{f,T_1,T_N}} \left(\frac{K}{S_{T_1}} - R_{f,T_1,T_N} \right)^k (K - S_{T_N})^+ \, dK \end{array} \right\} \end{aligned} \tag{I35}$$

Thus

$$\mathbb{M}^{*(k)}_{T_1,T_N} = \frac{k\,(k-1)\,R_{f,T_1,T_N}}{S^2_{T_1}} \left\{ \begin{array}{c} \int_{S_{T_1}R_{f,T_1,T_N}}^{\infty} \left(\frac{K}{S_{T_1}} - R_{f,T_1,T_N}\right)^k C_{T_N}[K]\,dK \\ + \int_0^{S_{T_1}R_{f,T_1,T_N}} \left(\frac{K}{S_{T_1}} - R_{f,T_1,T_N}\right)^k P_{T_N}[K]\,dK \end{array} \right\}, \tag{I36}$$

where $C_{T_N}[K]$ and $P_{T_N}[K]$ are the prices of call and put options with strike K and maturity T_N. Therefore, our inverse SDF can be written as a weighted sum of the excess market return, a collection of payoffs on calls and puts and a set of payoffs from strategies that buy call and put options expiring at T_N and sell them at T_1. These strategies arise due to intertemporal hedging in a multi-period economy. The loadings on these strategies, which depend on the coefficients $a_{k,t}$, embed information on aggregate demand for strategies that involve buying options with maturity T_N and selling them at T_1.

II . Proofs and derivations of Section II

A. *Contribution of intertemporal hedging in the model of Epstein and Zin (1989)*

The structure imposed on the utility function, independent of the return process, is central to deriving the implications of intertemporal hedging for the equity risk premium. In this appendix, we detail how intertemporal hedging contributes to the equity risk premium when assuming Epstein and Zin (1989) utility.

Assuming that return on the wealth portfolio is equal to the market return, the SDF takes the form

$$m_{t,T_N} = \delta^{\theta} \left(\frac{C_{t,T_N}}{C_t} \right)^{-\frac{\theta}{\psi}} R_{M,t,T_N}^{\theta-1}, \tag{II37}$$

where δ is the discount factor, C_{t,T_N} is the consumption between times t and T_N, ψ is the elasticity of intertemporal substitution (EIS), and

$$\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$$

with γ the risk aversion parameter.

Let us apply a Taylor-series expansion and use our methodology to demonstrate that intertemporal hedging directly contributes to the equity risk premium.

Consider the return of investing in asset k between t and T_1, and in the risk-free asset between T_1 and T_N: $R_{k,t,T_1} R_{f,T_1,T_N}$. Under no-arbitrage conditions, the conditional expectation of this return can be expressed as

$$\begin{aligned}
\mathbb{E}_t \left(R_{k,t,T_1} R_{f,T_1,T_N} \right) &= \mathbb{E}_t \left(\frac{m_{t,T_N}}{\mathbb{E}_t m_{t,T_N}} \frac{\mathbb{E}_t m_{t,T_N}}{m_{t,T_N}} R_{k,t,T_1} R_{f,T_1,T_N} \right) = \mathbb{E}_t^* \left(\frac{\mathbb{E}_t m_{t,T_N}}{m_{t,T_N}} R_{k,t,T_1} R_{f,T_1,T_N} \right) \\
&= \mathbb{E}_t^* \left(R_{f,T_1,T_N} R_{k,t,T_1} \mathbb{E}_{T_1}^* \left(\frac{\mathbb{E}_t m_{t,T_N}}{m_{t,T_N}} \right) \right) \\
&= R_{f,T_1,T_N} \mathbb{E}_t^* \left(R_{k,t,T_1} \mathbb{E}_{T_1}^* \left(\frac{\mathbb{E}_t m_{t,T_N}}{m_{t,T_N}} \right) \right) \qquad \text{(II38)}
\end{aligned}$$

Therefore we have

$$\mathbb{E}_t \left(R_{k,t,T_1} \right) = \mathbb{E}_t^* \left(R_{k,t,T_1} \frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}} \right) = \mathbb{E}_t^* \left(R_{k,t,T_1} \mathbb{E}_{T_1}^* \left(\frac{\mathbb{E}_t m_{t,T_N}}{m_{t,T_N}} \right) \right). \tag{II39}$$

The uniqueness of the SDF implies that:

$$\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}} = \mathbb{E}^*_{T_1}\left(\frac{\mathbb{E}_t m_{t,T_N}}{m_{t,T_N}}\right) \tag{II40}$$

Let us now calculate the right handside of Equation (II40) using the expression of the Epstein-Zin SDF:

$$m_{t,T_N} = \delta^\theta \left(\frac{C_{t,T_1}}{C_t}\right)^{-\frac{\theta}{\psi}} \left(\frac{C_{T_1,T_N}}{C_{t,T_1}}\right)^{-\frac{\theta}{\psi}} R^{\theta-1}_{M,t,T_1} R^{\theta-1}_{M,T_1,T_N}. \tag{II41}$$

Thus,

$$\frac{1}{m_{t,T_N}} = \delta^{-\theta} \left(\frac{C_{t,T_1}}{C_t}\right)^{\frac{\theta}{\psi}} \left(\frac{C_{T_1,T_N}}{C_{t,T_1}}\right)^{\frac{\theta}{\psi}} R^{1-\theta}_{M,t,T_1} R^{1-\theta}_{M,T_1,T_N}, \tag{II42}$$

and

$$\frac{\mathbb{E}_t m_{t,T_N}}{m_{t,T_N}} = \frac{1}{R_{f,t,T_N}}\delta^{-\theta} \left(\frac{C_{t,T_1}}{C_t}\right)^{\frac{\theta}{\psi}} \left(\frac{C_{T_1,T_N}}{C_{t,T_1}}\right)^{\frac{\theta}{\psi}} R^{1-\theta}_{M,t,T_1} R^{1-\theta}_{M,T_1,T_N}. \tag{II43}$$

Using $\mathbb{E}^*_t\left(\frac{\mathbb{E}_t m_{t,T_N}}{m_{t,T_N}}\right) = 1$ yields

$$\mathbb{E}^*_t\left(\left(\frac{C_{t,T_1}}{C_t}\right)^{\frac{\theta}{\psi}} \left(\frac{C_{T_1,T_N}}{C_{t,T_1}}\right)^{\frac{\theta}{\psi}} R^{1-\theta}_{M,t,T_1} R^{1-\theta}_{M,T_1,T_N}\right) = \delta^\theta R_{f,t,T_N} \tag{II44}$$

and hence

$$\frac{\mathbb{E}_t m_{t,T_N}}{m_{t,T_N}} = \frac{\left(\frac{C_{t,T_1}}{C_t}\right)^{\frac{\theta}{\psi}} \left(\frac{C_{T_1,T_N}}{C_{t,T_1}}\right)^{\frac{\theta}{\psi}} R^{1-\theta}_{M,t,T_1} R^{1-\theta}_{M,T_1,T_N}}{\mathbb{E}^*_t\left(\left(\frac{C_{t,T_1}}{C_t}\right)^{\frac{\theta}{\psi}} \left(\frac{C_{T_1,T_N}}{C_{t,T_1}}\right)^{\frac{\theta}{\psi}} R^{1-\theta}_{M,t,T_1} R^{1-\theta}_{M,T_1,T_N}\right)}. \tag{II45}$$

We then take the expectation at T_1:

$$\mathbb{E}^*_{T_1}\left(\frac{\mathbb{E}_t m_{t,T_N}}{m_{t,T_N}}\right)=\frac{\mathbb{E}^*_{T_1}\left(\left(\frac{C_{t,T_1}}{C_t}\right)^{\frac{\theta}{\psi}}\left(\frac{C_{T_1,T_N}}{C_{t,T_1}}\right)^{\frac{\theta}{\psi}}R^{1-\theta}_{M,t,T_1}R^{1-\theta}_{M,T_1,T_N}\right)}{\mathbb{E}^*_t\left(\mathbb{E}^*_{T_1}\left(\left(\frac{C_{t,T_1}}{C_t}\right)^{\frac{\theta}{\psi}}\left(\frac{C_{T_1,T_N}}{C_{t,T_1}}\right)^{\frac{\theta}{\psi}}R^{1-\theta}_{M,t,T_1}R^{1-\theta}_{M,T_1,T_N}\right)\right)}. \tag{II46}$$

Now, we perform an expansion series of $\left(\frac{C_{t,T_1}}{C_t}\right)^{\frac{\theta}{\psi}}\left(\frac{C_{T_1,T_N}}{C_{t,T_1}}\right)^{\frac{\theta}{\psi}}R^{1-\theta}_{M,t,T_1}R^{1-\theta}_{M,T_1,T_N}$ around[2]

$$\left(\frac{C_{t,T_1}}{C_t},\frac{C_{T_1,T_N}}{C_{t,T_1}},R_{M,t,T_1},R_{M,T_1,T_N}\right)=\left(1,1,R_{f,t,T_1},R_{f,T_1,T_N}\right). \tag{II47}$$

Taking the T_1- conditional expectation of this expansion-series produces

$$\begin{aligned}
&\mathbb{E}^*_{T_1}\left(\left(\frac{C_{t,T_1}}{C_t}\right)^{\frac{\theta}{\psi}}\left(\frac{C_{T_1,T_N}}{C_{t,T_1}}\right)^{\frac{\theta}{\psi}}R^{1-\theta}_{M,t,T_1}R^{1-\theta}_{M,T_1,T_N}\right)\\
&=A_0+A_1\left(\frac{C_{t,T_1}}{C_t}-1\right)+A_2\left(\mathbb{E}^*_{T_1}\frac{C_{T_1,T_N}}{C_{t,T_1}}-1\right)+A_3\left(R_{M,t,T_1}-R_{f,t,T_1}\right)\\
&+A_4\left(\frac{C_{t,T_1}}{C_t}-1\right)\left(R_{M,t,T_1}-R_{f,t,T_1}\right)+A_5\left(\frac{C_{t,T_1}}{C_t}-1\right)\left(\mathbb{E}^*_{T_1}\frac{C_{T_1,T_N}}{C_{t,T_1}}-1\right)\\
&+A_6\left(\frac{C_{T_1,T_N}}{C_{t,T_1}}-1\right)\left(R_{M,t,T_1}-R_{f,t,T_1}\right)+A_7\mathbb{E}^*_{T_1}\left(\frac{C_{T_1,T_N}}{C_{t,T_1}}-1\right)\left(R_{M,T_1,T_N}-R_{f,T_1,T_N}\right)\\
&+A_8\left(\frac{C_{t,T_1}}{C_t}-1\right)^2+A_9\mathbb{E}^*_{T_1}\left(\frac{C_{T_1,T_N}}{C_{t,T_1}}-1\right)^2+A_{10}\left(R_{M,t,T_1}-R_{f,t,T_1}\right)^2\\
&+A_{11}\mathbb{E}^*_{T_1}\left(R_{M,T_1,T_N}-R_{f,T_1,T_N}\right)^2
\end{aligned} \tag{II48}$$

[2]An expansion series around $\left(\mathbb{E}^*_t\frac{C_{t,T_1}}{C_t},\mathbb{E}^*_{T_1}\frac{C_{T_1,T_N}}{C_{t,T_1}},R_{f,t,T_1},R_{f,T_1,T_N}\right)$ has similar implications.

with

$$A_0 = R_{f,t,T_1}^{1-\theta} R_{f,T_1,T_N}^{1-\theta}, \qquad A_1 = A_2 = \frac{\theta}{\psi} R_{f,t,T_1}^{1-\theta} R_{f,T_1,T_N}^{1-\theta},$$
$$A_3 = (1-\theta)\, R_{f,t,T_1}^{-\theta} R_{f,T_1,T_N}^{1-\theta}, \qquad A_4 = A_6 = \frac{\theta\,(1-\theta)}{\psi} R_{f,t,T_1}^{1-\theta} R_{f,T_1,T_N}^{1-\theta}$$
$$A_5 = \left(\frac{\theta}{\psi}\right)^2 R_{f,t,T_1}^{1-\theta} R_{f,T_1,T_N}^{1-\theta}, \qquad A_7 = \frac{\theta\,(1-\theta)}{\psi} R_{f,t,T_1}^{1-\theta} R_{f,T_1,T_N}^{1-\theta},$$
$$A_8 = A_9 = \frac{1}{2}\left(\frac{\theta}{\psi}\right)\left(\frac{\theta}{\psi}-1\right) R_{f,t,T_1}^{1-\theta} R_{f,T_1,T_N}^{1-\theta}$$
$$A_{10} = \frac{1}{2}(\theta-1)\,\theta R_{f,t,T_1}^{-(\theta+1)} R_{f,T_1,T_N}^{1-\theta}, \qquad A_{11} = \frac{1}{2}(\theta-1)\,\theta R_{f,t,T_1}^{1-\theta} R_{f,T_1,T_N}^{-(\theta+1)} \tag{II49}$$

Hence the inverse SDF takes the form

$$\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}} = \frac{Z_{T_1}}{\mathbb{E}_t^* Z_{T_1}} \tag{II50}$$

with

$$\begin{aligned} Z_{T_1} = \;& 1 + (A_1/A_0)\left(\frac{C_{t,T_1}}{C_t}-1\right) + (A_2/A_0)\left(\mathbb{E}_{T_1}^* \frac{C_{T_1,T_N}}{C_{T_1}}-1\right) + (A_3/A_0)\left(R_{M,t,T_1}-R_{f,t,T_1}\right) \\ & + (A_4/A_0)\left(\frac{C_{t,T_1}}{C_t}-1\right)\left(R_{M,t,T_1}-R_{f,t,T_1}\right) + (A_5/A_0)\left(\frac{C_{t,T_1}}{C_t}-1\right)\left(\mathbb{E}_{T_1}^* \frac{C_{T_1,T_N}}{C_{T_1}}-1\right) \\ & + (A_6/A_0)\left(R_{M,t,T_1}-R_{f,t,T_1}\right)\mathbb{E}_{T_1}^*\left(\frac{C_{T_1,T_N}}{C_{T_1}}-1\right) \\ & + (A_7/A_0)\,\mathbb{E}_{T_1}^*\left(\frac{C_{T_1,T_N}}{C_{T_1}}-1\right)\left(R_{M,T_1,T_N}-R_{f,T_1,T_N}\right) + (A_8/A_0)\left(\frac{C_{t,T_1}}{C_t}-1\right)^2 \\ & + (A_9/A_0)\,\mathbb{E}_{T_1}^*\left(\frac{C_{T_1,T_N}}{C_{T_1}}-1\right)^2 + (A_{10}/A_0)\left(R_{M,t,T_1}-R_{f,t,T_1}\right)^2 \\ & + (A_{11}/A_0)\,\mathbb{VAR}_{T_1}^*\left(R_{M,T_1,T_N}\right). \end{aligned} \tag{II51}$$

The Epstein–Zin formulation of the SDF (II41) directly produces an SDF expressed as a function of market returns and consumption growth, without volatility playing any direct role. To highlight the impact of volatility on the SDF, a structure is often imposed on the return process. In contrast, in our approach, we do not impose any constraint on the return process. A Taylor-series expansion of marginal utility makes it transparent how market volatility affects the SDF, through the term $(A_{10}/A_0)\left(R_{M,t,T_1} - R_{f,t,T_1}\right)^2$. Similarly, market volatility after T_1 affects the SDF through the term $(A_{11}/A_0)\,\mathbb{VAR}^*_{T_1}\left(R_{M,T_1,T_N}\right)$.

Using this inverse SDF and applying (I2), the expected excess return on the market is

$$\mathbb{E}\left(R_{M,t,T_1} - R_{f,t,T_1}\right) = \frac{\mathbb{COV}^*_t\left(R_{M,t,T_1}, Z_{T_1}\right)}{\mathbb{E}^*_t Z_{T_1}}. \tag{II52}$$

The effect of intertemporal hedging on the equity risk premium happens through the term $\mathbb{COV}^*_t\left(R_{M,t,T_1}, \mathbb{VAR}^*_{T_1}\left(R_{M,T_1,T_N}\right)\right)$. This effect is given by

$$\frac{A_{11}}{A_0}\frac{\mathbb{COV}^*_t\left(R_{M,t,T_1}, \mathbb{VAR}^*_{T_1}\left(R_{M,T_1,T_N}\right)\right)}{\mathbb{E}^*_t Z_{T_1}}, \tag{II53}$$

with

$$\frac{A_{11}}{A_0} = \frac{1}{2}\frac{(\theta-1)\,\theta}{R^2_{f,T_1,T_N}} = \frac{1}{2}\frac{1}{R^2_{f,T_1,T_N}}\frac{\psi\gamma\,(\gamma-1)\left(\psi-\frac{1}{\gamma}\right)}{(\psi-1)^2} \tag{II54}$$

When θ is between 0 and 1, $(\theta-1)\,\theta < 0$, and intertemporal hedging increases the equity risk premium through (II53) if $\mathbb{COV}^*_t\left(R_{M,t,T_1}, \mathbb{VAR}^*_{T_1}\left(R_{M,T_1,T_N}\right)\right) < 0$.

In contrast, when $(\theta-1)\,\theta > 0$, intertemporal hedging decreases the equity risk premium through (II53) if $\mathbb{COV}^*_t\left(R_{M,t,T_1}, \mathbb{VAR}^*_{T_1}\left(R_{M,T_1,T_N}\right)\right) < 0$. This result holds true, in particular, when $\gamma > 1$ and $\psi > 1$, which implies that θ is negative and $A_{11} > 0$.

B. Contribution of intertemporal hedging in the model of Campbell and Cochrane (1999)

In the model of Campbell and Cochrane (1999) with external habit formation, consumption and wealth growth are linked through the surplus ratio and the return on aggregate wealth:

$$\frac{C_{t,T_N}}{C_t} = \frac{s_{t,T_N}}{s_t} R_{M,t,T_N} \tag{II55}$$

where s_t is the surplus consumption ratio at time t, and s_{t,T_N} is the change in surplus consumption ratio between t and T_N (habit effect).

Plugging this relation into the Epstein-Zin SDF links the SDF directly to market returns and habit dynamics:

$$m_{t,T_N} = \delta^{\theta} \left(\frac{s_{t,T_N}}{s_t} \right)^{-\frac{\theta}{\psi}} R_{M,t,T_N}^{\theta-1-\frac{\theta}{\psi}}, \tag{II56}$$

Campbell and Cochrane (1999) show that this representation of the SDF produces countercyclical risk aversion and time-varying volatility premia consistent with the data.

Using the same steps as in Appendix Section II.A produces

$$
\begin{aligned}
Z_{T_1} =& 1 + (B_1/A_0)\left(\frac{s_{t,T_1}}{s_t} - 1\right) + (B_2/A_0)\left(\mathbb{E}^*_{T_1}\frac{s_{T_1,T_N}}{s_{T_1}} - 1\right) \\
&+ (B_3/A_0)\left(R_{M,t,T_1} - R_{f,t,T_1}\right) + (B_4/A_0)\left(\frac{s_{t,T_1}}{s_t} - 1\right)\left(R_{M,t,T_1} - R_{f,t,T_1}\right) \\
&+ (B_5/A_0)\left(\frac{s_{t,T_1}}{s_t} - 1\right)\left(\mathbb{E}^*_{T_1}\frac{s_{T_1,T_N}}{s_{T_1}} - 1\right) + (B_6/A_0)\left(R_{M,t,T_1} - R_{f,t,T_1}\right)\mathbb{E}^*_{T_1}\left(\frac{s_{T_1,T_N}}{s_{T_1}} - 1\right) \\
&+ (B_7/A_0)\,\mathbb{E}^*_{T_1}\left(\frac{s_{T_1,T_N}}{s_{T_1}} - 1\right)\left(R_{M,T_1,T_N} - R_{f,T_1,T_N}\right) + (B_8/A_0)\left(\frac{s_{t,T_1}}{s_t} - 1\right)^2 \\
&+ (B_9/A_0)\,\mathbb{E}^*_{T_1}\left(\frac{s_{T_1,T_N}}{s_{T_1}} - 1\right)^2 + (B_{10}/A_0)\left(R_{M,t,T_1} - R_{f,t,T_1}\right)^2 \\
&+ (B_{11}/A_0)\,\mathbb{VAR}^*_{T_1}\left(R_{M,T_1,T_N}\right). \qquad \text{(II57)}
\end{aligned}
$$

with

$$
\begin{aligned}
&B_0 = R_{f,t,T_1}^{1-\theta+\frac{\theta}{\psi}} R_{f,T_1,T_N}^{1-\theta+\frac{\theta}{\psi}}, && B_1 = B_2 = \frac{\theta}{\psi} R_{f,t,T_1}^{1-\theta+\frac{\theta}{\psi}} R_{f,T_1,T_N}^{1-\theta+\frac{\theta}{\psi}}, \\
&B_3 = \left(1-\theta+\frac{\theta}{\psi}\right) R_{f,t,T_1}^{-\theta+\frac{\theta}{\psi}} R_{f,T_1,T_N}^{1-\theta+\frac{\theta}{\psi}}, && B_4 = B_7 = \frac{\theta\left(1-\theta+\frac{\theta}{\psi}\right)}{\psi} R_{f,t,T_1}^{1-\theta+\frac{\theta}{\psi}} R_{f,T_1,T_N}^{1-\theta+\frac{\theta}{\psi}} \\
&B_5 = \left(\frac{\theta}{\psi}\right)^2 R_{f,t,T_1}^{1-\theta+\frac{\theta}{\psi}} R_{f,T_1,T_N}^{1-\theta+\frac{\theta}{\psi}}, && B_6 = \frac{\theta\left(1-\theta+\frac{\theta}{\psi}\right)}{\psi} R_{f,t,T_1}^{1-\theta} R_{f,T_1,T_N}^{1-\theta}, \\
&B_8 = B_9 = \frac{1}{2}\left(\frac{\theta}{\psi}\right)\left(\frac{\theta}{\psi} - 1\right) R_{f,t,T_1}^{1-\theta+\frac{\theta}{\psi}} R_{f,T_1,T_N}^{1-\theta+\frac{\theta}{\psi}}, && B_{10} = \frac{1}{2}\left(1-\theta+\frac{\theta}{\psi}\right)\left(-\theta+\frac{\theta}{\psi}\right) R_{f,t,T_1}^{-\theta+\frac{\theta}{\psi}-1} \\
&B_{11} = \frac{1}{2}\left(1-\theta+\frac{\theta}{\psi}\right)\left(-\theta+\frac{\theta}{\psi}\right) R_{f,t,T_1}^{1-\theta+\frac{\theta}{\psi}} R_{f,T_1,T_N}^{-\theta+\frac{\theta}{\psi}-1}. && \text{(II58)}
\end{aligned}
$$

The contribution of intertemporal hedging to the equity risk premium happens through the term $\mathbb{COV}^*_t\left(R_{M,t,T_1}, \mathbb{VAR}^*_{T_1}\left(R_{M,T_1,T_N}\right)\right)$. This contribution is given by

$$
\frac{B_{11}}{A_0}\frac{\mathbb{COV}^*_t\left(R_{M,t,T_1}, \mathbb{VAR}^*_{T_1}\left(R_{M,T_1,T_N}\right)\right)}{\mathbb{E}^*_t Z_{T_1}}, \qquad \text{(II59)}
$$

with

$$\frac{B_{11}}{A_0} = \frac{1}{2} \frac{\left(\theta - \frac{\theta}{\psi} - 1\right)\left(\theta - \frac{\theta}{\psi}\right)}{R^2_{f,T_1,T_N}} = \frac{1}{2} \frac{\gamma\left(\gamma - 1\right)}{R^2_{f,T_1,T_N}}$$

Interestingly, the sign of $\frac{B_{11}}{A_0}$ does not depend on ψ, which simplifies out of the equation. Hence, in this model, assuming that the leverage effect $\mathbb{COV}^*_t\left(R_{M,t,T_1}, \mathbb{VAR}^*_{T_1}\left(R_{M,T_1,T_N}\right)\right) < 0$, if $\gamma > 1$, then $\gamma\left(\gamma - 1\right) > 0$ and hence intertemporal hedging decreases the equity risk premium through (II59). If, in turn, $\gamma < 1$, then intertemporal hedging increases the equity risk premium.

