From Tulip to Gold: Rational Bubbles in Consumable Assets

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March 16, 2026

Abstract

We study bubble dynamics when the speculative object is also a consumption good. In our model, an asset can serve both as a store of value and as a source of direct utility. This simple feature has important implications. First, even a small consumption value can make prices more stable by narrowing the range of self-fulfilling outcomes. Second, changes in consumption value can generate large price movements and episodes in which part of the asset is removed from circulation. We also allow supply to respond to prices, showing how costly production can generate richer price dynamics. The framework provides a unified way to think about bubbles in storable commodities, such as gold, and in digital assets, including cryptocurrencies.

JEL Classifications: D53, D91, E42, E44, G12

Keywords: Rational bubbles, consumable assets, storable commodities, endogenous asset supply, gold, cryptocurrencies

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1 Introduction

History offers many episodes in which market prices appeared to far exceed immediate "use value"—from tulips to currency-like objects and storable commodities such as cowry shells, metals, and spirits (Einzig, 1966; Hogendorn and Johnson, 1986; Garber, 1989; Goldgar, 2007). Yet some such objects prove more durable as stores of value than others, with gold the canonical example (Jevons, 1875; Menger, 1892; Bernstein and Volcker, 2012). As a recent illustration, gold reached record highs in late January 2026, briefly trading above roughly $5,500 per troy ounce after rising by more than 60% over 2025 (Figure 1, panel (a)).[1] This episode underscores gold's dual role: it is both an ornamental good and a store of value, often trading at valuations difficult to justify by contemporaneous use alone. While gold has direct use in ornamentation and industry, panel (b) suggests that recent demand was driven importantly by investment appetite—including bar-and-coin, ETF, and central-bank demand. Similar duality extends to other storable commodities (Machlup, 1969; Pindyck and Rotemberg, 1983; Roll, 1984; Pindyck, 2004) and, more recently, to digital assets that provide usage rights yet are also held for speculative resale (Cong, Li, and Wang, 2021, 2022; Sockin and Xiong, 2023; Han and Wang, 2025).

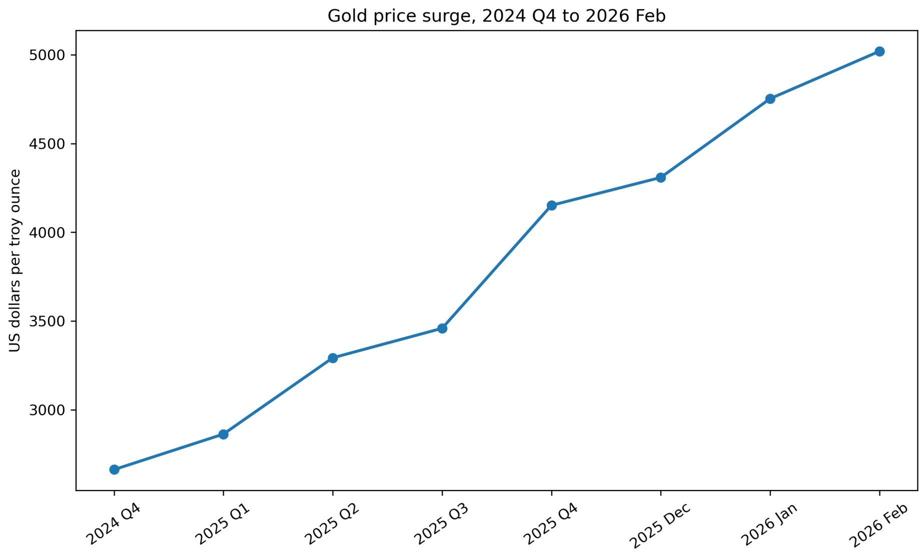
This paper studies bubble dynamics in precisely such environments—markets in which the bubble is also a consumption good. Our model emphasizes the interaction between these two roles. When the asset is consumed for utility, the aggregate outstanding quantity does not directly affect the utility generated by consumption.[2] When the asset serves as a store of value, by contrast, the outstanding quantity is central: for a given amount of intertemporal wealth transfer supported by the asset, halving the total outstanding inventory naturally doubles the value of each unit. This link makes asset consumption strategic substitutes: an agent has weaker incentives to consume the asset when others choose to consume, because aggregate consumption reduces the outstanding inventory and thereby raises the asset's value.

To illustrate the implications of this force, we develop an overlapping-generations model in which a durable asset can either be carried into old age as a store of value or consumed while young for direct utility. We show that this simple "consumable bubble" feature disciplines equilibrium dynamics by restricting the set of feasible equilibrium paths. Our model suggests that, although consumable-bubble assets may seemingly resemble pure bubble assets, the nature of bubble episodes differs in fundamental ways. In particular, episodes driven by consumable bubbles are more robust, in the sense that they are less sensitive to shifts in beliefs.

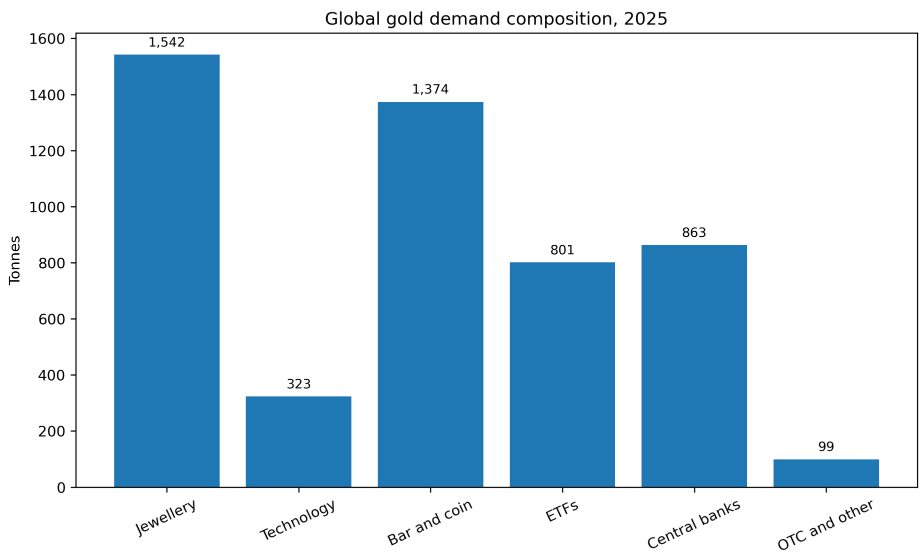
Specifically, the model delivers two main results. First, introducing a fixed consumption value—regardless of its magnitude—implies a unique equilibrium outcome. Because future generations always retain the option to consume the asset, any resale price must be consistent

[1] See, for example, "Gold's Glittering Run Pushes Prices Above $5,500," *Wall Street Journal*, Jan. 29, 2026.

[2] It may still affect the asset's value indirectly through the equilibrium pricing mechanism.



(a) Recent gold price dynamics. *Source*: World Bank, *Commodities Price Data (The Pink Sheet)*.



(b) Composition of global gold demand in 2025. *Source*: World Gold Council.

Figure 1: **Gold as a consumable bubble asset.** The top panel shows the recent surge in the gold price. The bottom panel shows the composition of global gold demand in 2025 across jewellery, technology, bar and coin investment, ETFs, central banks, and OTC/other demand.

with this outside option, which rules out a wide class of equilibria sustained solely by belief-driven valuations. Second, when the consumption value varies over time, the prospect of a high future consumption payoff increases the amount of wealth the asset can transfer across generations. In a two-state specification, a larger gap between the high and low consumption values raises both the asset price and the amount of wealth it transfers. The economic mechanism is analogous to a call option: the asset is primarily held as a store of value and is consumed only when its consumption value is high. Anticipation of this contingency raises the asset's continuation value and, with it, the amount of wealth transfer it can support. The model therefore predicts that sudden burn/bind/redeem episodes should be accompanied by sharp price movements, and that greater volatility in consumption value leads to a larger bubble sector.

Finally, we extend the framework to allow for endogenous asset supply. In the extension, new units of assets can be issued or produced in the future—through mining, minting, or platform issuance—in response to high current asset price. This extension introduces feedback between speculative demand, consumption utility, and quantity, providing a unified way to think about bubbles in markets ranging from precious metals and storable commodities to digital game tokens and collectibles.