C. Recursive utility with a general aggregator

In this section, we use a Taylor-series expansion to characterize the contribution of intertemporal hedging to the total equity risk premium without imposing a specific functional form on the aggregator and on the certainty equivalent, as it is done by Epstein and Zin (1989, 1991). We show that the direction of the relationship between intertemporal hedging and the equity risk premium is dictated by the curvature properties of the aggregator and certainty-equivalent functions. When both take a power form, the shape of these functions is determined by a single parameter, and intertemporal hedging reduces the equity risk premium under common preference parameter calibrations. Epstein and Zin used power forms for tractability.

Plugging the functional forms (20) in (18) and (19), the Epstein–Zin aggregator is given by

$$U_t = \left(\left(1 - \beta\right) C_t^{1-\frac{1}{\psi}} + \beta\left(\mu_t\left[U_{t+1}\right]\right)^{1-\frac{1}{\psi}}\right)^{\frac{1}{1-\frac{1}{\psi}}} \tag{II60}$$

and the certainty equivalent $\mu_t [U_{t+1}]$ is

$$\mu_t [U_{t+1}] = \left(\mathbb{E}_t \left[U_{t+1}^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}} . \tag{II61}$$

Both the certainty equivalent $\mu_t [U_{t+1}]$ and the aggregator U_t take a Constant Elasticity of Substitution (CES) form, similar to the CRRA specification.

Let us now consider the more general recursive preferences of Equations (18)-(19), and assume that the functional forms of $\vartheta [.]$, $g [.]$, and $\upsilon [.]$ are not known and $\vartheta [.] \neq \upsilon [.]$. Assuming regularity conditions on $\vartheta [.]$, $g [.]$, and $\upsilon [.]$, the SDF over the period $[t, T_N]$ is

$$\begin{aligned} m_{t,T_N} &\propto \beta \frac{g^{'} [C_{T_N}]}{g^{'} [C_t]} \frac{\vartheta^{'} [U_t]}{\vartheta^{'} [U_{T_N}]} \frac{\upsilon^{'} [U_{T_N}]}{\upsilon^{'} [\mu_t]} \\ &\propto \beta \frac{g^{'} [C_{T_N}]}{g^{'} [C_t]} \frac{\vartheta^{'} [\mu_t]}{\vartheta^{'} [U_{T_N}]} \frac{\upsilon^{'} [U_{T_N}]}{\upsilon^{'} [\mu_t]} = \frac{const}{Z_{T_N}} \end{aligned} \tag{II62}$$

with

$$Z_{T_N} = \frac{g^{'} [C_t]}{g^{'} [C_{T_N}]} \frac{\vartheta^{'} [U_{T_N}]}{\vartheta^{'} [\mu_t]} \frac{\upsilon^{'} [\mu_t]}{\upsilon^{'} [U_{T_N}]} \tag{II63}$$

From Equation (II62) we have

$$const = m_{t,T_N} Z_{T_N} = \mathbb{E}_t m_{t,T_N} Z_{T_N} = \mathbb{E}_t m_{t,T_N} \mathbb{E}_t^* Z_{T_N} \tag{II64}$$

Hence

$$\frac{\mathbb{E}_t m_{t,T_N}}{m_{t,T_N}} = \frac{Z_{T_N}}{\mathbb{E}_t^* Z_{T_N}}$$

Now, use Equation (II40) to infer

$$\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}} = \mathbb{E}^*_{T_1} \frac{\mathbb{E}_t m_{t,T_N}}{m_{t,T_N}} = \frac{\mathbb{E}^*_{T_1} Z_{T_N}}{\mathbb{E}^*_t Z_{T_N}}. \tag{II65}$$

Z_{T_N} depends on C_{T_N} and U_{T_N}. U_{T_N}, in turn, depends on the market return: $U_{T_N} = U_{T_N}[W_{T_N}]$ with $W_{T_N} = (W_t - C_t) R_{M,t,T_N}$. A second order Taylor expansion-series of $\frac{U_{T_N}[W_{T_N}]}{\mu_t}$ around $w_0 = (W_t - C_t) R_{f,t,T_N}$ produces

$$\frac{U_{T_N}}{\mu_t} = \underbrace{\frac{U_{T_N}[w_0]}{\mu_t}}_{\lambda_{0,t}} + \underbrace{\frac{(W_t - C_t) U'_{T_N}[w_0]}{\mu_t}}_{\lambda_{1,t}} (R_{M,t,T_N} - R_{f,t,T_N}) + \underbrace{\frac{(W_t - C_t)^2 U''_{T_N}[w_0]}{\mu_t}}_{\lambda_{2,t}} (R_{M,t,T_N} - R_{f,t,T_N})^2 \tag{II66}$$

which simplifies to

$$\frac{U_{T_N}}{\mu_t} = 1 + \lambda^*_{1,t} \left(R_{M,t,T_N} - R^*_{f,t,T_N}\right) + \lambda_{2,t} \left(R_{M,t,T_N} - R^*_{f,t,T_N}\right)^2 \tag{II67}$$

where R^*_{f,t,T_N} and $\lambda^*_{1,t}$ are chosen so that Equations (II66) and (II67) match:[3]

$$\begin{aligned} \lambda^*_{1,t} &= \lambda_{1,t} - 2\lambda_{2,t}(R_{f,t,T_N} - R^*_{f,t,T_N}) \quad \text{and} \\ R^*_{f,t,T_N} &= R_{f,t,T_N} - \frac{\lambda_{1,t} \pm \sqrt{\lambda^2_{1,t} - 4\lambda_{2,t}(\lambda_{0,t} - 1)}}{2\lambda_{2,t}}. \end{aligned} \tag{II68}$$

Now that we have rewritten U_{T_N} as a function of $R_{M,t,T_N} = R_{M,t,T_1} R_{M,T_1,T_N}$, as a result, Z_{T_N} depends on $(C_{T_N}, R_{M,t,T_1}, R_{M,T_1,T_N})$. A second-order Taylor

[3]We assume that (II66) can be rewritten into (II67) for tractability. The matching can be done if $\lambda^2_{1,t} - 4\lambda_{2,t}(\lambda_{0,t} - 1) \geq 0$, $\lambda_{2,t} < 0$ and the resulting $R^*_{f,t,T_N} \geq 0$. If these conditions are not satisfied, the calculations are more cumbersome but the same reasoning holds.

expansion-series of Z_{T_N} around $(C_{T_N}, R_{M,t,T_1}, R_{M,T_1,T_N}) = \left(C_t, R^*_{f,t,T_1}, R^*_{f,T_1,T_N}\right)$ yields

$$\begin{aligned} Z_{T_N} &= 1 + H + \lambda\left(R_{M,T_1,T_N} - R^*_{f,T_1,T_N}\right)^2 \\ &= 1 + H + \lambda\left(R_{M,T_1,T_N} - R_{f,T_1,T_N} + R_{f,T_1,T_N} - R^*_{f,T_1,T_N}\right)^2 \quad \text{(II69)} \end{aligned}$$

where H contains all the terms of the expansion-series except $\lambda\left(R_{M,T_1,T_N} - R_{f,T_1,T_N}\right)^2$. Since

$$\mathbb{E}^*_{T_1} Z_{T_N} = 1 + \lambda\left(R_{f,T_1,T_N} - R^*_{f,T_1,T_N}\right)^2 + \mathbb{E}^*_{T_1} H + \lambda \mathbb{E}^*_{T_1}\left(R_{M,T_1,T_N} - R_{f,T_1,T_N}\right)^2, \quad \text{(II70)}$$

we have

$$\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}} = \frac{\left\{1 + \lambda\left(R_{f,T_1,T_N} - R^*_{f,T_1,T_N}\right)^2 + \mathbb{E}^*_{T_1} H + \lambda \mathbb{E}^*_{T_1}\left(R_{M,T_1,T_N} - R_{f,T_1,T_N}\right)^2\right\}}{\left\{\begin{array}{c} 1 + \lambda\left(R_{f,T_1,T_N} - R^*_{f,T_1,T_N}\right)^2 \\ +\mathbb{E}^*_t\left(\mathbb{E}^*_{T_1} H\right) + \lambda \mathbb{E}^*_t\left(\mathbb{E}^*_{T_1}\left(R_{M,T_1,T_N} - R_{f,T_1,T_N}\right)^2\right) \end{array}\right\}}$$

Our goal is to determine λ because the contribution of leverage to the equity risk premium is $\lambda \mathbb{COV}^*_t\left(R_{M,t,T_1}, \mathbb{E}^*_{T_1}\left(R_{M,T_1,T_N} - R_{f,T_1,T_N}\right)^2\right)$. Let us set $(x, y) = (R_{M,t,T_1}, R_{M,T_1,T_N})$ and $(x^*_0, y^*_0) = \left(R^*_{f,t,T_1}, R^*_{f,T_1,T_N}\right)$.

$$\lambda = \frac{1}{2}\left\{\frac{\partial^2\left(\frac{\vartheta'\left[U_{T_N}\right]}{\vartheta'[\mu_t]}\right)}{\partial^2 R_{M,T_1,T_N}} + 2\frac{\partial\left(\frac{\vartheta'\left[U_{T_N}\right]}{\vartheta'[\mu_t]}\right)}{\partial R_{M,T_1,T_N}}\frac{\partial\left(\frac{\upsilon'[\mu_t]}{\upsilon'\left[U_{T_N}\right]}\right)}{\partial R_{M,T_1,T_N}} + \frac{\partial^2\left(\frac{\upsilon'[\mu_t]}{\upsilon'\left[U_{T_N}\right]}\right)}{\partial^2 R_{M,T_1,T_N}}\right\}_{(x,y)=\left(x_0^*,y_0^*\right)} \tag{II71}$$

$$\begin{aligned}
&= \frac{1}{2}\left\{\begin{array}{c} \frac{\vartheta'''\left[U_{T_N}\right]}{\vartheta'[\mu_t]}\left(U'_{T_N}\right)^2 + \frac{\vartheta''\left[U_{T_N}\right]}{\vartheta'[\mu_t]}U''_{T_N} - 2\frac{\vartheta''\left[U_{T_N}\right]}{\vartheta'[\mu_t]}\frac{\upsilon''\left[U_{T_N}\right]\upsilon'[\mu_t]}{\left(\upsilon'\left[U_{T_N}\right]\right)^2}\left(U'_{T_N}\right)^2 \\ -\frac{\upsilon'''\left[U_{T_N}\right]\upsilon'[\mu_t]}{\left(\upsilon'\left[U_{T_N}\right]\right)^2}\left(U'_{T_N}\right)^2 + 2\frac{\left(\upsilon''\left[U_{T_N}\right]\right)^2\upsilon'[\mu_t]}{\left(\upsilon'\left[U_{T_N}\right]\right)^3}\left(U'_{T_N}\right)^2 - \frac{\upsilon''\left[U_{T_N}\right]\upsilon'[\mu_t]}{\left(\upsilon'\left[U_{T_N}\right]\right)^2}U''_{T_N} \end{array}\right\}_{(x,y)=\left(x_0^*,y_0^*\right)} \\
&= \frac{1}{2}\left\{\left(\begin{array}{c} \frac{\vartheta'''[\mu_t]}{\vartheta'[\mu_t]} - 2\frac{\vartheta''[\mu_t]}{\vartheta'[\mu_t]}\frac{\upsilon''[\mu_t]}{\upsilon'[\mu_t]} \\ -\frac{\upsilon'''\left[U_{T_N}\right]}{\upsilon'[\mu_t]} + 2\frac{\left(\upsilon''[\mu_t]\right)^2}{\left(\upsilon'[\mu_t]\right)^2} \end{array}\right)\mu_t^2\lambda_{1,t}^{*2} + \left(\frac{\vartheta''[\mu_t]}{\vartheta'[\mu_t]} - \frac{\upsilon''[\mu_t]}{\upsilon'[\mu_t]}\right)2\mu_t\lambda_{2,t}\right\}_{(x,y)=\left(x_0^*,y_0^*\right)} \\
&= \left(\frac{\rho_t^\vartheta}{\left(\tau_t^\vartheta\right)^2} - \frac{1}{\tau_t^\upsilon}\frac{1}{\tau_t^\vartheta} + \frac{\left(1-\rho_t^\upsilon\right)}{\left(\tau_t^\upsilon\right)^2}\right)\lambda_{1,t}^{*2} + \left(\frac{1}{\tau_t^\upsilon} - \frac{1}{\tau_t^\vartheta}\right)\lambda_{2,t} \\
&= \left(\left(\frac{1}{\tau_t^\vartheta} - \frac{1}{\tau_t^\upsilon}\right)\frac{1}{\tau_t^\vartheta} + \frac{\left(1-\rho_t^\upsilon\right)}{\left(\tau_t^\upsilon\right)^2} - \frac{\left(1-\rho_t^\vartheta\right)}{\left(\tau_t^\vartheta\right)^2}\right)\lambda_{1,t}^{*2} + \left(\frac{1}{\tau_t^\upsilon} - \frac{1}{\tau_t^\vartheta}\right)\lambda_{2,t}
\end{aligned} \tag{II72}$$

where

$$\frac{1}{\tau_t^\vartheta} = -\frac{\mu_t\vartheta''[\mu_t]}{\vartheta'[\mu_t]},\ \rho_t^\vartheta = \frac{1}{2}\frac{\vartheta'[\mu_t]\,\vartheta'''[\mu_t]}{\left(\vartheta''[\mu_t]\right)^2},\ \frac{1}{\tau_t^\upsilon} = -\frac{\mu_t\upsilon''[\mu_t]}{\upsilon'[\mu_t]},\ \rho_t^\upsilon = \frac{1}{2}\frac{\upsilon'[\mu_t]\,\upsilon'''[\mu_t]}{\left(\upsilon''[\mu_t]\right)^2}$$

The sign of λ is therefore determined by the four preference parameters τ_t^ϑ, τ_t^υ, ρ_t^ϑ and ρ_t^υ, which capture the shape of the aggregator and the certainty equivalent. λ can be either positive or negative, depending on these four parameters.

In the case of Epstein-Zin utility,

$$\frac{U_{T_N}}{\mu_t} = (R_{M,t,T_N})^{\xi(\gamma,\psi)} \text{ with } \xi(\gamma,\psi) = \frac{1-\frac{1}{\psi}-\gamma}{\frac{1}{\psi}\left(\frac{1}{\psi}-\gamma\right)} \text{and}$$

$$\frac{1}{\tau_t^{\vartheta}} = \frac{1}{\psi},\ \frac{1}{\tau_t^{\upsilon}} = \gamma,\ \rho_t^{\vartheta} = \frac{\frac{1}{\psi}+1}{\frac{1}{\psi}} \text{ and } \rho_t^{\upsilon} = \frac{\gamma+1}{\gamma}.$$

This implies that λ only depends on two parameters, γ and ψ:

$$\lambda = \left(\frac{1}{2\psi}\left(\frac{1}{\psi}+1\right) - \frac{\gamma}{\psi} + \gamma^2\left(1-\frac{1}{2}\frac{(\gamma+1)}{\gamma}\right)\right)\xi(\gamma,\psi)^2 + \left(\gamma-\frac{1}{\psi}\right)\xi(\gamma,\psi)\left(\xi(\gamma,\psi)-1\right) \tag{II73}$$

Using standard calibrations yields

$$\gamma = 10,\ \psi = 2 \text{ and } \lambda \approx 180.5$$

$$\gamma = 10,\ \psi = 1 \text{ and } \lambda \approx 45.56$$

$$\gamma = 10,\ \psi = 3 \text{ and } \lambda \approx 404.55.$$

Since $\lambda > 0$ in these calibrations, under Epstein Zin preferences, intertemporal hedging decreases the equity risk premium. But in the general case, λ as given by Equation (II72) is not constrained to be positive.

D. *Recursive utility in the case of small departures from Epstein-Zin utility*

Assuming that regularity conditions are satisfied on $g[.]$, $\vartheta[.]$ and $\upsilon[.]$, we now consider the case in which these functions are first-order departures from the power forms used in Epstein Zin utility. Let us denote

$$g[x] = f_g[g_{EZ}[x]], \ \vartheta[x] = f_\vartheta[\vartheta_{EZ}[x]], \ \upsilon[x] = f_\upsilon[\upsilon_{EZ}[x]] \tag{II74}$$

where ϑ_{EZ}, g_{EZ} and υ_{EZ} are the function forms of Epstein Zin preferences, given in Equation (20). Since $\vartheta_{EZ}[1] = 1$, $g_{EZ}[1] = 1$ and $\upsilon_{EZ}[1] = 1$, we set $g[1] = 1$, $\vartheta[1] = 1$ and $\upsilon[1] = 1$ as well for simplicity. Now let us revisit the coefficient λ derived in Equation (II71), which captures the contribution of the leverage term to the equity risk premium. Notice that

$$\begin{aligned}
\frac{\partial\left(\frac{\vartheta'[U_{T_N}]}{\vartheta'[\mu_t]}\right)}{\partial R_{M,T_1,T_N}} &= \frac{\partial\left(\frac{f'_\vartheta[\vartheta_{EZ}[U_{T_N}]]\vartheta'_{EZ}[U_{T_N}]}{f'_\vartheta[\vartheta_{EZ}[\mu_t]]\vartheta'_{EZ}[\mu_t]}\right)}{\partial R_{M,T_1,T_N}} \\
&= \frac{f''_\vartheta[\vartheta_{EZ}[U_{T_N}]]\left(\vartheta'_{EZ}[U_{T_N}]\right)^2}{f'_\vartheta[\vartheta_{EZ}[\mu_t]]\vartheta'_{EZ}[\mu_t]}U'_{T_N} + \frac{f'_\vartheta[\vartheta_{EZ}[U_{T_N}]]\vartheta''_{EZ}[U_{T_N}]}{f'_\vartheta[\vartheta_{EZ}[\mu_t]]\vartheta'_{EZ}[\mu_t]}U'_{T_N}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left(\frac{\partial\left(\frac{\vartheta'[U_{T_N}]}{\vartheta'[\mu_t]}\right)}{\partial R_{M,T_1,T_N}}\right)_{(x,y)=\left(x_0^*,y_0^*\right)} &= \frac{\vartheta_{EZ}[\mu_t]f''_\vartheta[\vartheta_{EZ}[\mu_t]]}{f'_\vartheta[\vartheta_{EZ}[\mu_t]]}\frac{\vartheta'_{EZ}[\mu_t]\mu_t}{\vartheta_{EZ}[\mu_t]}\lambda^*_{1,t} + \frac{\mu_t\vartheta''_{EZ}[\mu_t]}{\vartheta'_{EZ}[\mu_t]}\lambda^*_{1,t} \\
&= -\frac{1}{\tau_t^{f_\vartheta}}(1-\frac{1}{\psi})\lambda^*_{1,t} - \frac{1}{\psi}\lambda^*_{1,t} \qquad \text{(II75)}
\end{aligned}$$

where

$$\frac{1}{\tau_t^{f_\vartheta}} = -\frac{\vartheta_{EZ}[\mu_t]f''_\vartheta[\vartheta_{EZ}[\mu_t]]}{f'_\vartheta[\vartheta_{EZ}[\mu_t]]}. \tag{II76}$$

Furthermore,

$$\frac{\partial^2\left(\frac{\vartheta'\left[U_{T_N}\right]}{\vartheta'[\mu_t]}\right)}{\partial^2 R_{M,T_1,T_N}} = \frac{\left\{\begin{array}{c}\left[f_\vartheta'''\left[\vartheta_{EZ}[U_{T_N}]\right]\left[\vartheta'_{EZ}[U_{T_N}]\right]^3 + 3\,f_\vartheta''\left[\vartheta_{EZ}[U_{T_N}]\right]\,\vartheta'_{EZ}[U_{T_N}]\vartheta''_{EZ}[U_{T_N}]\right. \\ \left. +f_\vartheta'\left[\vartheta_{EZ}[U_{T_N}]\right]\,\vartheta'''_{EZ}[U_{T_N}]\right]\,(U'_{T_N})^2 \\ +\left[f_\vartheta''\left[\vartheta_{EZ}[U_{T_N}]\right]\left(\vartheta'_{EZ}[U_{T_N}]\right)^2 + f_\vartheta'\left[\vartheta_{EZ}[U_{T_N}]\right]\,\vartheta''_{EZ}[U_{T_N}]\right]\,U''_{T_N}\end{array}\right\}}{f_\vartheta'\left[\vartheta_{EZ}[\mu_t]\right]\,\vartheta'_{EZ}[\mu_t]} \tag{II77}$$