The remainder of the paper is organized as follows. Section 2 situates our contribution within the related literature. Section 3 illustrates the main ideas in a simple model. Section 4 presents the baseline environment and defines equilibrium. Section 5 analyzes stochastic consumption value and the induced inventory adjustments. Section 6 introduces endogenous production and characterizes the resulting price–quantity feedback. Section 7 discusses empirical implications and concludes.

2 Literature

There is a large body of rational-bubble research showing that an asset can have positive value even in the absence of fundamentals because it facilitates intertemporal transfers in dynamically inefficient economies (Samuelson, 1958; Diamond, 1965). In such settings, rational bubbles can arise and coexist with fundamental equilibria, leaving equilibrium selection indeterminate. A canonical treatment in OLG models is Tirole (1985), who formalizes the conditions under which bubbles can be sustained under dynamic inefficiency and highlights the resulting multiplicity of price paths. More recent extensions (Blanchard, 1979; Blanchard and Watson, 1982; Weil, 1987; Han and Wang, 2025) emphasize stochastic bubble equilibria and the possibility of abrupt collapses driven by self-fulfilling shifts in beliefs, even in stationary environments. Our baseline motivating example builds directly on this tradition: in a pure-bubble OLG economy, there is a continuum of equilibrium price paths, including paths that drift toward the fundamental value. By contrast, introducing even a minimal consumption (or collateral) value can eliminate this multiplicity: equilibrium selection becomes pinned down, and the unique stationary bubble equilibrium emerges as the only feasible outcome.

Hirano and Toda (2025) show that when safe stores of value are scarce, a bubble may be necessary to clear asset markets and absorb desired savings, even in economies with longlived, dividend-paying assets (see also Hirano and Toda (2024) for a related review). Our framework is complementary. We likewise study an environment in which agents demand an intertemporal store of value, but we introduce an additional margin—direct consumption utility and the associated inventory-adjustment decision. In our model, this "use" margin not only influences the bubble's level by anchoring prices to a consumption outside option; it also refines equilibrium selection and generates discrete, state-dependent changes in circulating supply. Finally, whereas Hirano and Toda (2025) feature nonstationary bubbles, our setting—where the asset is consumable, as with commodities—admits a stationary bubble equilibrium and yields a unique equilibrium.

Because the asset in our model is both storable and consumable, the paper also connects to the commodity-storage literature, where prices reflect intertemporal scarcity, storage costs, and convenience yields. Classical contributions include Kaldor (1939), Working (1949), and the rational expectations storage models of Deaton and Laroque (1992, 1996). In commodity markets, shifts in demand for immediate use interact with inventory and can generate sharp price responses when stocks become scarce; in this sense, inventory is a key state variable (Brennan, 1958; Pindyck, 2004; Williams and Wright, 1991). Our mechanism differs in its microfoundations, but it is similar in spirit: the model predicts that periods of asset scarcity—together with burn/bind/redeem episodes—coincide with sharp price movements reminiscent of commodity-bubble dynamics, echoing the central intuition of storage models.

Our motivating discussion of gold and other precious metals connects to a long tradition emphasizing that monetary and quasi-monetary objects can derive value from both nonpecuniary "use" demand (ornamental, industrial, or status services) and their role as a store of value (e.g., Jevons, 1875; Menger, 1892; Einzig, 1966). In historical and macroeconomic perspectives, gold's durability as a valued asset is often linked to the presence of an outside option grounded in physical use and socially persistent demand, alongside its scarcity and storability (see, for example, Eichengreen (1992) for discussions). In commodity-finance work, precious metals are frequently treated as storable commodities whose prices reflect both investment demand and convenience yields, with inventory and scarcity shaping price dynamics (Fama and French, 1988; Routledge, Seppi, and Spatt, 2000). Metals such as copper, silver and platinum provide a complementary example because it has substantial industrial use demand (e.g., autocatalysts) in addition to investment demand, making shifts in fundamentals and effective scarcity especially salient for pricing (Henderson, Pearson, and Wang, 2015). Our model offers a unified conceptualization of these cases: when an asset has even a small and persistently plausible consumption/use value, equilibrium pricing becomes anchored by the option to consume, and bubble-like valuations become less sensitive to belief-driven regime shifts. This mechanism helps rationalize why precious metals such as gold and platinum may appear more "durable" as stores of value than purely pecuniary digital objects whose perceived future use demand is more fragile.

Finally, our paper also speaks to the emerging literature on cryptocurrencies, tokens, and digital collectibles, whose values combine use and speculation and whose platform design shapes both supply (issuance/mining/minting) and effective float (burning/binding/staking); see, for example, Easley, O'Hara, and Basu (2019); Prat and Walter (2021); Biais, Bisière, Bouvard, Casamatta, and Menkveld (2023). Recent theory treats crypto valuation as an equilibrium object with payment utility, adoption, and platform fundamentals (e.g., Cong, Li, and Wang, 2021, 2022; Sockin and Xiong, 2023; Han, Lee, and Li, 2025b). Related work examines how protocol rules translate into scarcity and prices, and shows that crypto prices co-move with network activity, platform usage, and speculative flows—consistent with time-varying "use" and resale components (see, for example, Liu and Tsyvinski (2021); Makarov and Schoar (2020); Han, Lee, and Li (2025a)). Our approach complements this literature by providing a parsimonious general-equilibrium mechanism in which consumption utility disciplines bubble indeterminacy and tightly links prices to use value and circulating supply. The model yields two key implications: first, usage value serves as an anchor that can support crypto valuations even when speculative demand accounts for most of the price; second, supply design and regulation matter because they shape expectations about future scarcity—and thus future value.

3 A Motivating Example

This section uses a simple example to illustrate how assigning even a small consumption value to an otherwise useless asset reduces the equilibrium indeterminacy of the pure bubble. We begin with a standard overlapping-generations (OLG) economy featuring a *pure* bubble, and then modify the environment so that agents can directly consume part of the asset and derive utility from doing so. We show that introducing consumption value, however small, sharply disciplines equilibrium price paths and leads to a more "stable" asset.

3.1 Standard OLG Economy with Pure Bubble

Consider a standard OLG model with a pure bubble. In each period, a unit mass of agents is born. Each agent lives for two periods and discounts second-period utility by $\beta \in (0, 1)$. Preferences over consumption of the endowment good are

$$U(c_y, c_o) = u(c_y) + \beta u(c_o),$$

where (c_y, c_o) denotes consumption when young and old, respectively. The function $u(\cdot)$ is differentiable, increasing, strictly concave and has elasticity of intertemporal substitution (EIS) at least one. Each agent is endowed with e_y units of the endowment good when young and e_o units when old. We assume that $u'(e_y) < u'(e_o)$ so that agents have an incentive to transfer

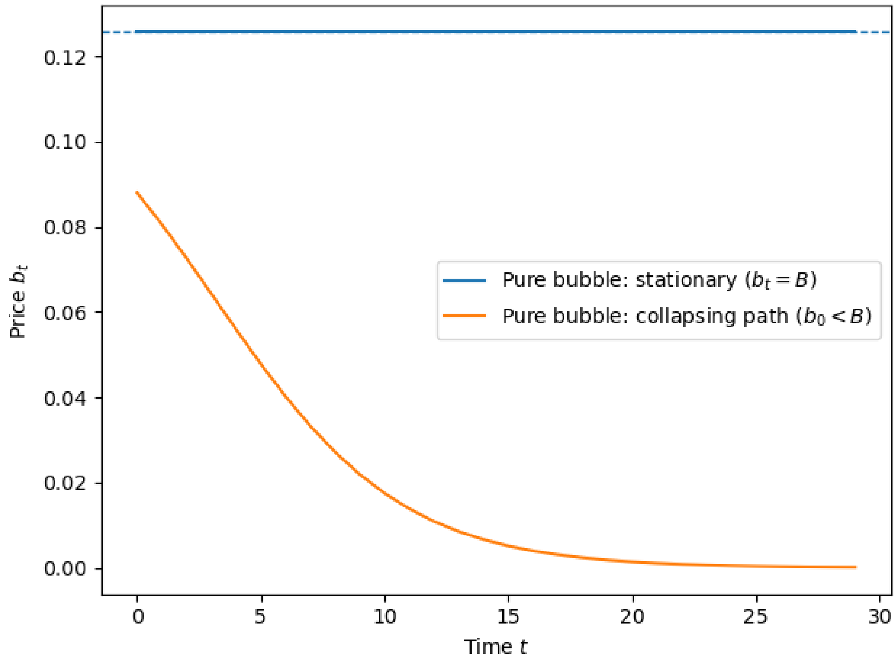


Figure 2: **Pure-bubble OLG: multiplicity of equilibrium price paths.** The horizontal dashed line is the stationary bubble price B that solves $u'(e_y - B) = \beta u'(e_o + B)$. Starting at $b_0 = B$ yields the stationary equilibrium $b_t \equiv B$. Starting at $b_0 < B$ generates a collapsing equilibrium path $\{b_t\}_{t\geq 0}$ that monotonically converges to 0, illustrating the standard indeterminacy of pure bubbles in overlapping-generations economies.

the endowment intertemporally to smooth consumption. However, the endowment good is perishable and cannot be stored across periods.