Hence,

$$\begin{aligned}\left(\frac{\partial^2\left(\frac{\vartheta'\left[U_{T_N}\right]}{\vartheta'[\mu_t]}\right)}{\partial^2 R_{M,T_1,T_N}}\right)_{(x,y)=\left(x_0^*,y_0^*\right)} &= \left\{\frac{f_\vartheta'''\left[\vartheta_{EZ}\left[\mu_t\right]\right]\left(\vartheta'_{EZ}\left[\mu_t\right]\right)^2}{f_\vartheta'\left[\vartheta_{EZ}\left[\mu_t\right]\right]}\mu_t^2 + \frac{3f_\vartheta''\left[\vartheta_{EZ}\left[\mu_t\right]\right]\vartheta''_{EZ}\left[\mu_t\right]}{f_\vartheta'\left[\vartheta_{EZ}\left[\mu_t\right]\right]}\mu_t^2\right\}\left(\lambda_{1,t}^*\right)^2 \\ &\quad + 2\mu_t\left[\frac{f_\vartheta''\left(\vartheta_{EZ}[\mu_t]\right)}{f_\vartheta'\left(\vartheta_{EZ}[\mu_t]\right)}\,\vartheta'_{EZ}[\mu_t] + \frac{\vartheta''_{EZ}[\mu_t]}{\vartheta'_{EZ}[\mu_t]}\right]\lambda_{2,t} + \frac{\vartheta'''_{EZ}\left[\mu_t\right]\mu_t^2}{\vartheta'_{EZ}\left[\mu_t\right]}\left(\lambda_{1,t}^*\right)^2 \\ &= \left\{\begin{array}{c}\frac{2(1-\frac{1}{\psi})^2\rho_t^{f_\vartheta}}{\left(\tau_t^{f_\vartheta}\right)^2} - \frac{3}{\psi}(1-\frac{1}{\psi})\frac{1}{\tau_t^{f_\vartheta}} \\ +\frac{1}{\psi}\left(1+\frac{1}{\psi}\right)\end{array}\right\}\left(\lambda_{1,t}^*\right)^2 \\ &\quad -\left[\frac{2}{\psi} + 2\left(1-\frac{1}{\psi}\right)\frac{1}{\tau_t^{f_\vartheta}}\right]\lambda_{2,t}\end{aligned} \tag{II78}$$

with

$$\rho_t^{f_\vartheta} = \frac{1}{2}\frac{f_\vartheta'''\left[\vartheta_{EZ}\left[\mu_t\right]\right]f_\vartheta'\left[\vartheta_{EZ}\left[\mu_t\right]\right]}{\left(f_\vartheta''\left[\vartheta_{EZ}\left[\mu_t\right]\right]\right)^2}. \tag{II79}$$

Similarly,

$$\begin{aligned}\frac{\partial\left(\frac{\upsilon'[\mu_t]}{\upsilon'\left[U_{T_N}\right]}\right)}{\partial R_{M,T_1,T_N}} &= \frac{\partial\left(\frac{f_\upsilon'[\upsilon_{EZ}[\mu_t]]\upsilon_{EZ}'[\mu_t]}{f_\upsilon'\left[\upsilon_{EZ}\left[U_{T_N}\right]\right]\upsilon_{EZ}'\left[U_{T_N}\right]}\right)}{\partial R_{M,T_1,T_N}} \\ &= -\frac{f_\upsilon'\left[\upsilon_{EZ}\left[\mu_t\right]\right]\upsilon_{EZ}'\left[\mu_t\right]\begin{pmatrix} f_\upsilon''\left[\upsilon_{EZ}\left[U_{T_N}\right]\right]\left(\upsilon_{EZ}'\left[U_{T_N}\right]\right)^2 U_{T_N}' \\ +f_\upsilon'\left[\upsilon_{EZ}\left[U_{T_N}\right]\right]\upsilon_{EZ}''\left[U_{T_N}\right]U_{T_N}'\end{pmatrix}}{\left(f_\upsilon'\left[\upsilon_{EZ}\left[U_{T_N}\right]\right]\upsilon_{EZ}'\left[U_{T_N}\right]\right)^2}\end{aligned}$$

and

$$\begin{aligned}\left(\frac{\partial\left(\frac{\upsilon'[\mu_t]}{\upsilon'\left[U_{T_N}\right]}\right)}{\partial R_{M,T_1,T_N}}\right)_{(x,y)=\left(x_0^*,y_0^*\right)} &= -\left\{\frac{\upsilon_{EZ}\left[\mu_t\right]f_\upsilon''\left[\upsilon_{EZ}\left[\mu_t\right]\right]}{f_\upsilon'\left[\upsilon_{EZ}\left[\mu_t\right]\right]}\rho + \frac{\upsilon_{EZ}''\left[\mu_t\right]\mu_t}{\upsilon_{EZ}'\left[\mu_t\right]}\right\}\lambda_{1,t}^* \\ &= \left\{\frac{1}{\tau_t^{f_\upsilon}}(1-\gamma)+\gamma\right\}\lambda_{1,t}^*. \qquad \text{(II80)}\end{aligned}$$

Recall that

$$\frac{\partial\left(\frac{\upsilon'[\mu_t]}{\upsilon'\left[U_{T_N}\right]}\right)}{\partial R_{M,T_1,T_N}} = -f_\upsilon'\left[\upsilon_{EZ}\left[\mu_t\right]\right]\upsilon_{EZ}'\left[\mu_t\right]\frac{f_\upsilon''\left[\upsilon_{EZ}\left[U_{T_N}\right]\right]U_{T_N}'}{\left(f_\upsilon'\left[\upsilon_{EZ}\left[U_{T_N}\right]\right]\right)^2} - f_\upsilon'\left[\upsilon_{EZ}\left[\mu_t\right]\right]\upsilon_{EZ}'\left[\mu_t\right]\frac{\upsilon_{EZ}''\left[U_{T_N}\right]U_{T}'}{f_\upsilon'\left[\upsilon_{EZ}\left[U_{T_N}\right]\right]\left(\upsilon_{EZ}'\right.}$$

Hence

$$
\begin{aligned}
&\frac{\partial^2\left(\frac{\upsilon'[\mu_t]}{\upsilon'\left[U_{T_N}\right]}\right)}{\partial^2 R_{M,T_1,T_N}} \\
= \quad & -f_\upsilon'\left[\upsilon_{EZ}\left[\mu_t\right]\right]\upsilon_{EZ}'\left[\mu_t\right]\frac{f_\upsilon'''\left[\upsilon_{EZ}\left[U_{T_N}\right]\right]\upsilon_{EZ}'\left[U_{T_N}\right]\left(U_{T_N}'\right)^2}{\left(f_\upsilon'\left[\upsilon_{EZ}\left[U_{T_N}\right]\right]\right)^2} - f_\upsilon'\left[\upsilon_{EZ}\left[\mu_t\right]\right]\upsilon_{EZ}'\left[\mu_t\right]\frac{f_\upsilon''\left[\upsilon_{EZ}\left[U_{T_N}\right]\right] [illegible]}{\left(f_\upsilon'\left[\upsilon_{EZ}\left[U_{T_N}\right] [illegible]\right.\right.} \\
& +2f_\upsilon'\left[\upsilon_{EZ}\left[\mu_t\right]\right]\upsilon_{EZ}'\left[\mu_t\right]\frac{\left(f_\upsilon''\left[\upsilon_{EZ}\left[U_{T_N}\right]\right]\right)^2\upsilon_{EZ}'\left[U_{T_N}\right]\left(U_{T_N}'\right)^2}{\left(f_\upsilon'\left[\upsilon_{EZ}\left[U_{T_N}\right]\right]\right)^3} \\
& -f_\upsilon'\left[\upsilon_{EZ}\left[\mu_t\right]\right]\upsilon_{EZ}'\left[\mu_t\right]\frac{\upsilon_{EZ}'''\left[U_{T_N}\right]\left(U_{T_N}'\right)^2+\upsilon_{EZ}''\left[U_{T_N}\right]U_{T_N}''}{f_\upsilon'\left[\upsilon_{EZ}\left[U_{T_N}\right]\right]\left(\upsilon_{EZ}'\left[U_{T_N}\right]\right)^2} \\
& +f_\upsilon'\left[\upsilon_{EZ}\left[\mu_t\right]\right]\upsilon_{EZ}'\left[\mu_t\right]\frac{f_\upsilon''\left[\upsilon_{EZ}\left[U_{T_N}\right]\right]\upsilon_{EZ}'\left[U_{T_N}\right]\upsilon_{EZ}''\left[U_{T_N}\right]\left(U_{T_N}'\right)^2}{\left(f_\upsilon'\left[\upsilon_{EZ}\left[U_{T_N}\right]\right]\right)^2\left(\upsilon_{EZ}'\left[U_{T_N}\right]\right)^2} \\
& +2f_\upsilon'\left[\upsilon_{EZ}\left[\mu_t\right]\right]\upsilon_{EZ}'\left[\mu_t\right]\frac{\upsilon_{EZ}'\left[U_{T_N}\right]\upsilon_{EZ}''\left[U_{T_N}\right]\upsilon_{EZ}''\left[U_{T_N}\right]\left(U_{T_N}'\right)^2}{f_\upsilon'\left[\upsilon_{EZ}\left[U_{T_N}\right]\right]\left(\upsilon_{EZ}'\left[U_{T_N}\right]\right)^4}
\end{aligned}
$$

and

$$
\begin{aligned}
\left(\frac{\partial^2\left(\frac{\upsilon'[\mu_t]}{\upsilon'\left[U_{T_N}\right]}\right)}{\partial^2 R_{M,T_1,T_N}}\right)_{(x,y)=\left(x_0^*,y_0^*\right)} = \quad & \left\{\begin{array}{c} -\left(\upsilon_{EZ}'\left[\mu_t\right]\right)^2\frac{f_\upsilon'''\left[\upsilon_{EZ}\left[\mu_t\right]\right]}{f_\upsilon'\left[\upsilon_{EZ}\left[\mu_t\right]\right]}+2\left(\upsilon_{EZ}'\left[\mu_t\right]\right)^2\frac{\left(f_\upsilon''\left[\upsilon_{EZ}\left[\mu_t\right]\right]\right)^2}{\left(f_\upsilon'\left[\upsilon_{EZ}\left[\mu_t\right]\right]\right)^2} \\ +\frac{f_\upsilon''\left[\upsilon_{EZ}\left[\mu_t\right]\right]\upsilon_{EZ}''\left[\mu_t\right]}{f_\upsilon'\left[\upsilon_{EZ}\left[\mu_t\right]\right]}+2\frac{\upsilon_{EZ}''\left[\mu_t\right]\upsilon_{EZ}''\left[\mu_t\right]}{\left(\upsilon_{EZ}'\left[\mu_t\right]\right)^2}-\frac{\upsilon_{EZ}'''\left[\mu_t\right]}{\left(\upsilon_{EZ}'\left[\mu_t\right]\right)} \end{array}\right\}\left(\mu_t\lambda_{1,[illegible]}^*\right. \\
& +\left\{-\upsilon_{EZ}'\left[\mu_t\right]\frac{f_\upsilon''\left[\upsilon_{EZ}\left[\mu_t\right]\right]}{f_\upsilon'\left[\upsilon_{EZ}\left[\mu_t\right]\right]}-\frac{\upsilon_{EZ}''\left[\mu_t\right]}{\left(\upsilon_{EZ}'\left[\mu_t\right]\right)}\right\}2\mu_t\lambda_{2,t}
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
\left(\frac{\partial^2\left(\frac{\upsilon'[\mu_t]}{\upsilon'\left[U_{T_N}\right]}\right)}{\partial^2 R_{M,T_1,T_N}}\right)_{(x,y)=\left(x_0^*,y_0^*\right)} &= \left\{\begin{array}{c} -\left(1-\gamma\right)^2 \frac{(\upsilon_{EZ}[\mu_t])^2 f_\upsilon'''[\upsilon_{EZ}[\mu_t]]}{f_\upsilon'[\upsilon_{EZ}[\mu_t]]} + 2\left(1-\gamma\right)^2 \frac{\left(\upsilon_{EZ}[\mu_t] f_\upsilon''[\upsilon_{EZ}[\mu_t]]\right)^2}{\left(f_\upsilon'[\upsilon_{EZ}[\mu_t]]\right)^2} \\ + \frac{f_\upsilon''[\upsilon_{EZ}[\mu_t]]\upsilon_{EZ}[\mu_t]}{f_\upsilon'[\upsilon_{EZ}[\mu_t]]}\left(-\gamma\right)\left(1-\gamma\right) + 2\gamma^2 - \gamma\left(\gamma+1\right) \end{array}\right\} \\
&\quad + \left\{-\left(1-\gamma\right)\frac{\upsilon_{EZ}\left[\mu_t\right] f_\upsilon''\left[\upsilon_{EZ}\left[\mu_t\right]\right]}{f_\upsilon'\left[\upsilon_{EZ}\left[\mu_t\right]\right]} + \gamma\right\} 2\lambda_{2,t} \\
&= \left\{2\frac{\left(1-\gamma\right)^2\left(1-\rho_t^{f_\upsilon}\right)}{\left(\tau_t^{f_\upsilon}\right)^2} - \frac{1}{\tau_t^{f_\upsilon}}\left(-\gamma\right)\left(1-\gamma\right) + 2\gamma^2 - \gamma\left(\gamma+1\right)\right\} \lambda_1^* \\
&\quad + \left\{\left(1-\gamma\right)\frac{1}{\tau_t^{f_\upsilon}} + \gamma\right\} 2\lambda_{2,t} \qquad \text{(II81)}
\end{aligned}
$$

with

$$
\rho_t^{f_\upsilon} = \frac{1}{2}\,\frac{f_\upsilon'''[\upsilon_{EZ}[\mu_t]]\;\, f_\upsilon'[\upsilon_{EZ}[\mu_t]]}{\left(f_\upsilon''[\upsilon_{EZ}[\mu_t]]\right)^2}.
$$

Finally, we combine (II75), (II78), (II80), and (II81), we simplify (II71) and obtain

$$
\lambda = \frac{1}{2}\left(D_{1,t}^* + D_0^*\right)\left(\lambda_{1,t}^*\right)^2 + \left(D_{2,t}^* + D_{3,t}^*\right)\lambda_{2,t} \qquad \text{(II82)}
$$

with

$$
\begin{aligned}
D_0^* &= -\gamma\frac{1}{\psi}+\gamma\left(\gamma-1\right)+\frac{1}{\psi}\left(1+\frac{1}{\psi}\right)\\
D_{1,t}^* &= \frac{2(1-\frac{1}{\psi})^2\rho_t^{f_\vartheta}}{\left(\tau_t^{f_\vartheta}\right)^2}+2\frac{\left(1-\gamma\right)^2\left(1-\rho_t^{f_\upsilon}\right)}{\left(\tau_t^{f_\upsilon}\right)^2}-\frac{1}{\tau_t^{f_\vartheta}}\left(\frac{3}{\psi}+\gamma\right)\left(1-\frac{1}{\psi}\right)\\
&\quad+\frac{1}{\tau_t^{f_\upsilon}}\left(\gamma\left(1-\gamma\right)-\frac{1}{\psi}\left(1-\gamma\right)\right)-\frac{1}{\tau_t^{f_\upsilon}}\frac{1}{\tau_t^{f_\vartheta}}(1-\frac{1}{\psi})\left(1-\gamma\right)\\
D_{2,t}^* &= (1-\gamma)\frac{1}{\tau_t^{f_\upsilon}}-\left(1-\frac{1}{\psi}\right)\frac{1}{\tau_t^{f_\vartheta}},\\
D_{3,t}^* &= \gamma-\frac{1}{\psi}.
\end{aligned}
$$

Using standard parameters,

$$
\begin{aligned}
&\gamma=10,\ \psi=2,\ D_{3,t}^*=9.5 \text{ and } D_0^*=\frac{343}{4}\\
&\gamma=10,\ \psi=1,\ D_{3,t}^*=9 \text{ and } D_0^*=82\\
&\gamma=10,\ \psi=3,\ D_{3,t}^*=\frac{29}{3} \text{ and } D_0^*=\frac{784}{9}.
\end{aligned}
$$

In the case of Epstein Zin specification, $\frac{1}{\tau_t^{f_\vartheta}}=\frac{1}{\tau_t^{f_\upsilon}}=0$ hence $D_{1,t}^*=D_{2,t}^*=0$. As $D_{3,t}^*$ and D_0^* are positive, $\lambda>0$. However, when $D_{1,t}^*$ and $D_{2,t}^*$ are different from zero, one cannot conclude that $\lambda>0$.

E. *Bound for the equity risk premium* (9)

Let us rewrite equation (I19) without truncation:

$$
f[x,y]=1+\sum_{k=1}^{\infty}\sum_{m=0}^{k}\theta_{k,m}\left(x-x_0\right)^m\left(y-y_0\right)^{k-m} \tag{II83}
$$

where

$$\theta_{k,m} = \frac{1}{m!\,(k-m)!}\,\frac{\partial^k f}{\partial x^m\,\partial y^{k-m}}\bigg|_{(x_0,y_0)}. \tag{II84}$$

We have, in particular, $\theta_{1,1} = \frac{a_{1,t}}{R_{f,t,T_1}}$, $\theta_{1,0} = \frac{a_{1,t}}{R_{f,T_1,T_N}}$, $\theta_{2,2} = \frac{a_{2,t}}{R^2_{f,t,T_1}}$, $\theta_{2,0} = \frac{a_{2,t}}{R^2_{f,T_1,T_N}}$ and $\theta_{2,1} = \frac{a_{1,t}+2a_{2,t}}{R_{f,t,T_N}}$.

The term that is truncated in the resulting second order expansion in equation (I24) is

$$\sum_{k=3}^{\infty}\sum_{m=0}^{k}\theta_{k,m}\underbrace{\left(R_{M,t,T_1}-R_{f,t,T_1}\right)^m\,\mathbb{E}^*_{T_1}\left(\left(R_{M,T_1,T_N}-R_{f,T_1,T_N}\right)^{k-m}\right)}_{\text{Priced risk factor}}. \tag{II85}$$

The $\theta_{k,m}$, $k \geq 3$, are functions of higher order preference coefficients. Thus, in our equity risk premium estimate, the truncated part of the numerator is

$$\sum_{k=3}^{\infty}\sum_{m=0}^{k}\theta_{k,m}\underbrace{\mathbb{E}^*_t\left(\left(R_{M,t,T_1}-R_{f,t,T_1}\right)^{m+1}\,\mathbb{E}^*_{T_1}\left(\left(R_{M,T_1,T_N}-R_{f,T_1,T_N}\right)^{k-m}\right)\right)}_{\text{Risk premium associated to risk factor}} \tag{II86}$$

and in the denominator it is

$$\sum_{k=3}^{\infty}\sum_{m=0}^{k}\theta_{k,m}\mathbb{E}^*_t\left(\left(R_{M,t,T_1}-R_{f,t,T_1}\right)^{m}\,\mathbb{E}^*_{T_1}\left(\left(R_{M,T_1,T_N}-R_{f,T_1,T_N}\right)^{k-m}\right)\right). \tag{II87}$$

Let us assume that Assumptions 1 and 2 hold. Under these assumptions, (II86) ≥ 0 and (II87) ≤ 0, i.e., the second-order equity risk premium estimate in Equation (9) is in fact a lower bound to the equity risk premium.

F. *Equity risk premium in a multi-period model with consumption*

We now introduce consumption in the representative agent problem. We assume that assumptions 1 and 2 are satisfied. In addition, we assume that the consumption-wealth ratio is positively related to the market return, and that the correlation of the square of the consumption wealth ratio and market return is negative (condition reminiscent of market coskewness). We show that under these conditions, our measure of expected excess return remains a lower bound to the true expected excess return.

To proceed, we start by having the representative agent solve the problem

$$\max_{\omega_t, c_t} \mathbb{E}_t \left\{ \max_{\omega_{T_1}, c_{T_1}} \left\{ \mathbb{E}_{T_1} u \left[W_{t,T_N} \right] \right\} \right\}, \tag{II88}$$

where the terminal wealth is

$$W_{t,T_N} = \left(1 - c_{T_1}\right) W_{T_1} \left(\omega_{T_1}^{\intercal} R_{T_1,T_N} \right) \text{ with } W_{T_1} = \left(1 - c_t\right) W_t \left(\omega_t^{\intercal} R_{t,T_1} \right) \tag{II89}$$

and c_t is the consumption wealth ratio. As in the case without consumption, the SDF is given by the identity (I18).

Let us define

$$cc_{tT_1} = \left(1 - c_{T_1}\right)\left(1 - c_t\right), \tag{II90}$$

$$\mathbf{x} = cc_{tT_1}, \qquad \mathbf{y} = R_{M,t,T_1}, \qquad \mathbf{z} = R_{M,T_1,T_N} \tag{II91}$$

$$\mathbf{x}_0 = 1, \qquad \mathbf{y}_0 = R_{f,t,T_1}, \qquad \mathbf{z}_0 = R_{f,T_1,T_N} \tag{II92}$$

and $\mathbf{X} = (\mathbf{x}, \mathbf{y}, \mathbf{z})$ and $\mathbf{X}_0 = (\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$. Note that $0 < cc_{tT_1} \leq 1$ since $0 < c_{T_1} \leq 1$ and $0 < c_t \leq 1$. Now, assume that the utility function is well-behaved and admits high-order derivatives that exist. Denote $\overline{W}_{t,T_N} = W_t R_{f,t,T_1} R_{f,T_1,T_N}$ and

$$\mathbf{G} = \frac{u' \left[\overline{W}_{t,T_N}\right]}{u' \left[W_{t,T_N}\right]} \tag{II93}$$

A second-order Taylor expansion of $\mathbf{G}$ around $\mathbf{X} = \mathbf{X}_0$ gives

$$\begin{aligned}
\mathbf{G} =& 1 - (\mathbf{x} - \mathbf{x}_0) \frac{W_t \mathbf{y}_0 \mathbf{z}_0 u'' \left[\overline{W}_{t,T_N}\right]}{u' \left[\overline{W}_{t,T_N}\right]} - (\mathbf{y} - \mathbf{y}_0) \frac{W_t \mathbf{z}_0 u'' \left[\overline{W}_{t,T_N}\right]}{u' \left[\overline{W}_{t,T_N}\right]} - (\mathbf{z} - \mathbf{z}_0) \frac{W_t \mathbf{y}_0 u'' \left[\overline{W}_{t,T_N}\right]}{u' \left[\overline{W}_{t,T_N}\right]} \\
&+ \frac{1}{2} W_t^2 \mathbf{y}_0^2 \mathbf{z}_0^2 \left(-\frac{u''' \left[\overline{W}_{t,T_N}\right]}{u' \left[\overline{W}_{t,T_N}\right]} + \frac{2 \left(u'' \left[\overline{W}_{t,T_N}\right]\right)^2}{\left(u' \left[\overline{W}_{t,T_N}\right]\right)^2} \right) (\mathbf{x} - \mathbf{x}_0)^2 \\
&+ \frac{1}{2} W_t^2 \mathbf{z}_0^2 \left(-\frac{u''' \left[\overline{W}_{t,T_N}\right]}{u' \left[\overline{W}_{t,T_N}\right]} + \frac{2 \left(u'' \left[\overline{W}_{t,T_N}\right]\right)^2}{\left(u' \left[\overline{W}_{t,T_N}\right]\right)^2} \right) (\mathbf{y} - \mathbf{y}_0)^2 \\
&+ \frac{1}{2} W_t^2 \mathbf{y}_0^2 \left(-\frac{u''' \left[\overline{W}_{t,T_N}\right]}{u' \left[\overline{W}_{t,T_N}\right]} + \frac{2 \left(u'' \left[\overline{W}_{t,T_N}\right]\right)^2}{\left(u' \left[\overline{W}_{t,T_N}\right]\right)^2} \right) (\mathbf{z} - \mathbf{z}_0)^2 \\
&+ W_t^2 \mathbf{y}_0 \mathbf{x}_0 \mathbf{z}_0^2 \left(-\frac{u''' \left[\overline{W}_{t,T_N}\right]}{u' \left[\overline{W}_{t,T_N}\right]} + \frac{2 \left(u'' \left[\overline{W}_{t,T_N}\right]\right)^2}{\left(u' \left[\overline{W}_{t,T_N}\right]\right)^2} \right) (\mathbf{x} - \mathbf{x}_0)(\mathbf{y} - \mathbf{y}_0) \\
&+ \left(\frac{\partial^2 \mathbf{G}}{\partial \mathbf{x} \partial \mathbf{z}} \right)_{\mathbf{X} = \mathbf{X}_0} (\mathbf{x} - \mathbf{x}_0)(\mathbf{z} - \mathbf{z}_0) + \left(\frac{\partial^2 \mathbf{G}}{\partial \mathbf{y} \partial \mathbf{z}} \right)_{\mathbf{X} = \mathbf{X}_0} (\mathbf{z} - \mathbf{z}_0)(\mathbf{y} - \mathbf{y}_0).
\end{aligned} \tag{II94}$$

Notice that

$$\mathbb{E}_{T_1}^* (\mathbf{z} - \mathbf{z}_0) = 0 \tag{II95}$$

and

$$\begin{aligned}
\mathbb{E}_{T_1}^* (\mathbf{x} - \mathbf{x}_0)(\mathbf{z} - \mathbf{z}_0) &= (\mathbf{x} - \mathbf{x}_0) \mathbb{E}_{T_1}^* (\mathbf{z} - \mathbf{z}_0) = 0, \\
\mathbb{E}_{T_1}^* (\mathbf{z} - \mathbf{z}_0)(\mathbf{y} - \mathbf{y}_0) &= (\mathbf{y} - \mathbf{y}_0) \mathbb{E}_{T_1}^* (\mathbf{z} - \mathbf{z}_0) = 0.
\end{aligned} \tag{II96}$$

We use these expressions to compute υ_{T_1}, defined as in (3):

$$\begin{aligned}
\upsilon_{T_1} =& \mathbb{E}^*_{T_1}\left(\frac{u'\left[\overline{W}_{t,T_N}\right]}{u'\left[W_{t,T_N}\right]}\right) = 1 - \frac{W_t \mathbf{y}_0 \mathbf{z}_0 u''\left[\overline{W}_{t,T_N}\right]}{u'\left[\overline{W}_{t,T_N}\right]}\left(\mathbf{x}-\mathbf{x}_0\right) - \left(\mathbf{y}-\mathbf{y}_0\right)\frac{W_t \mathbf{z}_0 u''\left[\overline{W}_{t,T_N}\right]}{u'\left[\overline{W}_{t,T_N}\right]} \\
&+ \frac{1}{2}W_t^2 \mathbf{y}_0^2 \mathbf{z}_0^2\left(-\frac{u'''\left[\overline{W}_{t,T_N}\right]}{u'\left[\overline{W}_{t,T_N}\right]} + \frac{2\left(u''\left[\overline{W}_{t,T_N}\right]\right)^2}{\left(u'\left[\overline{W}_{t,T_N}\right]\right)^2}\right)\left(\mathbf{x}-\mathbf{x}_0\right)^2 \\
&+ \frac{1}{2}W_t^2 \mathbf{z}_0^2\left(-\frac{u'''\left[\overline{W}_{t,T_N}\right]}{u'\left[\overline{W}_{t,T_N}\right]} + \frac{2\left(u''\left[\overline{W}_{t,T_N}\right]\right)^2}{\left(u'\left[\overline{W}_{t,T_N}\right]\right)^2}\right)\left(\mathbf{y}-\mathbf{y}_0\right)^2 \\
&+ \frac{1}{2}W_t^2 \mathbf{y}_0^2\left(-\frac{u'''\left[\overline{W}_{t,T_N}\right]}{u'\left[\overline{W}_{t,T_N}\right]} + \frac{2\left(u''\left[\overline{W}_{t,T_N}\right]\right)^2}{\left(u'\left[\overline{W}_{t,T_N}\right]\right)^2}\right)\mathbb{E}^*_{T_1}\left(\mathbf{z}-\mathbf{z}_0\right)^2 \\
&+ W_t^2 \mathbf{y}_0 \mathbf{x}_0 \mathbf{z}_0^2\left(-\frac{u'''\left[\overline{W}_{t,T_N}\right]}{u'\left[\overline{W}_{t,T_N}\right]} + \frac{2\left(u''\left[\overline{W}_{t,T_N}\right]\right)^2}{\left(u'\left[\overline{W}_{t,T_N}\right]\right)^2}\right)\left(\mathbf{x}-\mathbf{x}_0\right)\left(\mathbf{y}-\mathbf{y}_0\right)
\end{aligned} \tag{II97}$$

which simplifies to

$$\begin{aligned}
\upsilon_{T_1} =& 1 + \frac{1}{\tau_t}\mathbb{E}^*_{T_1}\left(cc_{tT_1}-1\right) + \frac{1}{\tau_t R_{f,t,T_1}}\left(\omega_t^\intercal R_{t,T_1} - R_{f,t,T_1}\right) + \frac{(1-\rho_t)}{\tau_t^2}\left(cc_{tT_1}-1\right)^2 \\
&+ \frac{(1-\rho_t)}{\tau_t^2 R_{f,t,T_1}^2}\left(\omega_t^\intercal R_{t,T_1} - R_{f,t,T_1}\right)^2 + \frac{(1-\rho_t)}{\tau_t^2 R_{f,T_1,T_N}^2}\mathbb{E}^*_{T_1}\left(\omega_{T_1}^\intercal R_{T_1,T_N} - R_{f,T_1,T_N}\right)^2 \\
&+ \frac{2(1-\rho_t)}{\tau_t^2 R_{f,t,T_1}}\mathbb{E}^*_{T_1}\left(cc_{tT_1}-1\right)\left(\omega_t^\intercal R_{t,T_1} - R_{f,t,T_1}\right).
\end{aligned} \tag{II98}$$

We then use $R_{M,t,T_1} = \omega_t^\intercal R_{t,T_1}$, $R_{M,T_1,T_N} = \omega_{T_1}^\intercal R_{T_1,T_N}$ and express the expected value of υ_{T_1} under the risk neutral measure as

$$\begin{aligned}
\mathbb{E}^*_t \upsilon_{T_1} =& 1 + \frac{1}{\tau_t}\mathbb{E}^*_t\left(cc_{tT_1}-1\right) + \frac{(1-\rho_t)}{\tau_t^2}\mathbb{E}^*_t\left(cc_{tT_1}-1\right)^2 \\
&+ \frac{(1-\rho_t)}{\tau_t^2 R_{f,t,T_1}^2}\mathbb{M}^{*(2)}_{t,T_1} + \frac{(1-\rho_t)}{\tau_t^2 R_{f,T_1,T_N}^2}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1,T_N} \\
&+ \frac{2(1-\rho_t)}{\tau_t^2 R_{f,t,T_1}}\mathbb{COV}^*_t\left(cc_{tT_1}, R_{M,t,T_1}\right).
\end{aligned} \tag{II99}$$

The expected excess market return is

$$\begin{aligned}\mathbb{E}_t\left(R_{M,t,T_1}-R_{f,t,T_1}\right) &= \mathbb{E}_t\left[\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}}\frac{m_{t,T_1}}{\mathbb{E}_t m_{t,T_1}}\left(R_{M,t,T_1}-R_{f,t,T_1}\right)\right] \qquad \text{(II100)}\\ &= \mathbb{E}_t^*\left[\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}}\left(R_{M,t,T_1}-R_{f,t,T_1}\right)\right]\\ &= \frac{\mathbb{COV}_t^*\left[\upsilon_{T_1},R_{M,t,T_1}\right]}{\mathbb{E}_t^*\upsilon_{T_1}}. \qquad \text{(II101)}\end{aligned}$$

Observe that

$$\begin{aligned}\mathbb{COV}_t^*\left[\upsilon_{T_1},R_{M,t,T_1}\right] =&1+\frac{1}{\tau_t}\mathbb{COV}_t^*\left(cc_{tT_1},R_{M,t,T_1}\right)+\frac{1}{\tau_t R_{f,t,T_1}}\mathbb{M}_{t,T_1}^{*(2)}\\ &+\frac{(1-\rho_t)}{\tau_t^2}\mathbb{COV}_t^*\left(\left(cc_{tT_1}-1\right)^2,R_{M,t,T_1}\right)+\frac{(1-\rho_t)}{\tau_t^2 R_{f,t,T_1}^2}\mathbb{M}_{t,T_1}^{*(3)}\\ &+\frac{(1-\rho_t)}{\tau_t^2 R_{f,T_1,T_N}^2}\mathbb{LEV}_{t,T_1,T_N}^*\\ &+\frac{2(1-\rho_t)}{\tau_t^2 R_{f,t,T_1}^2}\mathbb{E}_t^*\left(\left(cc_{tT_1}-1\right)\left(R_{M,t,T_1}-R_{f,t,T_1}\right)^2\right).\end{aligned} \qquad \text{(II102)}$$

Notice that $\mathbb{E}_t^*\left(\left(cc_{tT_1}-1\right)\left(\omega_t^{\intercal}R_{t,T_1}-R_{f,t,T_1}\right)^2\right)<0$ because $cc_{tT_1}-1<0$.

In theory, each factor risk premium in υ_{T_1} positively contributes to the equity risk premium, therefore each term in (II102) is positive. Thus one should expect

$$\mathbb{COV}_t^*\left(cc_{tT_1},R_{M,t,T_1}\right)>0 \text{ and } \mathbb{COV}_t^*\left(\left(cc_{tT_1}-1\right)^2,R_{M,t,T_1}\right)\leq 0. \qquad \text{(II103)}$$

Since $1-c_{T_1}=\frac{W_{T_1}-C_{T_1}}{W_{T_1}}$ is the fraction of wealth W_{T_1} invested at T_1, it follows that

$$\mathbb{COV}_t^*\left(cc_{tT_1}, R_{M,t,T_1}\right) = (1-c_t)\,\mathbb{COV}_t^*\left(\frac{W_{T_1}-C_{T_1}}{W_{T_1}}, R_{M,t,T_1}\right) \text{ and}$$
$$\mathbb{COV}_t^*\left((cc_{tT_1}-1)^2, R_{M,t,T_1}\right) = (1-c_t)^2\,\mathbb{COV}_t^*\left(\left(\frac{W_{T_1}-C_{T_1}}{W_{T_1}}\right)^2, R_{M,t,T_1}\right). \tag{II104}$$

The positive sign of $\mathbb{COV}_t^*\left(cc_{tT_1}, R_{M,t,T_1}\right)$ is motivated by the positive impact of wealth-consumption ratio on the market expected excess return. Conditions (II103) are reminiscent of the dependence between the wealth-consumption ratio and the return on the market under the physical measure. Under the physical measure, the wealth-consumption ratio is positively correlated to the market. Under Assumptions 1 and 2, and (II103), the covariance $\mathbb{COV}_t^*\left[\upsilon_{T_1}, R_{M,t,T_1}\right]$ is bounded:

$$\mathbb{COV}_t^*\left[\upsilon_{T_1}, R_{M,t,T_1}\right] \geq \frac{1}{R_{f,t,T_1}}\frac{1}{\tau_t}\mathbb{M}_{t,T_1}^{*(2)} + \frac{(1-\rho_t)}{R_{f,t,T_1}^2\tau_t^2}\mathbb{M}_{t,T_1}^{*(3)} + \frac{(1-\rho_t)}{R_{f,T_1,T_N}^2\tau_t^2}\mathbb{LEV}_{t,T_1,T_N}^*. \tag{II105}$$

Next, since $cc_{tT_1} \leq 1$, we use (II99) and (II103) to obtain

$$\mathbb{E}_t^*\upsilon_{T_1} \leq 1 + \frac{(1-\rho_t)}{R_{f,t,T_1}^2\tau_t^2}\mathbb{M}_{t,T_1}^{*(2)} + \frac{(1-\rho_t)}{R_{f,T_1,T_N}^2\tau_t^2}\mathbb{E}_t^*\mathbb{M}_{T_1,T_N}^{*(2)}. \tag{II106}$$

Therefore,

$$\frac{1}{\mathbb{E}_t^*\upsilon_{T_1}} \geq \frac{1}{1 + \frac{(1-\rho_t)}{R_{f,t,T_1}^2\tau_t^2}\mathbb{M}_{t,T_1}^{*(2)} + \frac{(1-\rho_t)}{R_{f,T_1,T_N}^2\tau_t^2}\mathbb{E}_t^*\mathbb{M}_{T_1,T_N}^{*(2)}}. \tag{II107}$$

Combining (II105) and (II107), the expected excess return is bounded by our equity risk premium estimate:

$$\mathbb{E}_t\left[R_{M,t,T_1} - R_{f,t,T_1}\right] \geq \frac{\frac{1}{R_{f,t,T_1}}\frac{1}{\tau_t}\mathbb{M}^{*(2)}_{t,T_1} + \frac{(1-\rho_t)}{R^2_{f,t,T_1}\tau_t^2}\mathbb{M}^{*(3)}_{t,T_1} + \frac{(1-\rho_t)}{R^2_{f,T_1,T_N}\tau_t^2}\mathbb{LEV}^*_{t,T_1,T_N}}{1 + \frac{(1-\rho_t)}{R^2_{f,t,T_1}\tau_t^2}\mathbb{M}^{*(2)}_{t,T_1} + \frac{(1-\rho_t)}{R^2_{f,T_1,T_N}\tau_t^2}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1,T_N}} \tag{II108}$$

This shows that under minimal conditions, our measure of expected excess return is a bound on the true expected excess return when consumption is taken into account.

G. Restricted bound (23)

Let us assume that Assumptions 3–4 hold. Under these conditions, $a_{1,t} \geq 1$ and $a_{2,t} \leq -1$. Hence,

$$\frac{a_{1,t}}{R_{f,t,T_1}}\mathbb{M}^{*(2)}_{t,T_1} \geq \frac{1}{R_{f,t,T_1}}\mathbb{M}^{*(2)}_{t,T_1} \tag{II109}$$

$$\frac{a_{2,t}}{R^2_{f,t,T_1}}\mathbb{M}^{*(3)}_{t,T_1} \geq \frac{1}{R^2_{f,t,T_1}}\mathbb{M}^{*(3)}_{t,T_1} \tag{II110}$$

$$\frac{a_{2,t}}{R^2_{f,T_1,T_N}}\mathbb{LEV}^*_{t,T_1,T_N} \geq \frac{1}{R^2_{f,T_1,T_N}}\mathbb{LEV}^*_{t,T_1,T_N} \tag{II111}$$

which shows that the numerator of (9) is larger than the numerator of (23). Furthermore,

$$\frac{a_{2,t}}{R^2_{f,t,T_1}}\mathbb{M}^{*(2)}_{t,T_1} \leq \frac{-1}{R^2_{f,t,T_1}}\mathbb{M}^{*(2)}_{t,T_1} \tag{II112}$$

$$\frac{a_{2,t}}{R^2_{f,T_1,T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1,T_N} \leq \frac{-1}{R^2_{f,T_1,T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1,T_N} \tag{II113}$$

which shows that the denominator of (9) is smaller than the denominator of (23).

The inequality follows.

III . Estimation of moments

In this Section, we provide closed-form solutions to the risk-neutral and physical moments used in our analysis. In many cases, we use the spanning formula of Carr and Madan (2001) and Bakshi and Madan (2000) to evaluate the risk-neutral expected value of a twice-differentiable function of the underlying asset price, $H\left(S_{T_1}\right)$ as

$$\begin{aligned}\mathbb{E}_t^* H\left[S_{T_1}\right] &= H\left[S_t R_{f,t,T_1}\right] + \mathbb{E}_t^* H_S\left[S_t R_{f,t,T_1}\right] S_t\left(R_{M,t,T_1} - R_{f,t,T_1}\right) \\ &\quad + R_{f,t,T_1}\left[\int_{S_t R_{f,t,T_1}}^{\infty} H_{SS}\left[K\right] C_t\left[K\right] dK + \int_0^{S_t R_{f,t,T_1}} H_{SS}\left[K\right] P_t\left[K\right] dK\right],\end{aligned} \tag{III1}$$

where H_S and H_{SS} are the first and second derivative of function $H(\cdot)$, respectively, and $C_t[K]$ and $P_t[K]$ are the prices of call and put options with strike K and maturity T_1 at time t. We evaluate the integral terms via numerical integration using the 1,000-point moneyness grid described in Section IV.B.

A. *Closed-form expressions for* $\mathbb{M}_{t,T_1}^{*(k)}$ *and* $\mathbb{M}_{t,T_1}^{*(k)}\left[\alpha\right]$

Let us set $H(x) = \left(\frac{x}{S_t} - R_{f,t,T_1}\right)^k$. To evaluate the risk-neutral moments of order k, $\mathbb{M}_{t,T_1}^{*(k)}$, we plug H in Equation (III1). The risk-neutral moments are therefore computed using call and put options with maturity T_1.

To compute the truncated moments $\mathbb{M}^{*(k)}_{t,T_1}[\alpha] = \mathbb{E}^*_t\left((R_{M,t,T_1} - R_{f,t,T_1})^k \mathbb{1}_{S_{T_1}<\alpha S_t}\right)$, we use $G(x) = H(x)\mathbb{1}_{x<\alpha S_t}$. Equation (III1) simplifies to

$$\mathbb{M}^{*(k)}_{t,T_1}[\alpha] = H[\alpha S_t]\,\mathbb{P}^*_t[S_{T_1} < \alpha S_t] - H_S[\alpha S_t]\,R_{f,t,T_1} P_t[\alpha S_t] + R_{f,t,T_1}\int_0^{\alpha S_t} H_{SS}[K]\,P_t[K]\,dK. \tag{III2}$$

B. *Representation of* $\mathbb{M}^{*(2)}_{T_1,T_N}$ *as a function of* S_{T_1}

Let us decompose the variance of market returns between T_1 and T_N into a portfolio of options, using the spanning formula of Carr and Madan (2001) and Bakshi and Madan (2000) with $H(x) = \left(\frac{x}{S_{T_1}} - R_{f,T_1,T_N}\right)^2$:

$$\mathbb{M}^{*(2)}_{T_1,T_N} = \mathbb{E}^*_{T_1}\left(\left(\frac{S_{T_N}}{S_{T_1}} - R_{f,T_1,T_N}\right)^2\right) \tag{III3}$$

$$= \frac{2R_{f,T_1,T_N}}{S^2_{T_1}}\left[\int_{R_{f,T_1,T_N}S_{T_1}}^{\infty} C_{T_1}[K]dK + \int_0^{R_{f,T_1,T_N}S_{T_1}} P_{T_1}[K]dK\right] \tag{III4}$$

where $C_{T_1}[K]$ and $P_{T_1}[K]$ denote the prices of call and put options with strike K and maturity T_N at time T_1. For each value of $K = K_0$, $C_{T_1}[K]$ and $P_{T_1}[K]$ can be expressed as functions of $S_{T_1}, R_{f,T_1,T_N}, T_N - T_1$ and $\sigma^2_{IV}(k_0)$, where $\sigma^2_{IV}(k_0)$ is the implied variance of the option at time T_1. The implied variance is evaluated at the log-moneyness $k_0 = \ln(K_0/F_{T_1,T_N})$, where F_{T_1,T_N} is the forward price.

Consider, for example, the arbitrage-free stochastic volatility inspired (raw) parameterization of the volatility surface proposed by Gatheral and Jacquier (2014). It writes the total implied variance evaluated at the log-moneyness k_0 as a function of a set of parameters $\{a, b, \xi, m, \sigma\}$:

$$\sigma_{IV}^2(k_0)(T_N - T_1) = a + b\left(\xi(k_0 - m) + \sqrt{(k_0 - m)^2 + \sigma^2}\right) \tag{III5}$$

where $a \in \mathbb{R}$, $b \geq 0$, $|\xi| < 1$, $m \in \mathbb{R}$, $\sigma > 0$, and $a + b\sigma\sqrt{1 - \xi^2} \geq 0$. Given these parameters, the option prices $C_{T_1}[K]$ and $P_{T_1}[K]$ are thus functions of S_{T_1}.

C. *Volatility dynamics implied by (32)*

To show that our formulation (32) is different from the GARCH (1,1) model, we use the closed-form expression of θ_{T_1,T_N} displayed in (35) to obtain

$$\mathbb{M}_{t,T_N}^{*(2)} = \theta_{T_1,T_N}\mathbb{M}_{t,T_1}^{*(4)} + 2R_{f,t,T_1}\theta_{T_1,T_N}\mathbb{M}_{t,T_1}^{*(3)} + R_{f,T_1,T_N}^2\left(\theta_{T_1,T_N} + 1\right)\mathbb{M}_{t,T_1}^{*(2)}. \tag{III6}$$

This proves that $\mathbb{M}_{t,T_N}^{*(2)}$ differs from the GARCH dynamics. To check similarities with the GARCH process, let us assume for illustration purposes that $\mathbb{M}_{t,T_1}^{*(3)} = 0$ and $\mathbb{M}_{t,T_1}^{*(4)} = 3\left(\mathbb{M}_{t,T_1}^{*(2)}\right)^2$. Then

$$\mathbb{M}_{t,T_N}^{*(2)} = 3\theta_{T_1,T_N}\left(\mathbb{M}_{t,T_1}^{*(2)}\right)^2 + R_{f,T_1,T_N}^2\left(\theta_{T_1,T_N} + 1\right)\mathbb{M}_{t,T_1}^{*(2)}. \tag{III7}$$

The propagation of volatility is therefore quadratic in the current variance, unlike the affine GARCH recursion in which it is linear.