In addition, there is a durable asset in unit supply, a "digital currency" that is intrinsically useless (it yields no dividends and provides no direct utility). Despite this, the asset may carry a positive price in equilibrium because it facilitates intertemporal endowment transfers. There exists a stationary equilibrium in which the asset has a constant price B (in units of the endowment good) where B is pinned down by the first-order condition

$$u'(e_y - B) = \beta u'(e_o + B). \tag{1}$$

One might be tempted to interpret B as the value of the asset. However, this is not the only equilibrium. There is also a fundamental equilibrium in which the asset is worthless. Moreover, there is a continuum of possible price paths $\{b_t\}_{t\geq 0}$ satisfying

$$b_t\, u'(e_y - b_t) = \beta\, b_{t+1}\, u'(e_o + b_{t+1}). \tag{2}$$

In particular, any initial asset price $b_0 < B$ generates a sequence of equilibrium asset prices $\{b_t\}$ that declines over time and converges to 0 in the long run. Moreover, these equilibria can be ranked on social welfare by b_0: all generations are unanimously better off in an economy with a higher b_0.

This multiplicity has two important implications. First, even with full knowledge of fundamentals (preferences and endowments), the asset price is difficult to predict because beliefs can select among many self-consistent equilibria. Second, the asset can be "unstable" in the sense that shifts in beliefs can move the economy from one equilibrium path to another. With the multiplicity of deterministic equilibria, we can construct stochastic equilibria à la Weil (1987) in which the asset price may collapse abruptly even when all other aspects of the economy remain unchanged.

3.2 OLG Economy with Consumable Bubble

To see how even a small consumption value disciplines equilibrium price dynamics and leads to a much more "stable" asset, consider a modification in which the intrinsically useless digital currency is replaced by gold that provides a small direct benefit from consumption.[3] To simplify exposition, assume the asset enters utility additively and the option of consumption is only available to young agents. Consuming each unit of the asset delivers a small utility $\epsilon > 0$. Specifically, if an agent consumes (c_y, c_o) units of the endowment good when young and old, and consumes c_a units of the asset when young, then his utility is

$$U(c_y, c_o, c_a) = u(c_y) + \epsilon c_a + \beta u(c_o).$$

Denote the asset price at time t to be b_t and the asset holding of a young agent at time t to be x_e when young and x_o when old. We have $c_y = e_y - b_t x_e$, $c_o = e_o + b_{t+1} x_o$ and $c_a = x_e - x_o$. In this economy, the asset now serves dual purposes, both as a consumption good and as a store of value. Yet there is an intrinsic conflict between the two functions: if the agent consumes more of the asset, he gets less asset for helping him transfer the endowment goods into the future.

We now characterize the equilibrium. Since the asset can be consumed, we need to additionally track the aggregate quantity of the asset. Let x_t denote the aggregate quantity of the asset available in period t. Heuristically, an equilibrium is a sequence of prices and quantities $\{b_t, x_t\}$ such that, given the asset price and the consumption value, each generation makes the optimal purchase and consumption decisions and markets clear each period.[4]

[3] Note that we interpret consumption in a broad way. Take gold as an example, if an agent makes a piece of jewellery out of a gold bar since the gold price is low and incurs a two-percent wastage, this is interpreted as the agent consuming two percent of the gold for aesthetic value.

[4] A formal definition is provided in the next section.

When the consumption value is small, there is a stationary equilibrium where the asset has a constant price B that coincides with its pure-bubble counterpart. In that equilibrium, the asset is never consumed since it is more valuable as a store-of-value. Both the equilibrium price and welfare coincide with those in the pure-bubble economy.

What about other equilibria? Consider first the analogue of an equilibrium where the asset is worthless. Since the asset in this economy provides consumption utility, its price cannot be zero in any equilibrium. Instead, we focus on another perspective of the fundamental equilibrium, which is that all agents live in autarky. Correspondingly, there is a fundamental equilibrium in this economy if the agent eventually consumes all the assets.

Specifically, suppose in period t the young generation consumes all available assets, a young agent at time t cannot benefit from retaining some assets for sale in period $t+1$. Clearly, whether this no-deviation condition holds depends on the counterfactual asset price at zero inventory. We show that this condition cannot hold under any "reasonable" counterfactual asset price and thus no fundamental equilibrium exists.

Consider a deviation by agent i in which he retains $\eta_i > 0$ units of the asset (instead of consuming them) and carries them into period $t+1$ to sell to the next generation. Since generation $t+1$ can always buy the asset purely for its consumption value and the aggregate quantity of the asset is zero after all other agents consume the asset, it must be

$$b_{t+1}u'(e_y) \geq \epsilon.$$

Otherwise, any young agent would strictly prefer to buy and consume the asset at the margin, implying excess demand.

If retaining η_i units is not profitable for agent i, the utility gain from consuming those units must weakly exceed the discounted utility gain from carrying them and selling them next period:

$$\epsilon\eta_i \ \geq \ \beta\big[u(e_o + b_{t+1}\eta_i) - u(e_o)\big].$$

Taking $\eta_i \to 0$ yields the necessary condition

$$\epsilon \ \geq \ \beta b_{t+1}u'(e_o).$$

Combining this with $b_{t+1} \geq \epsilon/u'(e_y)$ implies

$$u'(e_y) \geq \beta u'(e_o).$$

But this cannot be the case since the economy allows for a positive stationary bubble price $B > 0$, satisfying

$$u'(e_y - B) = \beta u'(e_o + B).$$

Further analysis implies that no other type of equilibrium exists either. Any equilibrium path that involves no bubble consumption must coincide with the stationary equilibrium: if the price ever starts below the stationary level, it drifts downward over time and eventually becomes low enough that young agents strictly prefer to buy the asset for consumption. Moreover, there is no equilibrium where the agent consumes part of the bubble in some periods. Specifically, if agents in generation t consume the asset, the marginal benefit of holding the asset and consuming the asset must be equal. That is,

$$\beta b_{t+1} u'(e_o + b_{t+1} x_{t+1}) = \epsilon.$$

However, the first-order condition and the market-clearing condition of the generation $t+1$ imply that

$$b_{t+1} u'(e_y - b_{t+1} x_{t+1}) \geq \epsilon.$$

Thus, $u'(e_y - b_{t+1}) \geq \beta u'(e_o + b_{t+1})$ and $b_{t+1} > B$. This is impossible since the period-t first-order condition then implies that $b_t > B$. With such a high asset price and low consumption value, the agent would rather use the asset to store the value of the endowment good. Intuitively, asset consumption reduces the outstanding quantity, making the remaining asset scarcer and therefore more valuable as a store-of-value; as other agents consume, the incentive for any given agent to not consume (and carry the asset forward) becomes stronger, invalidating an equilibrium where the agent consumes part of the asset.

As shown in this example, introducing even a small consumption value eliminates the multiplicity present in the pure-bubble economy and makes the stationary equilibrium the only equilibrium. This also suggests that a bubble asset with a small consumption value can be more stable than an intrinsically useless bubble asset in the following sense: For a bubble with consumption value, a "burst" can only be triggered by changes in fundamentals (e.g., preferences or endowments). For instance, if there is an unexpected shock such that the endowment of the old generation becomes larger, the incentive to transfer wealth through time decreases, and the bubble price will become lower. However, since there is a unique equilibrium, the asset price will not be affected by shifts in beliefs about the bubble itself. On the contrary, for a bubble with no intrinsic value with a continuum of equilibria, even when the fundamentals in the economy stay the same, the bubble price can be changed by a belief shock over the value of the bubble asset.

4 Model

Building on the motivating example, this section formally describes the model environment and defines equilibrium when the bubble asset can be consumed for utility. Relative to the motivating example, the model introduces several generalizations. In particular, we allow the consumption value to be random and time-varying. Moreover, in the motivating example, although the asset has consumption value, it is never consumed on the equilibrium path. In the full model, we show that when the consumption value is sufficiently high, the asset is consumed.

The agents' endowments and utility function are the same as in the previous section. The difference is that the (per-unit) consumption value, denoted by ϵ, is random and takes two realizations ϵ_h and ϵ_l with $\epsilon_h > \epsilon_l$. We assume that the consumption value is i.i.d. across periods and that its realization is observed by all agents at the beginning of each period.[5] In each period, ϵ_h (ϵ_l) is realized with probability p $(1-p)$. If $\epsilon_t = \epsilon_h$, then a young agent at time t obtains utility $\epsilon_h c_a$ from consuming c_a units of the bubble asset.

The timing within period t is as follows.

1. At the beginning of the period, generation t is born with e_y units of endowment goods. The consumption value realizes as $\epsilon_t = \epsilon_h$ with probability p and $\epsilon_t = \epsilon_l$ with probability $1-p$. Generation $t-1$ receives e_o units of endowment goods.

2. Generation $t-1$ sells all its assets in a centralized market. The price b_t is determined by market clearing so that demand equals the total supply x_{t-1}.

3. Generation $t-1$ and generation t consume their endowment goods. Generation t also chooses how much of the bubble asset to consume.

4. All leftover endowment goods fully depreciate, and generation $t-1$ exits the economy.

To define equilibrium, we need to specify agents' purchasing and consumption behavior. Taking prices as given, each agent makes optimal purchasing and consumption decisions, and the bubble price is then determined by market clearing. Moreover, as discussed in the previous section, agents can consume all bubble assets and revert to autarky. Whether this is profitable depends on the counterfactual bubble price if all bubbles were consumed.

We therefore first investigate the conditions that must be satisfied by this counterfactual price if consuming all bubble assets is on the equilibrium path. Suppose that generation t consumes the entire bubble asset in period t. Consider a deviation by a single agent who retains some of the asset to sell to the new generation in period $t+1$. Denote this (finite) counterfactual market price by b_{t+1}.[6] Since an individual agent's holdings only consist of an infinitesimal portion of the aggregate amount of the asset, the market price is unaffected by the amount the deviating agent retains. If the individual keeps η units of the asset, the no-deviation condition is

$$\epsilon_t \eta \geq \beta E_t \left[u(e_o + b_{t+1}\eta) - u(e_o) \right].$$

[5] This can be extended to a Markov setting with similar results.

[6] If the price were infinite, then the agent would clearly have no incentive to consume all of his assets.

Dividing both sides by η and taking $\eta \to 0$, we obtain

$$\epsilon_t \geq \beta E_t \left[b_{t+1} u'(e_o) \right]. \tag{3}$$

Next, we analyze the dynamics of b_{t+i} for $i \geq 1$. These prices are determined by the optimal decision of generation $t+i$. The tradeoff is similar, and the optimality condition becomes

$$b_{t+i} u'(e_y) = \max \left\{ \epsilon_{t+i}, \beta E_{t+i} \big(b_{t+i+1} u'(e_o) \big) \right\}, \tag{4}$$

where the left-hand side is the marginal cost of purchasing the asset and the right-hand side is the marginal benefit of consuming or holding it.

We now define equilibrium. Let x_t denote the total amount of the bubble asset at the beginning of period t, and let b_t denote the bubble price in period t. Let the history at time t be $H_t = \{b_{i-1}, x_i, \epsilon_i\}_{i=0}^{t}$, which collects all realized quantities up to period t after generation t is born, with $b_{-1} = 0$. Since the economy starts with one unit of the bubble asset, $x_0 = 1$. Under this notation, the amount of the bubble asset consumed by generation t is $x_t - x_{t+1}$.

Definition 1. *An equilibrium consists of a sequence of non-negative bubble prices $\{b_t(H_t)\}_{t=0}^{\infty}$ and a sequence of non-negative bubble inventories $\{x_{t+1}(H_t)\}_{t=0}^{\infty}$ such that:*

1. *Non-negative consumption: $x_{t+1} \leq x_t$.*
2. *Optimal purchasing: Given H_t, if $x_t > 0$,*

$$b_t u'(e_y - b_t x_t) = \max \Big\{ \beta E_t \big[b_{t+1} u'(e_o + b_{t+1} x_{t+1}) \big],\ \epsilon_t \Big\}.$$

3. *Optimal consumption: Given H_t, if $x_{t+1} > 0$,*

$$\beta E_t \big[b_{t+1} u'(e_o + b_{t+1} x_{t+1}) \big] \geq \epsilon_t.$$

   Equality holds when $x_t - x_{t+1} > 0$.

4. *No deviation to autarky: Given H_t, if $x_t > 0$ and $x_{t+1} = 0$, the continuation economy satisfies conditions (3) and (4).*

To capture the possibility that the asset is more valuable as a consumption good than as a store of value, we examine whether the consumption value is large enough that a stationary equilibrium without consumption cannot be sustained. Recall that, if a stationary equilibrium exists, the bubble size B is the unique solution to

$$u'(e_y - B) = \beta u'(e_o + B). \tag{5}$$

We say the asset has a *small* consumption value if

$$bu'(e_y - B) \geq \epsilon_h > \epsilon_l \geq 0,$$

since in this case, there is always a stationary equilibrium with no consumption. Conversely, we say the asset could have a *large* consumption value if

$$\epsilon_h > bu'(e_y - B) > \epsilon_l \geq 0.$$

We will show that equilibrium properties differ depending on whether the asset has a large consumption value.

Second, instead of considering dividend-paying assets as in the standard bubble literature, we analyze the possibility that the asset can be consumed. We argue that the bubble asset in our model is consistent with the traditional definition of bubble. Specifically, in the deterministic consumption value case where $\epsilon_l = \epsilon_h = \epsilon$, the asset might be considered as a dividend-paying asset that is able to deliver any non-negative dividend sequence as long as the sum of the dividends is less or equal to ϵx_0. Using the result from Montrucchio (2004) and Hirano and Toda (2025), the summation of the D/P ratio clearly converges and thus the asset has a bubble component.

Before turning to the equilibrium analysis, we highlight two modeling assumptions. First, we assume that the utility function features an elasticity of intertemporal substitution (EIS) weakly greater than one. This condition is satisfied, for example, by log utility and by CRRA preferences with coefficient of relative risk aversion $\gamma \leq 1$. Economically, EIS ≥ 1 implies that the substitution effect dominates the income effect: when investment opportunities improve, agents invest more and consume less. This assumption delivers a monotonicity property in asset price: holding fixed all other primitives and shocks, a lower asset price today implies a lower expected asset price in the future.[7] This assumption is commonly adopted for tractability; for instance, Achdou, Han, Lasry, Lions, and Moll (2022) uses it to establish equilibrium uniqueness in a different setting.

Second, rather than focusing on dividend-paying assets as in the standard bubble literature, we consider the situation where the asset can be consumed for utility. We emphasize that the resulting object is still a "bubble" in the traditional sense. In the special case with a deterministic consumption value, $\epsilon_l = \epsilon_h = \epsilon$, the asset can be interpreted as a dividend-paying claim capable of implementing any nonnegative dividend sequence, as long as the total dividends do not exceed ϵx_0. In the equilibrium, the sum of the dividend-price ratio is zero, and thus the asset is a bubble.

[7] Absent this restriction, there could be endogenous volatility in which a lower current asset price leads to a higher expected future price.