D. *Closed-form expressions for* $\mathbb{E}_t^* \mathbb{M}^{*(2)}_{T_{j-1},T_j}$ *and* $\mathbb{LEV}^*_{t,T_{j-1},T_j}$

To compute the risk neutral quantities, we use an approach similar to (32) by considering the decomposition:

$$\mathbb{M}^{*(2)}_{T_{j-1},T_j} = \theta_{T_{j-1},T_j} \left(R_{M,t,T_{j-1}} - R_{f,t,T_{j-1}}\right)^2 + \eta_{T_{j-1}} \tag{III8}$$

with $\mathbb{E}^* \left(\eta_{T_{j-1}} | R_{M,t,T_{j-1}}\right) = 0$. The loadings θ_{T_{j-1},T_j} can be computed at time t from options with maturities T_{j-1} and T_j:

$$\theta_{T_{j-1},T_j} = \frac{\mathbb{M}^{*(2)}_{t,T_j} - R^2_{f,T_{j-1},T_j} \mathbb{M}^{*(2)}_{t,T_{j-1}}}{\mathbb{E}_t^* \left(R^2_{M,t,T_{j-1}} \left(R_{M,t,T_{j-1}} - R_{f,t,T_{j-1}}\right)^2 \right)}. \tag{III9}$$

Our objective is to calculate the moments $\mathbb{E}_t^* \left((R_{M,t,T_1} - R_{f,t,T_1})^k \mathbb{M}^{*(2)}_{T_{j-1},T_j} \right)$. For $k = 0$, we obtain $\mathbb{E}_t^* (\mathbb{M}^{*(2)}_{T_{j-1},T_j})$ which appears in the denominator of the equity risk premium and of the physical variance. For $k = 1$, we obtain the additional term in the numerator of the equity risk premium, which arises due to intertemporal hedging. For $k = 2$, we obtain the additional term in the physical variance.

If $T_{j-1} = T_1$, using (III8) we get

$$\mathbb{E}_t^* \left((R_{M,t,T_1} - R_{f,t,T_1})^k \mathbb{M}^{*(2)}_{T_1,T_j} \right) = \theta_{T_1,T_j} \mathbb{M}^{*(k+2)}_{t,T_1}. \tag{III10}$$

For $T_{j-1} > 1$, using (III8) and

$$\mathbb{M}^{*(2)}_{T_1,T_{j-1}} = \theta_{T_1,T_{j-1}} \left(R_{M,t,T_1} - R_{f,t,T_1}\right)^2 + \eta_{T_1} \text{ with } \mathbb{E}_t^* \left(\eta_{T_1} | R_{M,t,T_1}\right) = 0, \tag{III11}$$

we obtain

$$\mathbb{E}_t^*\left((R_{M,t,T_1}-R_{f,t,T_1})^k\mathbb{M}^{*(2)}_{T_{j-1},T_j}\right) = \theta_{T_1,T_{j-1}}\theta_{T_{j-1},T_j}\left(\mathbb{M}^{*(k+4)}_{t,T_1}+2R_{f,t,T_1}\mathbb{M}^{*(k+3)}_{t,T_1}+R^2_{f,t,T_1}\mathbb{M}^{*(k+2)}_{t,T_1}\right) + \theta_{T_{j-1},T_j}R^2_{f,T_1,T_{j-1}}\mathbb{M}^{*(k+2)}_{t,T_1}. \quad \text{(III12)}$$

For $k=0$, (III12) collapses to

$$\mathbb{E}_t^*\mathbb{M}^{*(2)}_{T_{j-1},T_j} = \theta_{T_{j-1},T_j}\mathbb{M}^{*(2)}_{t,T_{j-1}}. \quad \text{(III13)}$$

For $k=1$,

$$\begin{aligned}\mathbb{LEV}^*_{t,T_{j-1},T_j} &= \mathbb{E}_t^*\left((R_{M,t,T_1}-R_{f,t,T_1})\mathbb{M}^{*(2)}_{T_{j-1},T_j}\right)\\ &= \theta_{T_{j-1},T_j}\left(\theta_{T_1,T_{j-1}}\mathbb{M}^{*(5)}_{t,T_1}+2\theta_{T_1,T_{j-1}}R_{f,t,T_1}\mathbb{M}^{*(4)}_{t,T_1}+(\theta_{T_1,T_{j-1}}R^2_{f,t,T_1}+R^2_{f,T_1,T_{j-1}})\mathbb{M}^{*(3)}_{t,T_1}\right).\end{aligned} \quad \text{(III14)}$$

For $k=2$,

$$\mathbb{E}_t^*\left((R_{M,t,T_1}-R_{f,t,T_1})^2\mathbb{M}^{*(2)}_{T_{j-1},T_j}\right) = \theta_{T_1,T_{j-1}}\theta_{T_{j-1},T_j}\left(\mathbb{M}^{*(6)}_{t,T_1}+2R_{f,t,T_1}\mathbb{M}^{*(5)}_{t,T_1}+R^2_{f,t,T_1}\mathbb{M}^{*(4)}_{t,T_1}\right) + \theta_{T_{j-1},T_j}R^2_{f,T_1,T_{j-1}}\mathbb{M}^{*(4)}_{t,T_1}. \quad \text{(III15)}$$

E. *Closed-form expression of* $\mathbb{E}_t^*\left(\mathbb{M}^{*(2)}_{T_{j-1},T_j}\mathbb{1}_{R_{M,t,T_1}<\alpha}\right)$

If $T_{j-1}=T_1$, using (III8) we obtain

$$\mathbb{E}_t^*\left[\mathbb{M}^{*(2)}_{T_{j-1},T_j}\mathbb{1}_{R_{M,t,T_1}<\alpha}\right] = \theta_{T_1,T_j}\mathbb{M}^{*(2)}_{t,T_1}[\alpha] \quad \text{(III16)}$$

In the model with single rebalancing time, $T_j = T_N$ and the intertemporal hedging component of the probability of a crash is given by

$$\mathbb{E}_t^*\left(\mathbb{M}^{*(2)}_{T_1,T_N}\mathbb{1}_{R_{M,t,T_1}<\alpha}\right) \;=\; \theta_{T_1,T_N}\mathbb{M}^{*(2)}_{t,T_1}\left[\alpha\right]. \tag{III17}$$

If $T_{j-1} > T_1$, we obtain

$$\begin{aligned}
\mathbb{E}_t^*\left[\mathbb{M}^{*(2)}_{T_{j-1},T_j}\mathbb{1}_{R_{M,t,T_1}<\alpha}\right] &= \theta_{T_{j-1},T_j}\mathbb{E}_t^*\left[\left(R_{M,t,T_{j-1}} - R_{f,t,T_{j-1}}\right)^2\mathbb{1}_{R_{M,t,T_1}<\alpha}\right] \\
&= \theta_{T_{j-1},T_j}\left\{\begin{array}{c}
\mathbb{E}_t^*\left(R^2_{M,t,T_1}\mathbb{1}_{R_{M,t,T_1}<\alpha}\left(\mathbb{M}^{*(2)}_{T_1,T_{j-1}} + R^2_{f,T_1,T_{j-1}}\right)\right) \\
-2R_{f,t,T_{j-1}}R_{f,T_1,T_{j-1}}\left(\mathbb{M}^{*(1)}_{t,T_1}\left[\alpha\right] + R_{f,t,T_1}\mathbb{M}^{*(0)}_{t,T_1}\left[\alpha\right]\right) \\
+R^2_{f,t,T_{j-1}}\mathbb{M}^{*(0)}_{t,T_1}\left[\alpha\right]
\end{array}\right\} \\
&= \theta_{T_{j-1},T_j}\left\{\begin{array}{c}
\theta_{T_1,T_{j-1}}\left(\mathbb{M}^{*(4)}_{t,T_1}[\alpha] + 2R_{f,t,T_1}\mathbb{M}^{*(3)}_{t,T_1}[\alpha] + R^2_{f,t,T_1}\mathbb{M}^{*(2)}_{t,T_1}[\alpha]\right) \\
+R^2_{f,T_1,T_{j-1}}\left(\mathbb{M}^{*(2)}_{t,T_1}[\alpha] + 2R_{f,t,T_1}\mathbb{M}^{*(1)}_{t,T_1}[\alpha] + R^2_{f,t,T_1}\mathbb{M}^{*(0)}_{t,T_1}[\alpha]\right) \\
-2R_{f,t,T_{j-1}}R_{f,T_1,T_{j-1}}\left(\mathbb{M}^{*(1)}_{t,T_1}\left[\alpha\right] + R_{f,t,T_1}\mathbb{M}^{*(0)}_{t,T_1}\left[\alpha\right]\right) \\
+R^2_{f,t,T_{j-1}}\mathbb{M}^{*(0)}_{t,T_1}\left[\alpha\right]
\end{array}\right\}
\end{aligned} \tag{III18}$$

IV . Out-of-sample performance over the sample used in Tetlock, McCoy, and Shah (2024)

In this section, we reproduce the out-of-sample performance results but on the sample from 1997 to 2021. Table IAI presents the summary statistics of the data over this time period and Table IAII reports the out-of-sample prediction performance.

Table IAI
Summary statistics for risk premia from 1997 to 2021

We report summary statistics for times series of risk premia predictions. We use weekly time series of overlapping values for horizons longer than 10 days. Each panel reports the results for a forecast horizon T_1 between one week and six months. All values are annualized and in percent. RP^{Log}_{t,T_1} is the lower bound of Martin (2017). $IERP_{t\to T_1}$ is the risk premium estimate of Tetlock, McCoy, and Shah (2024). RP_{t,T_1} is the second-order lower bound of Chabi-Yo and Loudis (2020). RP_{t,T_1,T_N} is the risk premium measure in Equation (12), with T_N = 1 and 2 years. Data are weekly from January 1997 to December 2021.

Prediction	Mean	Standard deviation	Skew.	Kurt.	10%	25%	50%	75%	90%
Panel A: One week									
RP^{Log}_{t,T_1}	3.43	4.87	6.29	59.08	1.01	1.36	2.05	3.55	6.19
$IERP_{t,T_1}$	8.37	13.40	7.20	75.05	2.35	3.12	4.75	8.38	14.55
RP_{t,T_1}	3.59	5.30	6.70	66.84	1.04	1.41	2.12	3.66	6.49
$RP_{t,T_1,1y}$	9.33	10.02	6.96	82.58	3.70	4.58	6.59	10.56	15.88
$RP_{t,T_1,2y}$	14.75	11.45	5.57	60.09	6.83	8.71	11.53	17.12	24.11
Panel B: One month									
RP^{Log}_{t,T_1}	4.29	4.38	4.57	34.44	1.32	1.82	3.21	5.10	7.85
$IERP_{t,T_1}$	8.74	11.11	5.69	47.26	2.46	3.66	5.88	9.82	15.32
RP_{t,T_1}	4.60	4.89	4.84	38.09	1.37	1.92	3.43	5.40	8.48
$RP_{t,T_1,1y}$	7.69	6.87	4.33	33.92	2.72	3.83	5.93	9.20	13.45
$RP_{t,T_1,2y}$	10.11	7.56	3.63	26.09	4.18	5.59	8.15	12.20	17.05
Panel C: One quarter									
RP^{Log}_{t,T_1}	4.35	3.39	3.39	21.22	1.68	2.15	3.51	5.32	7.52
$IERP_{t,T_1}$	9.11	8.81	4.45	31.42	3.26	4.56	6.80	10.51	15.85
RP_{t,T_1}	4.88	3.98	3.71	25.32	1.85	2.41	3.86	5.91	8.52
$RP_{t,T_1,1y}$	6.42	4.75	3.54	24.66	2.56	3.48	5.23	7.68	10.93
$RP_{t,T_1,2y}$	7.64	5.03	3.35	22.94	3.60	4.62	6.49	9.08	12.69
Panel D: Six months									
RP^{Log}_{t,T_1}	4.32	2.68	2.61	14.02	1.94	2.45	3.70	5.33	7.02
$IERP_{t,T_1}$	9.22	7.06	3.47	20.12	3.90	5.21	7.32	10.81	15.76
RP_{t,T_1}	5.02	3.23	2.81	15.99	2.20	2.84	4.28	6.10	8.31
$RP_{t,T_1,1y}$	6.18	3.78	2.57	13.96	2.82	3.66	5.29	7.38	10.35
$RP_{t,T_1,2y}$	7.05	4.02	2.48	13.35	3.50	4.46	6.11	8.42	11.49

Table IAII
Out-of-sample prediction and allocation performance from 1997 to 2021

We report the out-of-sample performance of different risk premium prediction methods. RP^{Log}_{t,T_1} is the lower bound of Martin (2017). $IERP_{t,T_1}$ is the Implied Equity Risk Premium of Tetlock, McCoy, and Shah (2024). RP_{t,T_1} is the second-order lower bound of Chabi-Yo and Loudis (2020). RP_{t,T_1,T_N} is the risk premia measure in Equation (12). We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (39)). The results in the last column are based on predicted returns obtained by averaging RP_{t,T_1,T_N} across T_N. For each prediction method, we test for the significance of the R^2_{OOS} difference relative to RP_{t,T_1} using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (41)). The physical variances are computed using option prices, using Equation (28). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to RP_{t,T_1} using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. Negative certainty equivalents are not reported. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively. Data are from January 1997 to December 2021.

Horizon T_1				RP_{t,T_1,T_N} with $T_N =$ (in months)										
(in months)	RP^{Log}_{t,T_1}	$IERP_{t,T_1}$	RP_{t,T_1}	1	2	3	4	5	6	9	12	18	24	Average across T_N
Panel A: Out-of-sample R^2														
10d	−0.41	−0.60	−0.38	−0.33	−0.21	−0.11	−0.00	0.10	0.21	0.46	0.65	0.91	0.95	0.43
1	0.92	1.39	1.08	-	1.25	1.52	1.61	1.70	1.79	1.97	2.11	2.30	2.34	1.95
2	1.51	2.18	1.96	-	-	2.23*	2.70*	2.87*	3.03*	3.42**	3.72**	4.20**	4.48*	3.43**
3	1.43	2.73	2.23	-	-	-	2.55**	3.14**	3.37**	3.90**	4.31**	4.95**	5.38**	4.04**
4	2.17	5.22	3.35	-	-	-	-	3.69**	4.31**	4.96**	5.45**	6.19**	6.69**	5.31**
5	3.07	8.01	4.65	-	-	-	-	-	5.00***	6.19***	6.73**	7.54**	8.09**	6.80***
6	3.40	9.42	5.28	-	-	-	-	-	-	6.23***	7.74***	8.59***	9.18***	8.02***
12	2.67	10.38	5.58	-	-	-	-	-	-	-	-	6.90***	9.03***	8.03***
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma = 3$														
10d	3.66	4.59	3.74	3.92*	4.42*	4.77*	5.17*	5.56*	5.95*	6.73*	7.28	7.30	7.88	6.80*
1	4.71	2.91	4.89	-	5.11*	5.58	5.80	6.03	6.28*	6.94	7.49	8.29	9.17	7.03*
2	4.99	3.54	5.39	-	-	5.67**	6.27**	6.53**	6.79**	7.53***	8.28**	9.75**	10.27**	7.67***
3	5.53	7.30	6.13	-	-	-	6.41**	6.99**	7.24**	7.99**	8.77**	10.48**	11.82**	8.41**
4	5.71	5.93	6.44	-	-	-	-	6.70***	7.25***	7.96**	8.66**	10.17**	11.39**	8.56**
5	5.42	4.31	6.16	-	-	-	-	-	6.35**	7.20**	7.75**	8.93**	10.05**	7.98**
6	5.58	2.91	6.59	-	-	-	-	-	-	7.31***	9.12***	10.69***	11.50**	9.77***
12	5.40	-	6.42	-	-	-	-	-	-	-	-	7.21	7.62	7.88

V . Validity and tightness tests

We follow the methodology of Back, Crotty, and Kazempour (2022) and test for the validity and tightness of these bounds. We recall below the methodology.

The test is based on the set of inequalities

$$\mathbb{E}_t[((R_{M,t,T_1} - R_{f,t,T_1}) - \tilde{r}_{M,t,T_1})z_t] \geq 0$$

for a vector z_t of nonnegative conditioning variables in the time-t information set. Back, Crotty, and Kazempour (2022) refer to $((R_{M,t,T_1} - R_{f,t,T_1}) - \tilde{r}_{M,t,T_1})$ as realized slackness. We consider $\tilde{r}_{M,t,T_1} \in \{RP^{Log}_{t,T_1}, RP_{t,T_1}, RP_{t,T_1,T^*_N}\}$. The vector z_t contains variables from Welch and Goyal (2008): Dividend Price Ratio (defined as the difference between the log of dividends and the log of the S&P 500 index price level, plus 5 to ensure a positive conditioning variable); Earnings Price Ratio (defined as the difference between the log of earnings and the log of the S&P 500 index price level, plus 5 to ensure a positive conditioning variable); Book-to-Market Ratio (the ratio of book value to market value for the Dow Jones Industrial Average); T-bill Rate (the 3-month Treasury bill rate); 1 + Term Spread (defined as the difference between the long-term yield from Ibbotson's and the 3-month T-bill rate, plus 1 to ensure a positive conditioning variable); Credit Spread (defined as difference between BAA and AAA-rated corporate bond yields); Stock Variance (defined as the sum of squared daily returns on the S&P 500); 1 + Net Equity Issuance (the ratio of 12-month moving sums of net issues by

NYSE-listed stocks to the total end-of-year market capitalizations of NYSE stocks, plus 1 to ensure a positive conditioning variable); 1 + Inflation (the Consumer Price Index, plus 1 to ensure a positive conditioning variable).

Denote by λ_0 the population mean of $\mathbb{E}_t[((R_{M,t,T_1} - R_{f,t,T_1}) - \tilde{r}_{M,t,T_1})z_t]$. To evaluate a bound's validity, we test the null hypothesis $\lambda_0 \geq 0$ against the alternative that λ_0 is unrestricted. To test tightness, we test the null hypothesis $\lambda_0 = 0$ against the alternative that $\lambda_0 \geq 0$.

Let $\bar{\lambda}$ denote the sample mean of $((R_{M,t,T_1} - R_{f,t,T_1}) - \tilde{r}_{M,t,T_1})z_t$, and Σ its sample covariance. Validity is tested using the statistic D_1, defined as

$$D_1 = \min_{\lambda \geq 0}(\lambda - \bar{\lambda})\Sigma^{-1}(\lambda - \bar{\lambda}) \tag{V19}$$

Under the null that $\lambda_0 \geq 0$, Back, Crotty, and Kazempour (2022) show that D_1 is asymptotically distributed as a mixture of chi-square distributions.

Under validity, tightness is tested using the statistic

$$D_2 = \hat{\lambda}'\Sigma^{-1}\hat{\lambda}, \tag{V20}$$

where $\hat{\lambda}$ is the vector of λ which reaches the minimum in equation (V19). Under the null that $\lambda_0 = 0$, D_2 is also asymptotically distributed as a mixture of chi-square distributions.

Following Back, Crotty, and Kazempour (2022), we calculate finite sample $p-$ values using Monte-Carlo simulations.

Table IAIII reports the results, for investment horizons T_N of one and two years. The statistic D_1 is zero for RP^{Log}_{t,T_1} and for RP_{t,T_1} for most forecast horizons T_1. It is positive for $RP_{t,T_1,1y}$ and $RP_{t,T_1,2y}$, but always below the 10% critical value. Validity is therefore never rejected. The statistic D_2 is the largest for RP^{Log}_{t,T_1}, smaller for RP_{t,T_1} , and even smaller for $RP_{t,T_1,1y}$ and $RP_{t,T_1,2y}$. Tightness is rejected for all bounds, except for $RP_{t,T_1,1y}$ and $RP_{t,T_1,2y}$ at $T_1 = 10$ days.

Table IAIII
Validity and tightness tests

We report test statistics for bound validity and bound tightness for different equity risk premium bounds. The test statistics D_1 and D_2 are defined in equation (V19) and (V20). The sample moments are means of weekly realized slackness and weekly realized slackness interacted with each of the variables from Welch and Goyal (2008) in z_t. The p-values are based on 100 Monte-Carlo simulations. Data are weekly from January 1996 to February 2023.

	RP_{t,T_1}^{Log}	RP_{t,T_1}	$RP_{t,T_1,1y}$	$RP_{t,T_1,2y}$
Panel A: One week				
D_1	0.00	0.00	0.41	1.44
p-value for validity	0.98	0.94	0.78	0.51
D_2	15.19	15.00	6.66	4.34
p-value for tightness	0.05	0.05	0.54	0.79
Panel B: One month				
D_1	0.00	0.66	2.09	4.24
p-value for validity	0.85	0.61	0.36	0.15
D_2	82.60	76.53	67.29	62.62
p-value for tightness	0.00	0.00	0.00	0.00
Panel C: Two months				
D_1	0.00	0.00	1.26	3.26
p-value for validity	0.86	0.87	0.46	0.18
D_2	225.41	221.85	195.53	175.32
p-value for tightness	0.00	0.00	0.00	0.00
Panel D: One quarter				
D_1	0.00	0.00	0.87	2.66
p-value for validity	0.83	0.75	0.45	0.24
D_2	405.06	395.76	356.34	320.61
p-value for tightness	0.00	0.00	0.00	0.00
Panel E: Six months				
D_1	0.00	0.00	1.00	2.48
p-value for validity	0.80	0.81	0.42	0.21
D_2	707.61	680.64	619.65	561.45
p-value for tightness	0.00	0.00	0.00	0.00

VI . Higher-order expansion

In this section, we investigate how higher-order moments contribute to the equity risk premium. We show that increasing the order of the approximation, thereby allowing for kurtosis preference, generates additional terms that contribute to the equity risk premium. For simplicity of notation, we assume that there is a single rebalancing date at T_1, but all the derivations can be extended in a straightforward way to multiple rebalancing dates.