5 Analysis

In this section, we analyze the model and characterize the unique equilibrium. In particular, if the asset's consumption value is always small, similar to the result in the previous section, the only equilibrium is a stationary equilibrium in which the asset is never consumed. If the asset could have a large consumption value, the asset will be consumed for the first and only time at the first realization of a large consumption value. Moreover, even before the asset is consumed, the possibility of large consumption value makes the asset a more valuable storage of value.

We begin by showing that, regardless of the magnitude of consumption value, there is no fundamental equilibrium in which the economy eventually falls into autarky. Equivalently, in no equilibrium does some generation consume the entire stock of the asset.

Proposition 1. *In any equilibrium, it cannot be the case that there exists a t such that $x_t = 0$ after some history.*

The intuition behind this result is that the asset plays two distinct roles: a consumption good and the store of value. These roles interact in a way that makes consuming everything unattractive. When the asset is consumed for utility, the aggregate supply matters only indirectly, through its effect on the price. By contrast, when the asset is used as a store of value, the aggregate supply directly determines its scarcity value: all else equal, the smaller the remaining stock, the higher the asset's value. In addition, a positive consumption value provides an anchor of price, so the asset is never valueless even without self-fulfilling coordination. Taken together, these forces imply that if others were to consume the entire stock, an individual would strictly prefer to retain his units to benefit from the induced scarcity. This rules out the existence of the fundamental equilibrium.

5.1 Bubble with small consumption value

We begin with the case in which the asset's consumption value is small, in the sense that

$$Bu'(e_y - B) \ \geq \ \epsilon_h \ > \ \epsilon_l \ \geq \ 0.$$

Under this condition, the stationary equilibrium in which $b_t = B$ and $x_{t+1} = 1$ after every history clearly exists. We show that it is, in fact, the only equilibrium.

Proposition 2. *If $Bu'(e_y - B) \geq \epsilon_h > \epsilon_l \geq 0$, $b_t = B$ and $x_t = 1$ for all t is the unique equilibrium of the economy.*

We sketch the argument here and relegate the formal proof to the Appendix. Clearly, the proposed price and asset supply constitute an equilibrium. To establish uniqueness, we first show that the equilibrium bubble size $b_t x_t$ cannot exceed B along any equilibrium path. Intuitively, if the bubble were ever larger than B, then, with positive probability, it would have to grow further in the future to satisfy agents' first-order conditions. Iterating this logic implies that the bubble would eventually exceed the entire economy, which is infeasible.

This bound implies uniqueness whenever the economy starts at price B. At this price level, no agent wishes to consume the asset. Moreover, if the lower next-period price (corresponding to $\epsilon_{t+1} \in \{\epsilon_l, \epsilon_h\}$) were below B, then the higher next-period price would have to exceed B. This cannot be the case, so the only possibility is $b_{t+1} = B$ under both realizations of ϵ_{t+1}. Iterating forward yields $b_t = B$ for all t.

It remains to rule out equilibria that begin with $b_0 < B$. Consider the equilibrium continuation that selects, in each period, the lower of the two candidate prices induced by $\epsilon_{t+1} \in \{\epsilon_l, \epsilon_h\}$. Starting from $b_0 < B$, this "low-price" path generates a decreasing price sequence, regardless of whether agents consume the asset. The sequence cannot converge to a strictly positive limit, because of the uniqueness of B. It cannot converge to zero either. Because $\epsilon_h > 0$ occurs with positive probability, the asset might deliver strictly positive consumption value, which leads to a strictly positive price. Thus, an equilibrium with $b_0 < B$ cannot exist, and the stationary equilibrium is unique.

Notice that we allow $\epsilon_l = 0$ in this proposition, so *certainty* of strictly positive consumption value is not required for uniqueness. What matters instead is the *certainty of the possibility* of positive consumption value: each generation must assign positive probability to every future generation facing a state in which consuming the asset yields strictly positive utility. This distinction suggests a potential interpretation for differences in stability across assets: precious metals arguably retain value partly because market participants view future "use" demand as persistently possible, whereas for cryptocurrencies such demand may be perceived as more fragile.

5.2 Bubble with large consumption value

Now we analyze the situation where $\epsilon_h > Bu'(e_y - B) > \epsilon_l \geq 0$. In this case, the stationary equilibrium where $b_t = B$ and $x_{t+1} = 1$ in all history does not exist. Otherwise, the generation with high consumption value would have the incentive to consume the asset. This observation suggests that after a certain history, the generation with high consumption value should consume the asset on the equilibrium path. As the aggregate supply of assets decreases (from consumption), the asset becomes more valuable as a store of value, while its value as a consumption good stays the same. We show that in the equilibrium, the generation that experiences high consumption for the first time will consume to the extent that the asset's marginal value at the store of value equals the high consumption value. From that period on, the economy enters a stationary equilibrium with a smaller aggregate asset supply than the initial level.

Proposition 3. *There is a unique equilibrium which takes the following form: Let t be the period where the consumption value ϵ_h is realized for the first time. The price of the asset in period t is $\tilde{b}_h$ satisfying $\tilde{b}_h u'(e_y - \tilde{b}_h) = \epsilon_h$. Each agent in generation t consumes $1 - x_h$ units of asset where x_h satisfies $\epsilon_h = \frac{B}{x_h} u'(e_y - B)$. Starting from period $t+1$, the agent never consumes the asset, and the asset price becomes $b_h = \frac{B}{x_h}$. Before period t, the asset price b_l satisfies*

$$b_l u'(e_y - b_l) = \beta[p\tilde{b}_h u'(e_o + \tilde{b}_h) + (1-p) b_l u'(e_o + b_l)]. \tag{6}$$

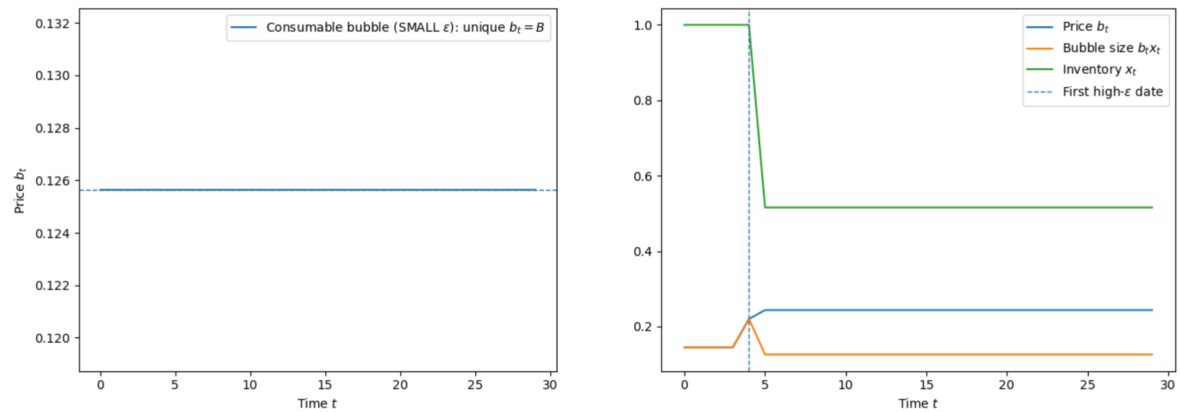
The equilibrium has the following properties:

- *The bubble price increases over time: $b_h > \tilde{b}_h > b_l > B$.*
- *The aggregate bubble size first increases and then decreases: $\tilde{b}_h > b_l > b_h x_h = B$.*

We sketch the proof here and relegate the formal argument to the appendix. Fix an arbitrary realized path of consumption values, let τ denote the first period in which any agent consumes the asset. The realization in period τ must be ϵ_h. Otherwise, if consumption occurred when the realization is ϵ_l, the implied lower continuation price (of the two) would decrease over time and converge to zero, which cannot occur in equilibrium. Hence, the first-order condition in period τ implies the asset price $\tilde{b}_h$ to satisfy $\tilde{b}_h\, u'(e_y - \tilde{b}_h) = \epsilon_h$. Moreover, τ must coincide with the first period along the realized history in which $\epsilon = \epsilon_h$. If instead consumption were after the first occurrence of ϵ_h, then the implied higher continuation price of the asset would diverge to infinity.

By Proposition A.1, in the first period when ϵ_h is realized, if consumption reduces aggregate asset level to x_h, the continuation equilibrium is the one characterized by the proposition and is unique. We next show that this is the only possibility. If $x_{\tau+1} > x_h$, then the first-order condition for generation τ implies that the higher continuation asset price diverges. If instead $x_{\tau+1} < x_h$, Proposition A.1 implies that the continuation asset price exceeds B/x_h, so the asset is strictly more valuable as a store of value than as a consumption good in period τ, and generation τ would not consume. Finally, given that consumption occurs at the first realization of ϵ_h, the equilibrium price path prior to τ is determined by the agents' first-order condition.

Several observations are worth highlighting. When the consumption value is low, its level is irrelevant for equilibrium outcomes because the asset is held purely as a store of value and is never consumed. By contrast, Proposition 3 implies that the consumption value matters for equilibrium pricing and bubble dynamics when it is high: the asset price is increasing in the realized consumption value in that state. Moreover, the *prospect* of a high consumption value raises the bubble size even before it materializes. Once the high consumption value is realized, the asset price rises further, but the bubble size decreases because aggregate asset holdings fall



(a) **Small use value** ($\epsilon_h \leq B\, u'(e_y - B)$): the unique equilibrium is stationary at $b_t = B$ and the asset is never consumed ($x_t \equiv 1$).

(b) **Large use value** ($\epsilon_h > B\, u'(e_y - B) > \epsilon_l$): the unique equilibrium features a one-time "burn/bind/redeem" at the first high-ϵ realization, reducing inventory from 1 to x_h and shifting the post-event price to $b_h = B/x_h$.

Figure 3: **Consumable bubbles.** Left: when the asset's use value is small, equilibrium coincides with the non-consumable (pure-bubble) outcome—price is constant at B and there is no consumption. Right: when the use value is large, equilibrium remains unique but becomes history-dependent through inventory: the first high-ϵ shock triggers partial consumption, permanently lowering supply to $x_h < 1$, so the bubble size $b_t x_t$ returns to B while the price level jumps to and stays at $b_h = B/x_h$.

after consumption reduces the total supply. Taken together, these forces imply that greater volatility in the asset's consumption value tends to generate larger bubbles.

Figure 3 contrasts the equilibrium dynamics of a consumable bubble across the two parameter regions. When the use value is small ($\epsilon_h \leq B\, u'(e_y - B)$), redeeming the asset for consumption is never privately optimal in equilibrium; the unique equilibrium replicates the pure-bubble allocation with constant price $b_t = B$ and constant inventory $x_t \equiv 1$. When the use value is large ($\epsilon_h > B\, u'(e_y - B) > \epsilon_l$), equilibrium remains unique but includes a one-time redemption episode at the first realization of the high use value. Before that date, the asset trades at the low-state price b_ℓ. At the first high-ϵ date, the equilibrium price jumps to $\tilde{b}_h$ and agents consume a fraction of the asset, reducing inventory from 1 to $x_h < 1$. After the event, the remaining inventory is carried forward ($x_t \equiv x_h$) and the price settles at $b_h = B/x_h$, so the bubble size $b_t x_t$ reverts to the stationary level B while the price stays elevated because supply has been permanently reduced by the burn/bind/redeem episode.

Figure 4 shows how transitory use-value shocks map into movements in the price decomposition. The variable ϵ_t shifts the immediate payoff from holding the asset as consumption, so a larger ϵ_t mechanically increases the portion of the price that is supported by contemporaneous use value. The premium Π_t isolates the residual component of the price that is not accounted

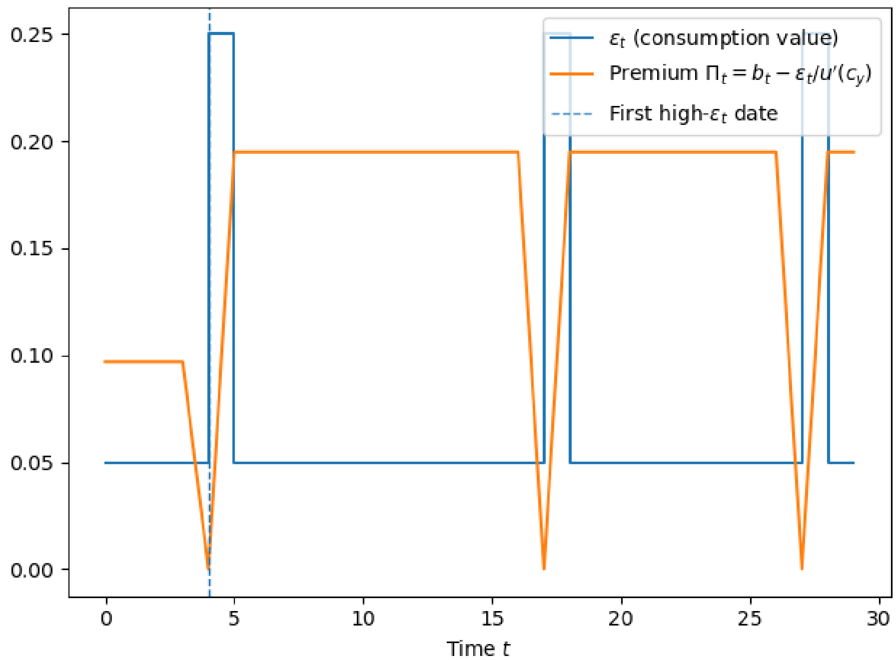


Figure 4: **Use-value shocks and the bubble premium in the large-use-value regime.** The step series plots the i.i.d. use value $\epsilon_t \in \{\epsilon_\ell, \epsilon_h\}$. The solid line plots the premium $\Pi_t = b_t - \frac{\epsilon_t}{u'(c_{y,t})}, \quad c_{y,t} = e_y - b_t x_t$, which subtracts the contemporaneous "fundamental" (use-value) component from the market price. The first high-ϵ realization triggers the burn/bind/redeem episode (vertical dashed line in Figure 3), after which prices and premia reflect the lower post-burn inventory $x_h < 1$.

for by current use value, scaled by the young's marginal utility $u'(c_{y,t})$. In the large-use-value region, the first arrival of the high state ϵ_h changes the state of the economy by inducing partial redemption: inventory falls from 1 to $x_h < 1$, which raises the post-event price level (since $b_h = B/x_h$) even though the bubble size $b_t x_t$ returns to B. After this one-time adjustment, subsequent fluctuations in ϵ_t primarily affect the decomposition between the use-value component and the premium, holding the post-burn inventory fixed.

6 Endogenous Asset Supply

In the baseline model, the asset is in a fixed aggregate supply (no production). In this section, we extend the model by allowing for endogenous asset supply and show that the main insights from the baseline economy continue to hold. In particular, a pure bubble asset without consumption value still admits equilibrium multiplicity, whereas introducing even a small consumption value uniquely selects an equilibrium in which the asset is used as a store of value. We also show that the equilibrium value of the asset is negatively related to its supply: when the asset is more difficult to produce, it stores more value in equilibrium.

We model asset creation through a stand-alone production sector that chooses next-period output as a function of the current-period asset price. Specifically, we assume that supply is governed by a reduced-form function $s(\cdot)$: if the period-t price is b_t, then $s(b_t)$ units of the asset are produced and become available in period $t+1$. Hence, the aggregate asset supply in the centralized market at $t+1$ is

$$x_{t+1} = x_t - c_t + s(b_t),$$

where c_t denotes the amount consumed by generation t. In the centralized market at $t+1$, generation t sells its remaining holdings $x_t - c_t$, and receives $b_{t+1}(x_t - c_t)$ units of the endowment good. The newly produced units increase the aggregate supply available at $t+1$ and, all else equal, put downward pressure on the equilibrium asset price.

We assume that $s(\cdot)$ is continuous and increasing, and that there exists $\underline{b} > 0$ such that $s(\underline{b}) = 0$ and $s(b) \to 0$ as $b \downarrow \underline{b}$. When the asset has consumption value, we focus on the case $\epsilon_h = \epsilon_l = \epsilon$ where ϵ is small, and impose $\underline{b} > \epsilon / u'(e_y)$. If an agent consumes the asset at time t, the optimality condition requires $\epsilon = b_t u'(e_y - b_t x_t)$. The restriction $\underline{b} > \epsilon / u'(e_y)$ implies that, at the lowest price that can induce production, the asset is still too expensive to be produced solely for its consumption value. The equilibrium definition is otherwise unchanged from the previous section.