A. One-period SDF

Under no-arbitrage assumptions, a third-order Taylor expansion-series produces a one-period SDF in a three-date (two-period) economy of the form

$$\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}} \approx \frac{1 + z_{T_1} + z^{\upsilon}_{T_1}}{\mathbb{E}^*_t \left(1 + z_{T_1} + z^{\upsilon}_{T_1}\right)}, \tag{VI1}$$

where

$$z_{T_1} = \frac{a_{1,t}}{R_{f,t,T_1}}(R_{M,t,T_1} - R_{f,t,T_1}) + \frac{a_{2,t}}{R^2_{f,t,T_1}}(R_{M,t,T_1} - R_{f,t,T_1})^2 + \frac{a_{3,t}}{R^3_{f,t,T_1}}(R_{M,t,T_1} - R_{f,t,T_1})^3,$$

$$z^{\upsilon}_{T_1} = \frac{a_{2,t}}{R^2_{f,T_1,T_N}}\mathbb{M}^{*(2)}_{T_1,T_N} + \frac{a_{3,t}}{R^3_{f,T_1,T_N}}\mathbb{M}^{*(3)}_{T_1,T_N} + \frac{a_{2,3,t}}{R_{f,t,T_1}R^2_{f,T_1,T_N}}(R_{M,t,T_1} - R_{f,t,T_1})\mathbb{M}^{*(2)}_{T_1,T_N}. \tag{VI2}$$

The coefficients are given by

$$a_{1,t} = \tfrac{1}{\tau_t}, \qquad a_{2,t} = \tfrac{(1-\rho_t)}{\tau_t^2}, \qquad a_{3,t} = \tfrac{(\kappa_t + 1 - 2\rho_t)}{\tau_t^3}, \tag{VI3}$$

and $a_{2,3,t} = 2a_{2,t} + 3a_{3,t}$. The risk aversion and skewness preference parameters are given by equations (7) and (8). The kurtosis tolerance parameters is given by

$$\kappa_t = \frac{1}{3!} \frac{u^{(4)}\left[W_t R_{f,t,T_N}\right]\left(u^{(1)}\left[W_t R_{f,t,T_N}\right]\right)^2}{\left(u^{(2)}\left[W_t R_{f,t,T_N}\right]\right)^3}. \tag{VI4}$$

Proof. Consider the partial derivatives

$$\begin{aligned} f_{xxy} &= \frac{2}{(x_0)^2 y_0} \frac{\left(W_t x_0 y_0 u''\right)^2}{(u')^2}\left(2 - \frac{u''' u'}{(u'')^2}\right) \\ &+ \frac{1}{(x_0)^2 y_0}\left\{6\frac{(W_t x_0 y_0)^3 u'' u' u'''}{(u')^3} - (W_t x_0 y_0)^3 \frac{u''''}{u'} - 6\frac{\left(W_t x_0 y_0 u''\right)^3}{(u')^3}\right\}, \\ f_{xxx} &= \frac{y_0^3}{x_0^3} f_{yyy} = \frac{1}{(x_0)^3}\left(6\frac{(W_t x_0 y_0)^3 u'' u'''}{(u')^2} - \frac{(W_t x_0 y_0)^3 u''''}{u'} - 6\frac{(W_t x_0 y_0)^3 (u'')^3}{(u')^3}\right) \end{aligned} \tag{VI5}$$

A third order Taylor expansion-series yields

$$\begin{aligned} f[x,y] &= f[x,y]^{2nd} \\ &+ \frac{1}{(x_0)^3}\frac{(\kappa_t + 1 - 2\rho_t)}{\tau_t^3}(x - x_0)^3 + \frac{1}{(y_0)^3}\frac{(\kappa_t + 1 - 2\rho_t)}{\tau_t^3}(y - y_0)^3 \\ &+ \frac{1}{(x_0)^2 y_0}\left(\frac{2(1 - \rho_t)}{\tau_t^2} + \frac{3(\kappa_t + 1 - 2\rho_t)}{\tau_t^3}\right)(x - x_0)^2 (y - y_0) \\ &+ \frac{1}{x_0 (y_0)^2}\left(\frac{2(1 - \rho_t)}{\tau_t^2} + \frac{3(\kappa_t + 1 - 2\rho_t)}{\tau_t^3}\right)(y - y_0)^2 (x - x_0) \end{aligned} \tag{VI6}$$

where $f[x,y]^{2nd}$ is the second order Taylor expansion-series in Equation (I23).

Replacing x, x_0, y, and y_0 by their expressions and using preference parameters a_1, a_2, and a_3 defined in Equation (VI3), we obtain,

$$\begin{aligned}
\mathbb{E}^*_{T_1}\left(f\left[x,y\right]\right) \quad = \quad & 1+\frac{a_{1,t}}{R_{f,t,T_1}}\left(R_{M,t,T_1}-R_{f,t,T_1}\right)+\frac{a_{1,t}}{R_{f,T_1,T_N}}\left(R_{f,T_1,T_N}-R_{f,T_1,T_N}\right) \\
& +\frac{a_{2,t}}{\left(R_{f,t,T_1}\right)^2}\left(R_{M,t,T_1}-R_{f,t,T_1}\right)^2+\frac{a_{2,t}}{\left(R_{f,T_1,T_N}\right)^2}\mathbb{E}^*_{T_1}\left(\left(R_{M,T_1,T_N}-R_{f,T_1,T_N}\right)^2\right) \\
& +\frac{a_{1,t}+2a_{2,t}}{R_{f,t,T_2}}\left(R_{M,t,T_1}-R_{f,t,T_1}\right)\left(R_{f,T_1,T_N}-R_{f,T_1,T_N}\right) \\
& +\frac{a_{3,t}}{\left(R_{f,t,T_1}\right)^3}\left(R_{M,t,T_1}-R_{f,t,T_1}\right)^3+\frac{a_{3,t}}{\left(R_{f,T_1,T_N}\right)^3}\mathbb{E}^*_{T_1}\left(\left(R_{M,T_1,T_N}-R_{f,T_1,T_N}\right)^3\right) \\
& +\frac{2a_{2,t}+3a_{3,t}}{\left(R_{f,t,T_1}\right)^2 R_{f,T_1,T_N}}\left(R_{M,t,T_1}-R_{f,t,T_1}\right)^2\left(R_{f,T_1,T_N}-R_{f,T_1,T_N}\right) \\
& +\frac{2a_{2,t}+3a_{3,t}}{R_{f,t,T_1}\left(R_{f,T_1,T_N}\right)^2}\mathbb{E}^*_{T_1}\left(\left(R_{M,T_1,T_N}-R_{f,T_1,T_N}\right)^2\right)\left(R_{M,t,T_1}-R_{f,t,T_1}\right) \quad \text{(VI7)}
\end{aligned}$$

which gives the desired result when interest rates are deterministic.

□

When (VI2) is removed from the SDF specification (VI1), which corresponds to a static SDF in a one-period economy, the equity risk premium reduces to the expected excess return of Chabi-Yo and Loudis (2020). We refer to their bound to as RP^{3rd}_{t,T_1}.

Using this third-order expansion, we next derive the equity risk premium.

B. *Equity risk premium*

With the third-order Taylor expansion-series approach, Equation (VI1) depends on new terms such as risk-neutral skewness and cross-term between risk-neutral volatility and market excess return, in addition to risk-neutral variance. These terms introduce additional high-order leverage effects in the expected excess return decomposition. To find a closed-form expression for the equity risk premium in terms of risk-neutral moments and high-order leverages, we first define high-order leverage effects under the risk-neutral measure as:

$$
\begin{aligned}
\mathbb{LES}^*_{t,T_1,T_N} &= \mathbb{COV}^*_t\left(R_{M,t,T_1} - R_{f,t,T_1}, \mathbb{M}^{*(3)}_{T_1,T_N}\right), && \text{(VI8)}\\
\mathbb{LEK}^*_{t,T_1,T_N} &= \mathbb{COV}^*_t\left((R_{M,t,T_1} - R_{f,t,T_1})^2, \mathbb{M}^{*(2)}_{T_1,T_N}\right). && \text{(VI9)}
\end{aligned}
$$

We then show how the equity risk premium depends on these terms in the following Proposition.

Proposition 1: *Up to the third-order Taylor expansion-series of the inverse marginal utility, the one-period expected excess market return obeys the following decomposition*

$$
\mathbb{E}_t\left(R_{M,t,T_1} - R_{f,t,T_1}\right) = \frac{\frac{a_{1,t}}{R_{f,t,T_1}}\mathbb{M}^{*(2)}_{t,T_1} + \frac{a_{2,t}}{R^2_{f,t,T_1}}\mathbb{M}^{*(3)}_{t,T_1} + \frac{a_{3,t}}{R^3_{f,t,T_1}}\mathbb{M}^{*(4)}_{t,T_1} + IH^{3rd}_{t,T_1,T_N}}{1 + \sum_{k=2}^{3}\frac{a_{k,t}}{R^k_{f,t,T_1}}\mathbb{M}^{*(k)}_{t,T_1} + \sum_{k=2}^{3}\frac{a_{k,t}}{R^k_{f,T_1,T_N}}\mathbb{E}^*_t\mathbb{M}^{*(k)}_{T_1,T_N} + \frac{a_{2,3,t}}{R_{f,t,T_1}R^2_{f,T_1,T_N}}\mathbb{LEV}^*_{t,T_1,T_N}}, \quad \text{(VI10)}
$$

where the intertemporal hedging component IH^{3rd}_{t,T_1,T_N} is given by

$$
IH^{3rd}_{t,T_1,T_N} = \frac{a_{2,t}}{R^2_{f,T_1,T_N}}\mathbb{LEV}^*_{t,T_1,T_N} + \frac{a_{3,t}}{R^3_{f,T_1,T_N}}\mathbb{LES}^*_t + \frac{a_{2,3,t}}{R_{f,t,T_1}R^2_{f,T_1,T_N}}\left(\mathbb{LEK}^*_{t,T_1,T_N} + \mathbb{M}^{*(2)}_{t,T_1}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1,T_N}\right), \quad \text{(VI11)}
$$

$a_{2,3,t} = 2a_{2,t} + 3a_{3,t}$ *and the risk-neutral quantities* $\mathbb{LEV}^*_{t,T_1,T_N}$, $\mathbb{M}^{*(k)}_{T_i,T_j}$, $\mathbb{LES}^*_t$ *and* $\mathbb{LEK}^*_{t,T_1,T_N}$ *are defined in Equations (10), (11), (VI8), and (VI9), respectively.*

Proof. Using the inverse SDF (VI1), we obtain

$$\begin{aligned}
\mathbb{E}_t\left(R_{t,T_1} - R_{f,t,T_1}\right) &= \mathbb{COV}_t^*\left(\frac{1 + z_{T_1} + z_{T_1}^v}{1 + \mathbb{E}_t^* z_{T_1} + \mathbb{E}_t^* z_{T_1}^v}, \left(R_{M,t,T_1} - R_{f,t,T_1}\right)\right) \\
&= \frac{\mathbb{COV}_t^*\left(z_{T_1}, r_{M,t,T_1}\right) + \mathbb{COV}_t^*\left(z_{T_1}^v, r_{M,t,T_1}\right)}{1 + \mathbb{E}_t^* z_{T_1} + \mathbb{E}_t^* z_{T_1}^v}.
\end{aligned}$$

Setting $r_{M,t,T_1} = R_{M,t,T_1} - R_{f,t,T_1}$ and using (VI.A) and (VI2), it follows that

$$\begin{aligned}
\mathbb{E}_t^* z_{T_1} &= \frac{a_{2,t}}{R_{f,t,T_1}^2}\mathbb{E}_t^* r_{M,t,T_1}^2 + \frac{a_{3,t}}{R_{f,t,T_1}^3}\mathbb{E}_t^* r_{M,t,T_1}^3 \\
\mathbb{E}_t^* z_{T_1}^v &= \frac{a_{2,t}}{R_{f,T_1,T_N}^2}\mathbb{E}_t^* \mathbb{M}_{T_1,T_N}^{*(2)} + \frac{a_{3,t}}{R_{f,T_1,T_N}^3}\mathbb{E}_t^* \mathbb{M}_{T_1,T_N}^{*(3)} + \frac{a_{2,3,t}}{R_{f,t,T_1} R_{f,T_1,T_N}^2}\mathbb{E}_t^* r_{M,t,T_1} \mathbb{M}_{T_i,T_j}^{*(2)}. \quad \text{(VI12)}
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathbb{E}_t^* z_{T_1}\left(R_{M,t,T_1} - R_{f,t,T_1}\right) &= \frac{a_{1,t}}{R_{f,t,T_1}}\mathbb{E}_t^* r_{M,t,T_1}^2 + \frac{a_{2,t}}{R_{f,t,T_1}^2}\mathbb{E}_t^* r_{M,t,T_1}^3 + \frac{a_{3,t}}{R_{f,t,T_1}^3}\mathbb{E}_t^* r_{M,t,T_1}^4 \\
&= \frac{a_{1,t}}{R_{f,t,T_1}}\mathbb{M}_{t,T_1}^{*(2)} + \frac{a_{2,t}}{R_{f,t,T_1}^2}\mathbb{M}_{t,T_1}^{*(3)} + \frac{a_{3,t}}{R_{f,t,T_1}^3}\mathbb{M}_{t,T_1}^{*(4)} \quad \text{(VI13)}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}_t^* z_{T_1}^v\left(R_{M,t,T_1} - R_{f,t,T_1}\right) &= \frac{a_{2,t}}{R_{f,T_1,T_N}^2}\mathbb{COV}_t^*\left(r_{M,t,T_1}, \mathbb{M}_{T_1,T_N}^{*(2)}\right) + \frac{a_{3,t}}{R_{f,T_1,T_N}^3}\mathbb{COV}_t^*\left(r_{M,t,T_1}, \mathbb{M}_{T_1,T_N}^{*(3)}\right) \\
&\quad + \frac{a_{2,3,t}}{R_{f,t,T_1} R_{f,T_1,T_N}^2}\left(\mathbb{COV}_t^*\left(r_{M,t,T_1}^2, \mathbb{M}_{T_1,T_N}^{*(2)}\right) + \mathbb{M}_{t,T_1}^{*(2)}\mathbb{E}_t^* \mathbb{M}_{T_1,T_N}^{*(2)}\right) \text{(VI14)}
\end{aligned}$$

This ends the proof. □

Furthermore, under Assumptions 1 and 2, our measure of the equity risk premium remains a lower bound for expected future market returns. The proof is similar to the one in the case of the second order approximation, see

Appendix II.E.

C. Physical variance

PROPOSITION IA1: *Up to a third-order expansion-series, under no-arbitrage conditions, the conditional variance of returns under the physical measure is a function of risk neutral return moments:*

$$\begin{aligned}\mathbb{V}ar_t &\equiv \mathbb{E}_t\left(R_{M,t,T_1} - \mathbb{E}_t R_{M,t,T_1}\right)^2 \\ &= \mathbb{E}_t\left(R_{M,t,T_1} - R_{f,t,T_1}\right)^2 - \left(\mathbb{E}_t\left(R_{M,t,T_1} - R_{f,t,T_1}\right)\right)^2 \end{aligned} \tag{VI15}$$

where $\mathbb{E}_t\left(R_{M,t,T_1} - R_{f,t,T_1}\right)$ *is given by Equation* (VI10) *and*

$$\mathbb{E}_t\left(R_{M,t,T_1} - R_{f,t,T_1}\right)^2 = \frac{\begin{array}{r}\mathbb{M}^{*(2)}_{t,T_1} + \frac{a_{1,t}}{R_{f,t,T_1}}\mathbb{M}^{*(3)}_{t,T_1} + \frac{a_{2,t}}{R^2_{f,t,T_1}}\mathbb{M}^{*(4)}_{t,T_1} + \frac{a_{3,t}}{R^3_{f,t,T_1}}\mathbb{M}^{*(5)}_{t,T_1} \\ + \frac{a_{2,t}}{R^2_{f,T_1,T_N}}\mathbb{E}^*_t\left((R_{M,t,T_1} - R_{f,t,T_1})^2\mathbb{M}^{*(2)}_{T_1,T_N}\right) \\ + \frac{a_{3,t}}{R^3_{f,T_1,T_N}}\mathbb{E}^*_t\left((R_{M,t,T_1} - R_{f,t,T_1})^2\mathbb{M}^{*(3)}_{T_1,T_N}\right) \\ + \frac{a_{2,3,t}}{R_{f,t,T_1}R^2_{f,T_1,T_N}}\mathbb{E}^*_t\left((R_{M,t,T_1} - R_{f,t,T_1})^3\mathbb{M}^{*(2)}_{T_1,T_N}\right)\end{array}}{\begin{array}{r}1 + \frac{a_{2,t}}{R^2_{f,t,T_1}}\mathbb{M}^{*(2)}_{t,T_1} + \frac{a_{3,t}}{R^2_{f,t,T_1}}\mathbb{M}^{*(3)}_{t,T_1} + \frac{a_{2,t}}{R^2_{f,T_1,T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1,T_N} \\ + \frac{a_{3,t}}{R^3_{f,T_1,T_N}}\mathbb{E}^*_t\mathbb{M}^{*(3)}_{T_1,T_N} + \frac{a_{2,3,t}}{R_{f,t,T_1}R^2_{f,T_1,T_N}}\mathbb{E}^*_t\left((R_{M,t,T_1} - R_{f,t,T_1})\mathbb{M}^{*(2)}_{T_1,T_N}\right)\end{array}}. \tag{VI16}$$

Proof. The result follows from applying the third-order approximation of Equation (VI1) and substituting it into

$$\mathbb{E}_t\left(R_{M,t,T_1} - R_{f,t,T_1}\right)^2 = \mathbb{E}^*_t\left(\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}}\left(R_{M,t,T_1} - R_{f,t,T_1}\right)^2\right). \tag{VI17}$$

□

D. *Conditional crash probability*

We next express the conditional probability of a crash using a third-order Taylor expansion series for the inverse marginal utility. To derive this probability, we define additional truncated moments as

$$\mathbb{M}^{*}_{t,s}[\alpha] = \mathbb{E}^{*}_{t}\left(\mathbb{M}^{*(3)}_{T_1,T_N}\mathbb{1}_{R_{M,t,T_1}<\alpha}\right), \tag{VI18}$$

$$\mathbb{M}^{*}_{t,sv}[\alpha] = \mathbb{E}^{*}_{t}\left(r_{M,t,T_1}\mathbb{M}^{*(2)}_{T_1,T_N}\mathbb{1}_{R_{M,t,T_1}<\alpha}\right). \tag{VI19}$$

Proposition 2: *Up to the third-order expansion-series of the inverse marginal utility, the conditional crash probability,* $\Pi^{3rd}_{t,T_1,T_N}[\alpha] = P_t\left(R_{M,t,T_1} < \alpha\right)$*, in a two-period (three-date) economy is*

$$\Pi^{3rd}_{t,T_1,T_N}[\alpha] = \frac{\left\{\begin{array}{c} \mathbb{M}^{*(0)}_{t,T_1}[\alpha] + \sum_{k=2}^{3}\frac{a_{k,t}}{R^{k}_{f,t,T_1}}\mathbb{M}^{*(k)}_{t,T_1}[\alpha] \\ +\frac{a_{2,t}}{R^{2}_{f,T_1,T_N}}\mathbb{M}^{*}_{t,v}[\alpha] + \frac{a_{3,t}}{R^{3}_{f,T_1,T_N}}\mathbb{M}^{*}_{t,s}[\alpha] + \frac{a_{2,3,t}}{R_{f,t,T_1}R^{2}_{f,T_1,T_N}}\mathbb{M}^{*}_{t,sv}[\alpha] \end{array}\right\}}{1+\sum_{k=2}^{3}\frac{a_{k,t}}{R^{k}_{f,t,T_1}}\mathbb{M}^{*(k)}_{t,T_1} + \sum_{k=2}^{3}\frac{a_{k,t}}{R^{k}_{f,T_1,T_N}}\mathbb{E}^{*}_{t}\mathbb{M}^{*(k)}_{T_1,T_N} + \frac{a_{2,3,t}}{R_{f,t,T_1}R^{2}_{f,T_1,T_N}}\mathbb{E}^{*}_{t}R_{M,t,T_1}\mathbb{M}^{*(2)}_{T_1,T_N}} \tag{VI20}$$

where $a_{2,3,t} = 2a_{2,t} + 3a_{3,t}$.

Proof. The probability of crash is

$$\Pi^{3rd}_{t,T_1,T_N}[\alpha] = \mathbb{E}^{*}_{t}\left(\frac{\mathbb{E}_t m_{t,T_1}}{m_{t,T_1}}\mathbb{1}_{R_{M,t,T_1}<\alpha}\right)$$

We then replace the inverse SDF by its expression and obtain

$$\begin{aligned}
\Pi^{3rd}_{t,T_1,T_N}[\alpha] &= \frac{\mathbb{E}^*_t\left(\left(1+z_{T_1}+z^{\upsilon}_{T_1}\right)\mathbb{1}_{R_{M,t,T_1}<\alpha}\right)}{1+\mathbb{E}^*_t z_{T_1}+\mathbb{E}^*_t z^{\upsilon}_{T_1}} \\
&= \frac{\mathbb{E}^*_t\left(\mathbb{1}_{R_{M,t,T_1}<\alpha}\right)+\mathbb{E}^*_t\left(z_{T_1}\mathbb{1}_{R_{M,t,T_1}<\alpha}\right)+\mathbb{E}^*_t\left(z^{\upsilon}_{T_1}\mathbb{1}_{R_{M,t,T_1}<\alpha}\right)}{1+\mathbb{E}^*_t z_{T_1}+\mathbb{E}^*_t z^{\upsilon}_{T_1}} \\
&= \frac{\left\{\begin{array}{c} \mathbb{M}^{*(0)}_{t,T_1}[\alpha]+\sum_{k=2}^{3}\frac{a_{k,t}}{R^k_{f,t,T_1}}\mathbb{M}^{*(k)}_{t,T_1}[\alpha]+ \\ \frac{a_{2,t}}{R^2_{f,T_1,T_N}}\mathbb{M}^*_{t,\upsilon}[\alpha]+\frac{a_{3,t}}{R^3_{f,T_1,T_N}}\mathbb{M}^*_{t,s}[\alpha]+\frac{a_{2,3,t}}{R_{f,t,T_1}R^2_{f,T_1,T_N}}\mathbb{M}^*_{t,s\upsilon}[\alpha] \end{array}\right\}}{1+\sum_{k=2}^{3}\frac{a_{k,t}}{R^k_{f,t,T_1}}\mathbb{M}^{*(k)}_{t,T_1}+\sum_{k=2}^{3}\frac{a_{k,t}}{R^k_{f,T_1,T_N}}\mathbb{E}^*_t\mathbb{M}^{*(k)}_{T_1,T_N}+\frac{a_{2,3,t}}{R_{f,t,T_1}R^2_{f,T_1,T_N}}\mathbb{E}^*_t R_{M,t,T_1}\mathbb{M}^{*(2)}_{T_1,T_N}}
\end{aligned}$$

This ends the proof. □

E. *Estimation of risk-neutral moments*

Let us first derive a closed-form expression for $\mathbb{LEK}^*_t$.

$$\begin{aligned}
\mathbb{LEK}^*_{t,T_1,T_N} &= \mathbb{COV}^*_t\left(\left(R_{M,t,T_1}-R_{f,t,T_1}\right)^2,\mathbb{M}^{*(2)}_{T_1,T_N}\right) \\
&= \mathbb{E}^*_t\left(\left(R_{M,t,T_1}-R_{f,t,T_1}\right)^2\mathbb{M}^{*(2)}_{T_1,T_N}\right)-\mathbb{M}^{*(2)}_{t,T_1}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1,T_N} \qquad \text{(VI21)}
\end{aligned}$$

Using the projection in equation (32), we obtain

$$\mathbb{LEK}^*_{t,T_1,T_N}=\theta_{T_1,T_N}\left(\mathbb{M}^{*(4)}_{t,T_1}-\left(\mathbb{M}^{*(2)}_{t,T_1}\right)^2\right) \qquad \text{(VI22)}$$

Let us now derive closed-form expressions for $\mathbb{E}^*_t\mathbb{M}^{*(3)}_{T_1,T_N}$ and $\mathbb{LES}^*_t$.