6.1 Pure Bubble

When the asset has no consumption value, an equilibrium can be summarized by a price-quantity sequence $\{(b_t, x_t)\}_{t=1}^{\infty}$ satisfying:

1. $x_1 = 1$.
2. $b_t u'(e_y - b_t x_t) = \beta\, b_{t+1} u'(e_o + b_{t+1} x_t)$.
3. $x_{t+1} = x_t + s(b_t)$.

Clearly, the fundamental equilibrium $b_t = 0$ for all t always exists. We next study equilibria with strictly positive asset prices. If $b_1 < \underline{b}$, $s(b_1) = 0$ and, as shown in previous analysis, $\{b_t\}$ is decreasing. Consequently $b_t < \underline{b}$ and $s(b_t) = 0$ for all t. In this case, the equilibrium paths coincide with the baseline model without asset supply. We therefore focus on the case $\underline{b} < b_1 \leq B$.[8] To characterize equilibrium paths, we call an initial value b_1 *feasible* if there exists an equilibrium sequence starting from $(b_1, 1)$. The following lemma shows that there exists a feasible initial price, and if an initial price is feasible, any lower initial price is also feasible.

[8] For an equilibrium to exist, it must be that $b_1 \leq B$.

Lemma 1. *There exists a feasible $\tilde{b}_1 > \underline{b}$. Moreover, if $\tilde{b}_1$ is feasible, then any $b_1 < \tilde{b}_1$ is also feasible. Finally, along the equilibrium paths we have $b_t < \tilde{b}_t$ and $x_t \leq \tilde{x}_t$ for all t, with strict inequality when $\tilde{b}_1 > \underline{b}$.*

Sketch of proof Given feasibility of $\tilde{b}_1$, the first-order condition pins down a unique continuation price b_2 as a function of (b_1, x_1). By monotonicity of $b_2 u'(e_o + b_2 x_1)$, $b_1 < \tilde{b}_1$ yields $0 < b_2 < \tilde{b}_2$. Since $s(\cdot)$ is increasing, $x_2 \leq \tilde{x}_2$. Iterating forward establishes the stated inequalities. Existence of some feasible $b_1 > \underline{b}$ follows by continuity: when $b_1 = \underline{b}$, we have $s(b_1) = 0$ and the induced continuation price satisfies $b_2 < \underline{b}$, which generates a valid equilibrium path; continuity then implies that a slightly larger b_1 remains feasible.

The lemma implies that there are infinitely many positive-price equilibrium paths, ordered by the initial price b_1. As in the baseline model, these equilibrium paths can also be ranked by the bubble size $b_t x_t$: if $\tilde{b}_1 > \hat{b}_1$, then the corresponding paths satisfy $\tilde{b}_t \tilde{x}_t > \hat{b}_t \hat{x}_t$ for all t. Since bubble size cannot exceed B, we can study the maximal equilibrium path, which plays the role of the stationary equilibrium in the baseline model.

Consider the largest equilibrium path, in the sense of the supremum over feasible b_1. Note first that $b_1 < B$: if $b_1 = B$, then the first-order condition implies $b_2 = B > \underline{b}$, and hence $b_2 x_2 > B$, which is impossible in the equilibrium. By continuity, there exists a $b_1 < B$ such that $b_2 x_2 = B$. If $b_2 \leq \underline{b}$, then the asset production ceases from period 2 onward and the continuation equilibrium is stationary with constant $x_t = x_2$ and $b_t = B/x_2$. If instead $b_2 > \underline{b}$, then the asset production would imply $x_3 > x_2$ and hence $b_3 x_3 > B$, violating feasibility. Continuity then implies that we can further lower b_1 so that the first time the bubble size hits B occurs at $t = 3$. This would induce the largest equilibrium path if $b_3 \leq \underline{b} < b_2$. Iterating this procedure pins down the maximal equilibrium path for any $\underline{b}$. Specifically, we have the following result.

Proposition 4. *If the asset has no consumption value, in addition to the fundamental equilibrium, the model admits a continuum of positive-price equilibrium paths with an attainable maximal equilibrium path. These equilibria are ordered by the initial price b_1 in the sense that a larger b_1 generates a uniformly larger bubble trajectory: if $b_1 > \tilde{b}_1$, then $b_t > \tilde{b}_t$ and $x_t > \tilde{x}_t$. Along the maximal equilibrium path, the bubble size reaches B in finite time and thereafter remains at B with an asset price $b_t \leq \underline{b}$. On all other equilibrium paths, the bubble price and size stay positive and converge to zero.*

This proposition shows that allowing for endogenous asset supply preserves the main qualitative insights from the baseline model. In particular, there exists an equilibrium in which the bubble price eventually stabilizes. In any other equilibrium, the economy is *asymptotically bubbleless*: the bubble price and its size converge to zero in the long run. The key difference is that, with endogenous supply, even though the bubble price still declines over time along these

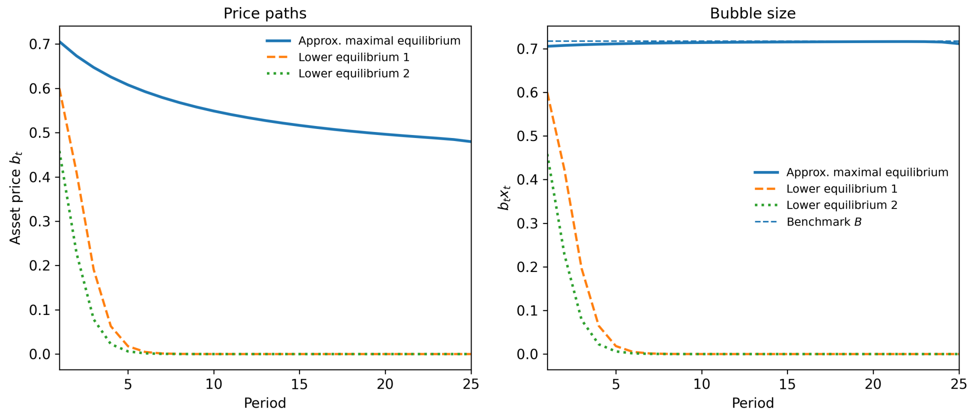


Figure 5: **Endogenous supply with a pure bubble.** The left panel shows a continuum of positive-price equilibrium paths. The maximal equilibrium path reaches a steady state in finite time, whereas lower-price equilibria converge to zero. The right panel plots bubble size $b_t x_t$. Along lower-price equilibria, bubble size need not decline monotonically because endogenous supply can temporarily offset the fall in price.

paths, the *bubble size* need not fall monotonically, since the additional supply of the asset may offset the price decline and make the bubble larger.

Figure 5 illustrates how endogenous production preserves the central multiplicity result of the pure-bubble economy while changing the transitional dynamics. As in the fixed-supply benchmark, there is a continuum of positive-price equilibrium paths in addition to the fundamental equilibrium. The maximal equilibrium path reaches the largest sustainable bubble size B in finite time and then becomes stationary. By contrast, lower-price equilibria eventually converge to zero. A notable difference from the fixed-supply case is that bubble size, $b_t x_t$, need not decline monotonically along non-maximal paths: although the asset price falls, endogenous supply can temporarily increase inventory and partly offset the price decline. Thus, with endogenous supply, the bubble may shrink in price terms while still expanding temporarily in aggregate size.

6.2 Consumable Bubble

In this section, we examine whether introducing a small consumption value restricts the set of equilibrium paths. We show that, as in the baseline model, when the asset has a small consumption value, the equilibrium is unique and coincides with the maximal equilibrium path. We then study how the equilibrium price varies with the ease of supplying the asset. In particular, we show that an asset that is easier to supply commands a lower equilibrium price.