We can write $\mathbb{E}_t^*\mathbb{M}_{T_1,T_N}^{*(3)}$ and $\mathbb{LES}_t^*$ respectively as,

$$\begin{aligned}
\mathbb{E}_t^*\mathbb{M}_{T_1,T_N}^{*(3)} &= \mathbb{E}_t^*\left((R_{M,T_1,T_N} - R_{f,T_1,T_N})^3\right) \\
&= \mathbb{E}_t^*\left(R_{M,T_1,T_N}^3 - R_{f,T_1,T_N}^3\right) - 3R_{f,T_1,T_N}\mathbb{E}_t^*\mathbb{M}_{T_1,T_N}^{*(2)}, \quad \text{(VI23)}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{LES}_t^* &= \mathbb{COV}_t^*\left(R_{M,t,T_1}, \mathbb{M}_{T_1,T_N}^{*(3)}\right) \\
&= \mathbb{COV}_t^*\left(R_{M,t,T_1}, \mathbb{E}_{T_1}^*\left(R_{M,T_1,T_N}^3 - R_{f,T_1,T_N}^3\right) - 3R_{f,T_1,T_N}\mathbb{M}_{T_1,T_N}^{*(2)}\right) \\
&= \mathbb{COV}_t^*\left(R_{M,t,T_1}, \mathbb{E}_{T_1}^*\left(R_{M,T_1,T_N}^3 - R_{f,T_1,T_N}^3\right)\right) - 3R_{f,T_1,T_N}\mathbb{COV}_t^*\left(R_{M,t,T_1}, \mathbb{M}_{T_1,T_N}^{*(2)}\right) \\
&= \mathbb{COV}_t^*\left(R_{M,t,T_1}, \mathbb{E}_{T_1}^*\left(R_{M,T_1,T_N}^3 - R_{f,T_1,T_N}^3\right)\right) - 3R_{f,T_1,T_N}\mathbb{LEV}_t^*. \quad \text{(VI24)}
\end{aligned}$$

Let us assume that the term $\mathbb{E}_{T_1}^*\left(R_{M,T_1,T_N}^3 - R_{f,T_1,T_N}^3\right)$ is a nonlinear function g of $R_{M,t,T_1} - R_{f,t,T_1}$:

$$\mathbb{E}_{T_1}^*\left(R_{M,T_1,T_N}^3\right) - R_{f,T_1,T_N}^3 = \gamma_{T_1,T_N} g[R_{M,t,T_1} - R_{f,t,T_1}] + \upsilon_{T_1}, \quad \text{(VI25)}$$

with $\mathbb{E}_t^*\left(\upsilon_{T_1}|R_{M,t,T_1}\right) = \mathbb{E}_t^*\left(\upsilon_{T_1}\right) = 0$. Multiplying both sides of Equation (VI25) by R_{M,t,T_1}^3 and taking the time-t risk-neutral expectation, we obtain,

$$\gamma_{T_1,T_N} = \frac{\mathbb{M}_{t,T_N}^{*(3)} + 3R_{f,t,T_N}\mathbb{M}_{t,T_N}^{*(2)} - R_{f,T_1,T_N}^3\left(\mathbb{M}_{t,T_1}^{*(3)} + 3R_{f,t,T_1}\mathbb{M}_{t,T_1}^{*(2)}\right)}{\mathbb{E}_t^*\left(R_{M,t,T_1}^3 g[R_{M,t,T_1} - R_{f,t,T_1}]\right)}. \quad \text{(VI26)}$$

If we use $g[R_{M,t,T_1} - R_{f,t,T_1}] = R_{M,t,T_1}^3$, we obtain

$$\gamma_{T_1,T_N} = \frac{\mathbb{M}_{t,T_N}^{*(3)} + 3R_{f,t,T_N}\mathbb{M}_{t,T_N}^{*(2)} - R_{f,T_1,T_N}^3\left(\mathbb{M}_{t,T_1}^{*(3)} + 3R_{f,t,T_1}\mathbb{M}_{t,T_1}^{*(2)}\right)}{\mathbb{E}_t^*\left(R_{M,t,T_1}^6\right)}. \quad \text{(VI27)}$$

Taking the expectation of (VI25) under the risk neutral measure,

$$\mathbb{E}_t^*\left(R_{M,T_1,T_N}^3\right) - R_{f,T_1,T_N}^3 = \gamma_{T_1,T_N}\mathbb{E}_t^*\left(R_{M,t,T_1}^3\right). \tag{VI28}$$

Multiplying both sides of Equation (VI25) by R_{M,t,T_1} and taking the time-t risk-neutral expectation

$$\mathbb{E}_t^*\left(R_{M,t,T_1}R_{M,T_1,T_N}^3\right) = R_{f,t,T_1}R_{f,T_1,T_N}^3 + \gamma_{T_1,T_N}\mathbb{E}_t^*\left(R_{M,t,T_1}^4\right). \tag{VI29}$$

Therefore, using Equations (VI23) and (VI24) we obtain $\mathbb{E}_t^*\mathbb{M}_{T_1,T_N}^{*(3)}$ and $\mathbb{LES}_t^*$ as,

$$\mathbb{E}_t^*\mathbb{M}_{T_1,T_N}^{*(3)} = \gamma_{T_1,T_N}\mathbb{E}_t^*\left(R_{M,t,T_1}^3\right) - 3R_{f,T_1,T_N}\mathbb{E}_t^*\mathbb{M}_{T_1,T_N}^{*(2)},$$

and

$$\mathbb{LES}_t^* \;=\; \gamma_{T_1,T_N}\mathbb{E}_t^*\left(R_{M,t,T_1}^4\right) - R_{f,t,T_1}\mathbb{E}_t^*\mathbb{M}_{T_1,T_N}^{*(3)} - 3R_{f,t,T_1}R_{f,T_1,T_N}\mathbb{E}_t^*\mathbb{M}_{T_1,T_N}^{*(2)} - 3R_{f,T_1,T_N}\mathbb{LEV}_t^*.$$

To compute the physical variance, we also need the following moments which we obtain using a similar approach:

$$\mathbb{E}_t^*\left(R_{M,t,T_1} - R_{f,t,T_1}\right)^3\left(R_{M,T_1,T_N} - R_{f,T_1,T_N}\right)^2 = \mathbb{E}_t^*\left(R_{M,t,T_1} - R_{f,t,T_1}\right)^3\mathbb{E}_{T_1}^*\left(R_{M,T_1,T_N} - R_{f,T_1,T_N}\right)^2. \tag{VI30}$$

Using expression (32)

$$\mathbb{E}_{T_1}^*\left(R_{M,T_1,T_N} - R_{f,T_1,T_N}\right)^2 = \theta_{T_1,T_N}\left(R_{M,t,T_1} - R_{f,t,T_1}\right)^2 + \epsilon_{T_1}, \tag{VI31}$$

it follows that

$$\mathbb{E}_t^* \left(R_{M,t,T_1} - R_{f,t,T_1}\right)^3 \left(R_{M,T_1,T_N} - R_{f,T_1,T_N}\right)^2 = \theta_{T_1,T_N} \mathbb{E}_t^* \left(R_{M,t,T_1} - R_{f,t,T_1}\right)^5 . \tag{VI32}$$

Finally,

$$\begin{aligned}\mathbb{E}_t^* \left(R_{M,t,T_1} - R_{f,t,T_1}\right)^2 \left(R_{M,T_1,T_N} - R_{f,T_1,T_N}\right)^3 = \; & \mathbb{E}_t^* \left(R_{M,t,T_1} - R_{f,t,T_1}\right)^2 R_{M,T_1,T_N}^3 \\ & - R_{f,T_1,T_N}^3 \mathbb{E}_t^* \left(R_{M,t,T_1} - R_{f,t,T_1}\right)^2 \\ & + 3R_{f,T_1,T_N}^2 \mathbb{E}_t^* \left(R_{M,t,T_1} - R_{f,t,T_1}\right)^2 R_{M,T_1,T_N} \\ & - 3R_{f,T_1,T_N} \mathbb{E}_t^* \left(R_{M,t,T_1} - R_{f,t,T_1}\right)^2 R_{M,T_1,T_N}^2 .\end{aligned} \tag{VI33}$$

This expression simplifies to

$$\begin{aligned}\mathbb{E}_t^* \left(R_{M,t,T_1} - R_{f,t,T_1}\right)^2 \left(R_{M,T_1,T_N} - R_{f,T_1,T_N}\right)^3 = \; & \mathbb{E}_t^* \left(R_{M,t,T_1} - R_{f,t,T_1}\right)^2 \mathbb{E}_{T_1}^* R_{M,T_1,T_N}^3 \\ & + R_{f,T_1,T_N}^3 \mathbb{M}_{t,T_1}^{*(2)} \\ & - 3R_{f,T_1,T_N} \mathbb{E}_t^* \left(R_{M,t,T_1} - R_{f,t,T_1}\right)^2 \mathbb{M}_{T_1,T_N}^{*(2)} .\end{aligned} \tag{VI34}$$

Since

$$\mathbb{E}_{T_1}^* R_{M,T_1,T_N}^3 = \gamma_{T_1,T_N} R_{M,t,T_1}^3 , \tag{VI35}$$

it follows that

$$\begin{aligned}\mathbb{E}_t^*\left(R_{M,t,T_1}-R_{f,t,T_1}\right)^2\left(R_{M,T_1,T_N}-R_{f,T_1,T_N}\right)^3 = \;& \gamma_t\mathbb{E}_t^*\left(\left(R_{M,t,T_1}-R_{f,t,T_1}\right)^2 R^3_{M,t,T_1}\right)\\ &+ R^3_{f,T_1,T_N}\mathbb{M}^{*(2)}_{t,T_1}\\ &- 3R_{f,T_1,T_N}\mathbb{E}_t^*\left(\left(R_{M,t,T_1}-R_{f,t,T_1}\right)^2\mathbb{M}^{*(2)}_{T_1,T_N}\right)\end{aligned} \tag{VI36}$$

where expression $\mathbb{E}_t^*\left(\left(R_{M,t,T_1}-R_{f,t,T_1}\right)^2\mathbb{M}^{*(2)}_{T_1,T_N}\right)$ can be derived as follows:

$$\left(R_{M,t,T_1}-R_{f,t,T_1}\right)^2\mathbb{M}^{*(2)}_{T_1,T_N} = \theta_{T_1,T_N}\left(R_{M,t,T_1}-R_{f,t,T_1}\right)^4+\left(R_{M,t,T_1}-R_{f,t,T_1}\right)^2\varepsilon_{T_1} \tag{VI37}$$

and

$$\begin{aligned}\mathbb{E}_t^*\left(\left(R_{M,t,T_1}-R_{f,t,T_1}\right)^2\mathbb{M}^{*(2)}_{T_1,T_N}\right) &= \theta_{T_1,T_N}\mathbb{E}_t^*\left(R_{M,t,T_1}-R_{f,t,T_1}\right)^4+\mathbb{E}_t^*\left(R_{M,t,T_1}-R_{f,t,T_1}\right)^2\varepsilon_{T_1}\\ &= \theta_{T_1,T_N}\mathbb{E}_t^*\left(R_{M,t,T_1}-R_{f,t,T_1}\right)^4\end{aligned} \tag{VI38}$$

F. The case with consumption: Third-order Taylor expansion-series

Assume that (i) Assumptions 1 and 2 hold, (ii) the consumption-wealth ratio is positively related to the market return and (iii) the correlation between the square of the consumption wealth ratio and the market return is negative (condition reminiscent of market co-skewness). Under these assumptions, our measure of expected excess return is a lower bound to the true measure of market expected excess return.

A third-order Taylor expansion of $\frac{u'\left[\overline{W}_{t,T_N}\right]}{u'\left[W_{t,T_N}\right]}$ around $\mathbf{X}=\mathbf{X}_0$ gives

$$
\begin{aligned}
\mathbf{G} \;=\;& 1+(\mathbf{x}-\mathbf{x}_0)\frac{1}{\mathbf{x}_0}\frac{1}{\tau_t}+(\mathbf{y}-\mathbf{y}_0)\frac{1}{\mathbf{y}_0}\frac{1}{\tau_t}+(\mathbf{z}-\mathbf{z}_0)\frac{1}{\mathbf{z}_0}\frac{1}{\tau_t}+\frac{1}{\mathbf{x}_0^2}\frac{(1-\rho_t)}{\tau_t^2}(\mathbf{x}-\mathbf{x}_0)^2 \\
&+\frac{1}{\mathbf{y}_0^2}\frac{(1-\rho_t)}{\tau_t^2}(\mathbf{y}-\mathbf{y}_0)^2+\frac{1}{\mathbf{z}_0^2}\frac{(1-\rho_t)}{\tau_t^2}(\mathbf{z}-\mathbf{z}_0)^2+\frac{1}{\mathbf{x}_0\mathbf{y}_0}\left(\frac{2(1-\rho_t)}{\tau_t^2}\right)(\mathbf{x}-\mathbf{x}_0)(\mathbf{y}-\mathbf{y}_0) \\
&+\left(\frac{\partial^2\mathbf{G}}{\partial\mathbf{x}\partial\mathbf{z}}\right)_{\mathbf{x}=\mathbf{x}_0}(\mathbf{x}-\mathbf{x}_0)(\mathbf{z}-\mathbf{z}_0)+\left(\frac{\partial^2\mathbf{G}}{\partial\mathbf{y}\partial\mathbf{z}}\right)_{\mathbf{x}=\mathbf{x}_0}(\mathbf{z}-\mathbf{z}_0)(\mathbf{y}-\mathbf{y}_0) \\
&+\frac{1}{\mathbf{x}_0^3}\frac{(\kappa_t-2\rho_t+1)}{\tau_t^3}(\mathbf{x}-\mathbf{x}_0)^3+\frac{1}{\mathbf{z}_0^3}\frac{(\kappa_t-2\rho_t+1)}{\tau_t^3}(\mathbf{z}-\mathbf{z}_0)^3+\frac{1}{\mathbf{y}_0^3}\frac{(\kappa_t-2\rho_t+1)}{\tau_t^3}(\mathbf{y}-\mathbf{y}_0)^3 \\
&+\frac{3}{3!}\frac{1}{\mathbf{x}_0^2\mathbf{y}_0}\left(\frac{4(1-\rho_t)}{\tau_t^2}+\frac{6(\kappa_t-2\rho_t+1)}{\tau_t^3}\right)(\mathbf{x}-\mathbf{x}_0)^2(\mathbf{y}-\mathbf{y}_0) \\
&+\frac{3}{3!}\frac{1}{\mathbf{y}_0^2\mathbf{x}_0}\left(\frac{4(1-\rho_t)}{\tau_t^2}+\frac{6(\kappa_t-2\rho_t+1)}{\tau_t^3}\right)(\mathbf{y}-\mathbf{y}_0)^2(\mathbf{x}-\mathbf{x}_0) \\
&+\frac{3}{3!}\frac{1}{\mathbf{z}_0^2\mathbf{x}_0}\left(\frac{4(1-\rho_t)}{\tau_t^2}+\frac{6(\kappa_t-2\rho_t+1)}{\tau_t^3}\right)(\mathbf{z}-\mathbf{z}_0)^2(\mathbf{x}-\mathbf{x}_0) \\
&+\frac{3}{3!}\frac{1}{\mathbf{z}_0^2\mathbf{y}_0}\left(\frac{4(1-\rho_t)}{\tau_t^2}+\frac{6(\kappa_t-2\rho_t+1)}{\tau_t^3}\right)(\mathbf{z}-\mathbf{z}_0)^2(\mathbf{y}-\mathbf{y}_0) \\
&+6\frac{1}{3!}\left(\frac{\partial^3\mathbf{G}}{\partial\mathbf{x}\partial\mathbf{y}\partial\mathbf{z}}\right)_{\mathbf{x}=\mathbf{x}_0}(\mathbf{z}-\mathbf{z}_0)(\mathbf{y}-\mathbf{y}_0)(\mathbf{x}-\mathbf{x}_0) \\
&+3\frac{1}{3!}\left(\frac{\partial^3\mathbf{G}}{\partial^2\mathbf{x}\partial\mathbf{z}}\right)_{\mathbf{x}=\mathbf{x}_0}(\mathbf{x}-\mathbf{x}_0)^2(\mathbf{z}-\mathbf{z}_0)+3\frac{1}{3!}\left(\frac{\partial^3\mathbf{G}}{\partial^2\mathbf{y}\partial\mathbf{z}}\right)_{\mathbf{x}=\mathbf{x}_0}(\mathbf{y}-\mathbf{y}_0)^2(\mathbf{z}-\mathbf{z}_0) \qquad \text{(VI39)}
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\upsilon_{T_1} &= \mathbb{E}^*_{T_1}\mathbf{G} \\
&= 1 + (\mathbf{x}-\mathbf{x}_0)\frac{1}{\mathbf{x}_0}\frac{1}{\tau_t} + (\mathbf{y}-\mathbf{y}_0)\frac{1}{\mathbf{y}_0}\frac{1}{\tau_t} + \frac{1}{\mathbf{x}_0^2}\frac{(1-\rho_t)}{\tau_t^2}(\mathbf{x}-\mathbf{x}_0)^2 \\
&\quad + \frac{1}{\mathbf{y}_0^2}\frac{(1-\rho_t)}{\tau_t^2}(\mathbf{y}-\mathbf{y}_0)^2 + \frac{1}{\mathbf{z}_0^2}\frac{(1-\rho_t)}{\tau_t^2}\mathbb{E}^*_{T_1}(\mathbf{z}-\mathbf{z}_0)^2 \\
&\quad + \frac{1}{\mathbf{x}_0\mathbf{y}_0}\left(\frac{2(1-\rho_t)}{\tau_t^2}\right)(\mathbf{x}-\mathbf{x}_0)(\mathbf{y}-\mathbf{y}_0) \\
&\quad + \frac{1}{\mathbf{x}_0^3}\frac{(\kappa_t-2\rho_t+1)}{\tau_t^3}(\mathbf{x}-\mathbf{x}_0)^3 + \frac{1}{\mathbf{z}_0^3}\frac{(\kappa_t-2\rho_t+1)}{\tau_t^3}\mathbb{E}^*_{T_1}(\mathbf{z}-\mathbf{z}_0)^3 \\
&\quad + \frac{1}{\mathbf{y}_0^3}\frac{(\kappa_t-2\rho_t+1)}{\tau_t^3}(\mathbf{y}-\mathbf{y}_0)^3 \\
&\quad + \frac{3}{3!}\frac{1}{\mathbf{x}_0^2\mathbf{y}_0}\left(\frac{4(1-\rho_t)}{\tau_t^2} + \frac{6(\kappa_t-2\rho_t+1)}{\tau_t^3}\right)(\mathbf{x}-\mathbf{x}_0)^2(\mathbf{y}-\mathbf{y}_0) \\
&\quad + \frac{3}{3!}\frac{1}{\mathbf{y}_0^2\mathbf{x}_0}\left(\frac{4(1-\rho_t)}{\tau_t^2} + \frac{6(\kappa_t-2\rho_t+1)}{\tau_t^3}\right)(\mathbf{y}-\mathbf{y}_0)^2(\mathbf{x}-\mathbf{x}_0) \\
&\quad + \frac{3}{3!}\frac{1}{\mathbf{z}_0^2\mathbf{x}_0}\left(\frac{4(1-\rho_t)}{\tau_t^2} + \frac{6(\kappa_t-2\rho_t+1)}{\tau_t^3}\right)(\mathbf{x}-\mathbf{x}_0)\mathbb{E}^*_{T_1}(\mathbf{z}-\mathbf{z}_0)^2 \\
&\quad + \frac{3}{3!}\frac{1}{\mathbf{z}_0^2\mathbf{y}_0}\left(\frac{4(1-\rho_t)}{\tau_t^2} + \frac{6(\kappa_t-2\rho_t+1)}{\tau_t^3}\right)(\mathbf{y}-\mathbf{y}_0)\mathbb{E}^*_{T_1}(\mathbf{z}-\mathbf{z}_0)^2
\end{aligned} \tag{VI.40}$$

Using Equation (6) in the main text of the paper, it follows that

$$\begin{aligned}
\upsilon_{T_1} &= \mathbb{E}^*_{T_1}\mathbf{G} \\
&= 1 + (\mathbf{x}-\mathbf{x}_0)\frac{1}{\mathbf{x}_0}a_{1,t} + (\mathbf{y}-\mathbf{y}_0)\frac{1}{\mathbf{y}_0}\frac{1}{\tau_t} + \frac{1}{\mathbf{x}_0^2}a_{2,t}(\mathbf{x}-\mathbf{x}_0)^2 \\
&\quad + \frac{1}{\mathbf{y}_0^2}a_{2,t}(\mathbf{y}-\mathbf{y}_0)^2 + \frac{1}{\mathbf{z}_0^2}a_{2,t}\mathbb{E}^*_{T_1}(\mathbf{z}-\mathbf{z}_0)^2 + \frac{2}{\mathbf{x}_0\mathbf{y}_0}a_{2,t}(\mathbf{x}-\mathbf{x}_0)(\mathbf{y}-\mathbf{y}_0) \\
&\quad + \frac{1}{\mathbf{x}_0^3}a_{3,t}(\mathbf{x}-\mathbf{x}_0)^3 + \frac{1}{\mathbf{z}_0^3}a_{3,t}\mathbb{E}^*_{T_1}(\mathbf{z}-\mathbf{z}_0)^3 + \frac{1}{\mathbf{y}_0^3}a_{3,t}(\mathbf{y}-\mathbf{y}_0)^3 \\
&\quad + \frac{6}{3!}\frac{1}{\mathbf{x}_0^2\mathbf{y}_0}a_{2,3,t}(\mathbf{x}-\mathbf{x}_0)^2(\mathbf{y}-\mathbf{y}_0) + \frac{6}{3!}\frac{1}{\mathbf{y}_0^2\mathbf{x}_0}a_{2,3,t}(\mathbf{y}-\mathbf{y}_0)^2(\mathbf{x}-\mathbf{x}_0) \\
&\quad + \frac{6}{3!}\frac{1}{\mathbf{z}_0^2\mathbf{x}_0}a_{2,3,t}(\mathbf{x}-\mathbf{x}_0)\mathbb{E}^*_{T_1}(\mathbf{z}-\mathbf{z}_0)^2 + \frac{6}{3!}\frac{1}{\mathbf{z}_0^2\mathbf{y}_0}a_{2,3,t}(\mathbf{y}-\mathbf{y}_0)\mathbb{E}^*_{T_1}(\mathbf{z}-\mathbf{z}_0)^2
\end{aligned} \tag{VI.41}$$

We then compute the expected value $\mathbb{E}_t^* \upsilon_{T_1}$ to obtain

$$\begin{aligned}
\mathbb{E}_t^* \upsilon_{T_1} &= 1 + (\mathbf{x} - \mathbf{x}_0) \frac{1}{\mathbf{x}_0} a_{1,t} + \frac{1}{\mathbf{x}_0^2} a_{2,t} \mathbb{E}_t^* (\mathbf{x} - \mathbf{x}_0)^2 \\
&+ \frac{1}{\mathbf{y}_0^2} a_{2,t} \mathbb{E}_t^* (\mathbf{y} - \mathbf{y}_0)^2 + \frac{1}{\mathbf{z}_0^2} a_{2,t} \mathbb{E}_t^* \mathbb{E}_{T_1}^* (\mathbf{z} - \mathbf{z}_0)^2 + \frac{2}{\mathbf{x}_0 \mathbf{y}_0} a_{2,t} \mathbb{E}_t^* (\mathbf{x} - \mathbf{x}_0) (\mathbf{y} - \mathbf{y}_0) \\
&+ \frac{1}{\mathbf{x}_0^3} a_{3,t} \mathbb{E}_t^* (\mathbf{x} - \mathbf{x}_0)^3 + \frac{1}{\mathbf{z}_0^3} a_{3,t} \mathbb{E}_t^* \mathbb{E}_{T_1}^* (\mathbf{z} - \mathbf{z}_0)^3 + \frac{1}{\mathbf{y}_0^3} a_{3,t} \mathbb{E}_t^* (\mathbf{y} - \mathbf{y}_0)^3 \\
&+ \frac{1}{\mathbf{y}_0^2 \mathbf{x}_0} a_{2,3,t} \mathbb{E}_t^* (\mathbf{y} - \mathbf{y}_0)^2 (\mathbf{x} - \mathbf{x}_0) + \frac{1}{\mathbf{z}_0^2 \mathbf{x}_0} a_{2,3,t} \mathbb{E}_t^* (\mathbf{x} - \mathbf{x}_0) \mathbb{E}_{T_1}^* (\mathbf{z} - \mathbf{z}_0)^2 \\
&+ \frac{1}{\mathbf{z}_0^2 \mathbf{y}_0} a_{2,3,t} \mathbb{COV}_t^* \left(\mathbf{y}, \mathbb{E}_{T_1}^* (\mathbf{z} - \mathbf{z}_0)^2 \right). \qquad \text{(VI42)}
\end{aligned}$$

Notice that $\mathbf{x} - \mathbf{x}_0 \leq 0$, and the following inequalities hold: $a_{1,t} > 0$, $a_{2,t} \leq 0$, $a_{3,t} \geq 0$, $a_{2,3,t} \geq 0$. Furthermore, $\mathbb{E}_t^* (\mathbf{x} - \mathbf{x}_0)^3 \leq 0$, $\mathbb{E}_t^* (\mathbf{y} - \mathbf{y}_0)^3 \leq 0$, $\mathbb{E}_t^* (\mathbf{x} - \mathbf{x}_0)^3 \leq 0$ and

$$\begin{aligned}
&\mathbb{E}_t^* (\mathbf{x} - \mathbf{x}_0) (\mathbf{y} - \mathbf{y}_0) = \mathbb{COV}_t^* (\mathbf{x} - \mathbf{x}_0, \mathbf{y}) \geq 0 \\
&\mathbb{E}_t^* (\mathbf{y} - \mathbf{y}_0)^2 (\mathbf{x} - \mathbf{x}_0) \leq 0 \text{ (because } (\mathbf{x} - \mathbf{x}_0) \leq 0) \\
&\mathbb{E}_t^* (\mathbf{x} - \mathbf{x}_0) \mathbb{E}_{T_1}^* (\mathbf{z} - \mathbf{z}_0)^2 \leq 0 \text{ (because } (\mathbf{x} - \mathbf{x}_0) \leq 0) \\
&\mathbb{COV}_t^* \left(\mathbf{y}, \mathbb{E}_{T_1}^* (\mathbf{z} - \mathbf{z}_0)^2 \right) = \mathbb{LEV}_t^* \leq 0. \qquad \text{(VI43)}
\end{aligned}$$

This allows us to bound $\mathbb{E}_t^* \upsilon_{T_1}$ as

$$\begin{aligned}
\mathbb{E}_t^* \upsilon_{T_1} &\leq 1 + \frac{1}{\mathbf{y}_0^2} a_{2,t} \mathbb{E}_t^* (\mathbf{y} - \mathbf{y}_0)^2 + \frac{1}{\mathbf{z}_0^2} a_{2,t} \mathbb{E}_t^* \mathbb{E}_{T_1}^* (\mathbf{z} - \mathbf{z}_0)^2 \\
&+ \frac{1}{\mathbf{z}_0^3} a_{3,t} \mathbb{E}_t^* \mathbb{E}_{T_1}^* (\mathbf{z} - \mathbf{z}_0)^3 + \frac{1}{\mathbf{y}_0^3} a_{3,t} \mathbb{E}_t^* (\mathbf{y} - \mathbf{y}_0)^3 \\
&+ \frac{1}{\mathbf{z}_0^2 \mathbf{y}_0} a_{2,3,t} \mathbb{COV}_t^* \left(\mathbf{y}, \mathbb{E}_{T_1}^* (\mathbf{z} - \mathbf{z}_0)^2 \right). \qquad \text{(VI44)}
\end{aligned}$$

As a result,

$$\frac{1}{\mathbb{E}_t^* \upsilon_{T_1}} \geq \frac{1}{\left\{\begin{array}{c} 1 + \frac{1}{\mathbf{y}_0^2} a_{2,t} \mathbb{E}_t^* \left(\mathbf{y} - \mathbf{y}_0\right)^2 + \frac{1}{\mathbf{z}_0^2} a_{2,t} \mathbb{E}_t^* \mathbb{E}_{T_1}^* \left(\mathbf{z} - \mathbf{z}_0\right)^2 \\ + \frac{1}{\mathbf{z}_0^3} a_{3,t} \mathbb{E}_t^* \mathbb{E}_{T_1}^* \left(\mathbf{z} - \mathbf{z}_0\right)^3 + \frac{1}{\mathbf{y}_0^3} a_{3,t} \mathbb{E}_t^* \left(\mathbf{y} - \mathbf{y}_0\right)^3 \\ + \frac{1}{\mathbf{z}_0^2 \mathbf{y}_0} a_{2,3,t} \mathbb{COV}_t^* \left(\mathbf{y}, \mathbb{E}_{T_1}^* \left(\mathbf{z} - \mathbf{z}_0\right)^2\right) \end{array}\right\}}. \tag{VI45}$$

Next, our goal is to bound $\mathbb{COV}_t^* \left(\upsilon_{T_1}, \mathbf{y} - \mathbf{y}_0\right) = \mathbb{COV}_t^* \left(\upsilon_{T_1}, R_{M,t,T_1} - R_{f,t,T_1}\right)$. We use (VI41) to compute this covariance as

$$\begin{aligned}
& \mathbb{COV}_t^* \left(\upsilon_{T_1}, \mathbf{y} - \mathbf{y}_0\right) \\
= \; & \frac{1}{\mathbf{y}_0} \frac{1}{\tau_t} \mathbb{VAR}_t^* \left(\mathbf{y}\right) + \frac{1}{\mathbf{x}_0^2} a_{2,t} \mathbb{COV}_t^* \left(\left(\mathbf{x} - \mathbf{x}_0\right)^2, \mathbf{y} - \mathbf{y}_0\right) \\
& + \frac{1}{\mathbf{y}_0^2} a_{2,t} \mathbb{COV}_t^* \left(\left(\mathbf{y} - \mathbf{y}_0\right)^2, \mathbf{y} - \mathbf{y}_0\right) + \frac{1}{\mathbf{z}_0^2} a_{2,t} \mathbb{COV}_t^* \left(\mathbb{E}_{T_1}^* \left(\mathbf{z} - \mathbf{z}_0\right)^2, \mathbf{y} - \mathbf{y}_0\right) \\
& + \frac{2}{\mathbf{x}_0 \mathbf{y}_0} a_{2,t} \mathbb{COV}_t^* \left(\left(\mathbf{x} - \mathbf{x}_0\right)\left(\mathbf{y} - \mathbf{y}_0\right), \mathbf{y} - \mathbf{y}_0\right) \\
& + \frac{1}{\mathbf{x}_0^3} a_{3,t} \mathbb{COV}_t^* \left(\left(\mathbf{x} - \mathbf{x}_0\right)^3, \mathbf{y} - \mathbf{y}_0\right) + \frac{1}{\mathbf{z}_0^3} a_{3,t} \mathbb{COV}_t^* \left(\mathbb{E}_{T_1}^* \left(\mathbf{z} - \mathbf{z}_0\right)^3, \mathbf{y} - \mathbf{y}_0\right) \\
& + \frac{1}{\mathbf{y}_0^3} a_{3,t} \mathbb{COV}_t^* \left(\left(\mathbf{y} - \mathbf{y}_0\right)^3, \mathbf{y} - \mathbf{y}_0\right) \\
& + \frac{6}{3!} \frac{1}{\mathbf{x}_0^2 \mathbf{y}_0} a_{2,3,t} \mathbb{COV}_t^* \left(\left(\mathbf{x} - \mathbf{x}_0\right)^2 \left(\mathbf{y} - \mathbf{y}_0\right), \mathbf{y} - \mathbf{y}_0\right) \\
& + \frac{6}{3!} \frac{1}{\mathbf{y}_0^2 \mathbf{x}_0} a_{2,3,t} \mathbb{COV}_t^* \left(\left(\mathbf{y} - \mathbf{y}_0\right)^2 \left(\mathbf{x} - \mathbf{x}_0\right), \mathbf{y} - \mathbf{y}_0\right) \\
& + \frac{6}{3!} \frac{1}{\mathbf{z}_0^2 \mathbf{x}_0} a_{2,3,t} \mathbb{COV}_t^* \left(\left(\mathbf{x} - \mathbf{x}_0\right) \mathbb{E}_{T_1}^* \left(\mathbf{z} - \mathbf{z}_0\right)^2, \mathbf{y} - \mathbf{y}_0\right) \\
& + \frac{6}{3!} \frac{1}{\mathbf{z}_0^2 \mathbf{y}_0} a_{2,3,t} \mathbb{COV}_t^* \left(\left(\mathbf{y} - \mathbf{y}_0\right) \mathbb{E}_{T_1}^* \left(\mathbf{z} - \mathbf{z}_0\right)^2, \mathbf{y} - \mathbf{y}_0\right).
\end{aligned}$$

Notice that

$$\mathbb{COV}_t^* \left(\left(\mathbf{x} - \mathbf{x}_0\right)\left(\mathbf{y} - \mathbf{y}_0\right), \mathbf{y} - \mathbf{y}_0\right) = \mathbb{E}_t^* \left(\mathbf{x} - \mathbf{x}_0\right)\left(\mathbf{y} - \mathbf{y}_0\right)^2 \leq 0 \text{ (since } \mathbf{x} \leq \mathbf{x}_0\text{)},$$

and

$$\mathbb{COV}_t^*\left((\mathbf{x}-\mathbf{x}_0)^2(\mathbf{y}-\mathbf{y}_0),\mathbf{y}-\mathbf{y}_0\right)=\mathbb{E}_t^*(\mathbf{x}-\mathbf{x}_0)^2(\mathbf{y}-\mathbf{y}_0)^2\geq 0.$$

We assume

$$\mathbb{COV}_t^*\left((\mathbf{x}-\mathbf{x}_0)^2,\mathbf{y}-\mathbf{y}_0\right)\leq 0 \tag{VI46}$$

and

$$\begin{aligned}
\mathbb{COV}_t^*\left((\mathbf{x}-\mathbf{x}_0)^3,\mathbf{y}-\mathbf{y}_0\right) &\geq 0, &\text{(VI47)}\\
\mathbb{COV}_t^*\left((\mathbf{y}-\mathbf{y}_0)^2(\mathbf{x}-\mathbf{x}_0),\mathbf{y}-\mathbf{y}_0\right) &\geq 0, &\text{(VI48)}\\
\mathbb{COV}_t^*\left((\mathbf{x}-\mathbf{x}_0)\,\mathbb{E}_{T_1}^*(\mathbf{z}-\mathbf{z}_0)^2,\mathbf{y}-\mathbf{y}_0\right) &\geq 0. &\text{(VI49)}
\end{aligned}$$

These conditions are reminiscent of the sign of coskewness and cokurtosis when random variables of interest are return. While $\mathbf{y}-\mathbf{y}_0$ and $\mathbf{z}-\mathbf{z}_0$ are realized excess returns, $\mathbf{x}-\mathbf{x}_0$ is a function of wealth-consumption ratio (see (II90)-(II92)). Because coskewness is negative (see Harvey and Siddique (2000)) and cokurtosis is positive (Dittmar (2002)) and the wealth-consumption ratio is positively correlated to the market return, one should expect (VI47)-(VI49) to hold.

Under conditions (VI46)-(VI49), it follows that

$$
\begin{aligned}
\mathbb{COV}_t^*\left(\upsilon_{T_1}, \mathbf{y}-\mathbf{y}_0\right) \geq & \frac{1}{\mathbf{y}_0}\frac{1}{\tau_t}\mathbb{VAR}_t^*\left(\mathbf{y}\right)+\frac{1}{\mathbf{y}_0^2}a_{2,t}\mathbb{E}_t^*\left(\mathbf{y}-\mathbf{y}_0\right)^3 \\
& +\frac{1}{\mathbf{z}_0^2}a_{2,t}\mathbb{COV}_t^*\left(\mathbb{E}_{T_1}^*\left(\mathbf{z}-\mathbf{z}_0\right)^2, \mathbf{y}-\mathbf{y}_0\right) \\
& +\frac{1}{\mathbf{z}_0^3}a_{3,t}\mathbb{COV}_t^*\left(\mathbb{E}_{T_1}^*\left(\mathbf{z}-\mathbf{z}_0\right)^3, \mathbf{y}-\mathbf{y}_0\right) \\
& +\frac{1}{\mathbf{y}_0^3}a_{3,t}\mathbb{COV}_t^*\left(\left(\mathbf{y}-\mathbf{y}_0\right)^3, \mathbf{y}-\mathbf{y}_0\right) \\
& +\frac{1}{\mathbf{z}_0^2\mathbf{y}_0}a_{2,3,t}\mathbb{COV}_t^*\left(\left(\mathbf{y}-\mathbf{y}_0\right)\mathbb{E}_{T_1}^*\left(\mathbf{z}-\mathbf{z}_0\right)^2, \mathbf{y}-\mathbf{y}_0\right).
\end{aligned}
\tag{VI50}
$$

Combining (VI45) and (VI50) leads to

$$
\begin{aligned}
\mathbb{E}_t\left(R_{M,t,T_1}-R_{f,t,T_1}\right) &= \frac{\mathbb{COV}_t^*\left[\upsilon_{T_1}, R_{M,t,T_1}\right]}{\mathbb{E}_t^*\upsilon_{T_1}} \\
&\geq \frac{\left\{\begin{array}{c}
\frac{1}{\mathbf{y}_0}\frac{1}{\tau_t}\mathbb{VAR}_t^*\left(\mathbf{y}\right)+\frac{1}{\mathbf{y}_0^2}a_{2,t}\mathbb{E}_t^*\left(\mathbf{y}-\mathbf{y}_0\right)^3 \\
\frac{1}{\mathbf{z}_0^2}a_{2,t}\mathbb{COV}_t^*\left(\mathbb{E}_{T_1}^*\left(\mathbf{z}-\mathbf{z}_0\right)^2, \mathbf{y}-\mathbf{y}_0\right) \\
+\frac{1}{\mathbf{z}_0^3}a_{3,t}\mathbb{COV}_t^*\left(\mathbb{E}_{T_1}^*\left(\mathbf{z}-\mathbf{z}_0\right)^3, \mathbf{y}-\mathbf{y}_0\right) \\
+\frac{1}{\mathbf{y}_0^3}a_{3,t}\mathbb{COV}_t^*\left(\left(\mathbf{y}-\mathbf{y}_0\right)^3, \mathbf{y}-\mathbf{y}_0\right)
\end{array}\right\}}{\left\{\begin{array}{c}
1+\frac{1}{\mathbf{y}_0^2}a_{2,t}\mathbb{E}_t^*\left(\mathbf{y}-\mathbf{y}_0\right)^2+\frac{1}{\mathbf{z}_0^2}a_{2,t}\mathbb{E}_t^*\mathbb{E}_{T_1}^*\left(\mathbf{z}-\mathbf{z}_0\right)^2 \\
+\frac{1}{\mathbf{z}_0^3}a_{3,t}\mathbb{E}_t^*\mathbb{E}_{T_1}^*\left(\mathbf{z}-\mathbf{z}_0\right)^3+\frac{1}{\mathbf{y}_0^3}a_{3,t}\mathbb{E}_t^*\left(\mathbf{y}-\mathbf{y}_0\right)^3 \\
+\frac{1}{\mathbf{z}_0^2\mathbf{y}_0}a_{2,3,t}\mathbb{COV}_t^*\left(\mathbf{y}, \mathbb{E}_{T_1}^*\left(\mathbf{z}-\mathbf{z}_0\right)^2\right)
\end{array}\right\}}
\end{aligned}
$$

which simplifies to

$$\mathbb{E}_t\left(R_{M,t,T_1}-R_{f,t,T_1}\right)\geq\frac{\left(\begin{array}{c}\frac{1}{\mathbf{y}_0}\frac{1}{\tau_t}\mathbb{M}^{*(2)}_{t,T_1}+\frac{1}{\mathbf{y}_0^2}a_{2,t}\mathbb{M}^{*(3)}_{t,T_1}+\frac{1}{\mathbf{y}_0^3}a_{3,t}\mathbb{M}^{*(4)}_{t,T_1}\\+\frac{1}{\mathbf{z}_0^2}a_{2,t}\mathbb{LEV}^*_t+\frac{1}{\mathbf{z}_0^3}a_{3,t}\mathbb{LES}^*_t\end{array}\right)}{\left(\begin{array}{c}1+\frac{1}{\mathbf{y}_0^2}a_{2,t}\mathbb{M}^{*(2)}_{t,T_1}+\frac{1}{\mathbf{z}_0^2}a_{2,t}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1,T_N}+\frac{1}{\mathbf{y}_0^3}a_{3,t}\mathbb{M}^{*(3)}_{t,T_1}\\+\frac{1}{\mathbf{z}_0^3}a_{3,t}\mathbb{E}^*_t\mathbb{M}^{*(3)}_{T_1,T_N}+\frac{1}{\mathbf{z}_0^2\mathbf{y}_0}a_{2,3,t}\mathbb{LEV}^*_t\end{array}\right)}. \tag{VI51}$$

We, thereafter, replace $\mathbf{y}_0$ and $\mathbf{z}_0$ by their expressions

$$\mathbb{E}_t\left(R_{M,t,T_1}-R_{f,t,T_1}\right)\geq\frac{\left(\begin{array}{c}\frac{1}{R_{f,t,T_1}}\frac{1}{\tau_t}\mathbb{M}^{*(2)}_{t,T_1}+\frac{1}{R^2_{f,t,T_1}}a_{2,t}\mathbb{M}^{*(3)}_{t,T_1}+\frac{1}{R^3_{f,t,T_1}}a_{3,t}\mathbb{M}^{*(4)}_{t,T_1}\\+\frac{1}{R^2_{f,T_1,T_N}}a_{2,t}\mathbb{LEV}^*_t+\frac{1}{R^3_{f,T_1,T_N}}a_{3,t}\mathbb{LES}^*_t\end{array}\right)}{\left(\begin{array}{c}1+\frac{1}{R^2_{f,t,T_1}}a_{2,t}\mathbb{M}^{*(2)}_{t,T_1}+\frac{1}{R^2_{f,T_1,T_N}}a_{2,t}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1,T_N}+\frac{1}{R^3_{f,t,T_1}}a_{3,t}\mathbb{M}^{*(3)}_{t,T_1}\\+\frac{1}{R^3_{f,T_1,T_N}}a_{3,t}\mathbb{E}^*_t\mathbb{M}^{*(3)}_{T_1,T_N}+\frac{1}{R^2_{f,T_1,T_N}R_{f,t,T_1}}a_{2,3,t}\mathbb{LEV}^*_t\end{array}\right)}. \tag{VI52}$$

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