Proposition 5. *When the asset has a small consumption value ϵ, the equilibrium is unique, and the equilibrium bubble size reaches B in the long run. If the long-run asset price of the*

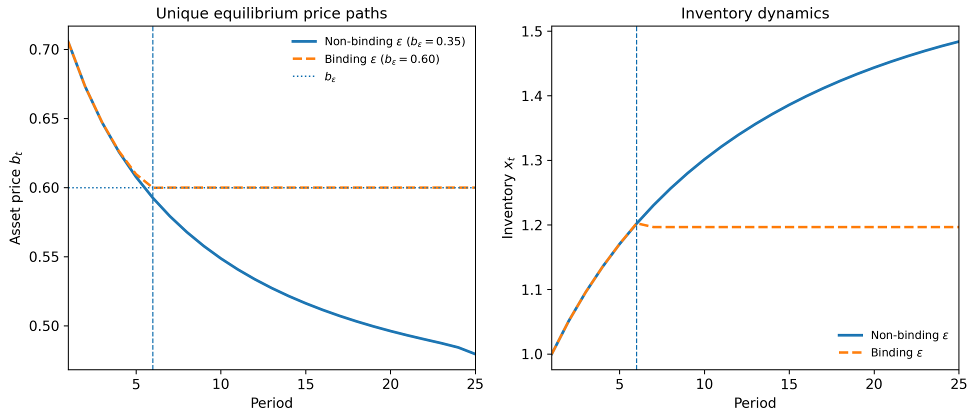


Figure 6: **Endogenous supply with a small consumption value.** The left panel shows the unique equilibrium price path. When the consumption value is non-binding, the unique equilibrium coincides with the maximal pure-bubble path. When the consumption value is binding, the economy converges to a higher long-run price b_ϵ, with a one-time consumption event before reaching steady state. The right panel shows the corresponding inventory dynamics.

corresponding maximal equilibrium path without consumption value is greater than or equal to $b_\epsilon = \frac{\epsilon}{u'(e_y - B)}$*, the equilibrium price trajectory is identical to the maximal equilibrium path, and the agent never consumes. Otherwise, the equilibrium price trajectory is higher than the corresponding maximal equilibrium path and the long-run asset price of the asset with consumption value is* b_ϵ*. The first generation of agents reaching the steady state consumes. In both cases, the asset price decreases over time.*

To provide intuition, recall that $\underline{b} > \frac{\epsilon}{u'(e_y)}$. This implies that, when the asset price is low, no additional supply enters the market and the economy is effectively identical to one with a fixed asset supply. In particular, there is no equilibrium in which a generation consumes its entire holdings.[9] Moreover, by Proposition 4, when the asset has no consumption value, the price remains high only along the maximal equilibrium path. The problem therefore, reduces to comparing the asset price on the maximal equilibrium path once it stabilizes with the price level implied by the consumption value. If the consumption value is low, then the maximal equilibrium from the zero-consumption-value economy is still an equilibrium when the asset has a consumption value, and agents never consume the asset along the equilibrium path. Conversely, if the consumption value is high, the equilibrium price path lies above the maximal path (in terms of prices) from the economy without consumption value, and agents consume the asset once before reaching the steady state.

Figure 5 illustrates how endogenous production preserves the central multiplicity result of

[9]Without this assumption, when the consumption value is high, there could be equilibrium where all the newly-produced assets are consumed in each period.

the pure-bubble economy while changing the transitional dynamics. As in the fixed-supply benchmark, there is a continuum of positive-price equilibrium paths in addition to the fundamental equilibrium. The maximal equilibrium path reaches the largest sustainable bubble size B in finite time and then becomes stationary. By contrast, lower-price equilibria eventually converge to zero. A notable difference from the fixed-supply case is that bubble size, $b_t x_t$, need not decline monotonically along non-maximal paths: although the asset price falls, endogenous supply can temporarily increase inventory and partly offset the price decline. Thus, with endogenous supply, the bubble may shrink in price terms while still expanding temporarily in aggregate size.

It is also interesting to discuss the relationship between an asset's supply function and its price trajectory as a store-of-value. A natural interpretation of why some durable commodities appear to be more "bubble-prone" than others is that their aggregate supply responds differently to price incentives. Gold is commonly viewed as scarce: it is hard to extract and refine. Even large price increases translate into only gradual expansions of production. By contrast, industrial metals such as copper are much easier to supply through scalable extraction and recycling. In the model, this distinction can be captured directly by the supply function $s(\cdot)$. The following proposition shows that, as a bubble, the scarcer asset has a higher initial price. Moreover, at any fixed level of inventory, the scarcer asset is always more expensive along the equilibrium path.

Proposition 6. *In an economy with the asset supply function s, assume that when $b > \underline{b}$ is close to $\underline{b}$, $s(b) \geq m(b - \underline{b})$ where $m \geq -\frac{u'(e_y - B)}{\beta u''(e_o + B)\underline{b}^2}$ and $u'''(e_o + B) > 0$. In this economy, the asset price stabilizes in finitely many periods.*

Consider another otherwise identical economy with asset supply function $\tilde{s}$, respectively. If $s(x) > \tilde{s}(x)$ for all x, then on the corresponding equilibrium paths, $b_1 < \tilde{b}_1$. Moreover, for any indices of periods t and τ, if $\tilde{x}_\tau \leq x_t$, it must be $\tilde{b}_\tau > b_t$.

Note that this proposition does not rule out the possibility that the asset with a lower supply function ends up stabilizing at a lower price due to the accumulated inventory becoming large eventually. The next proposition shows that, by putting more structures on the supply function, we can characterize the situation where the economy with a scarcer asset always has a higher asset price.

Proposition 7. *Consider two otherwise identical economies with asset supply functions s and $\tilde{s}$ satisfying $\tilde{s}(x) < \theta s(x)$ and $\tilde{\underline{b}} > \underline{b}$. Keeping all the assumptions in Proposition 4, additionally assume that there exists M such that $M|z_1 - z_2| \geq |s(z_1) - s(z_2)|$ for all $z_1, z_2 \in (\underline{b}, B)$. Then there exists a $\bar{\theta} < 1$ such that if $\theta < \bar{\theta} < 1$, $b_t < \tilde{b}_t$ for all t.*

Figure 7 illustrates the comparative statics of the supply function. An asset with a less elastic supply response accumulates inventory more slowly and therefore supports a higher

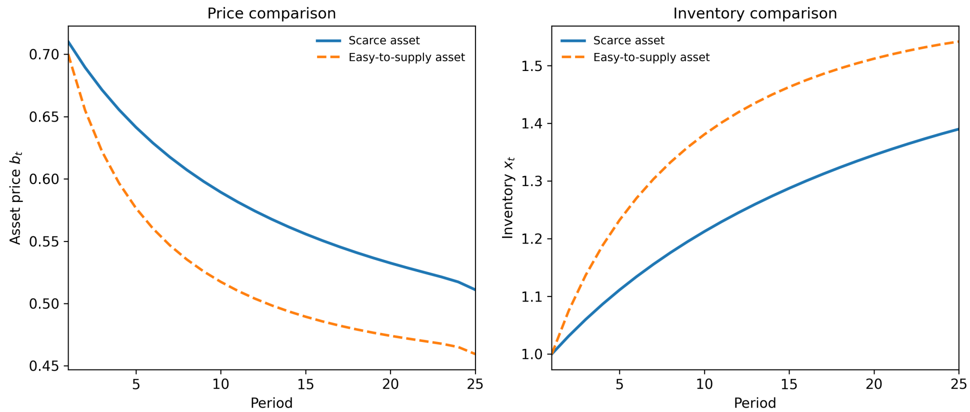


Figure 7: **Comparative statics with respect to supply elasticity.** The scarcer asset has a higher price path and accumulates less inventory than the easier-to-supply asset. This illustrates that assets with less elastic supply can support more value as stores of wealth.

equilibrium price path. By contrast, when supply responds strongly to price, additional production expands the asset stock more rapidly and depresses its value as a store of wealth. This provides a simple interpretation for why some assets appear more "bubble-prone" than others. Assets such as gold, whose supply responds only gradually even to large price increases, can sustain higher valuations than otherwise similar assets whose supply can be expanded quickly. The figure therefore visualizes the main economic message of Propositions 6 and 7: scarcity strengthens the asset's store-of-value role.

7 Conclusion

In this paper we study rational bubbles when the bubble asset is also a consumption good. The possible consumption value disciplines the bubble dynamics. When the consumption value is fixed, although in the equilibrium the asset still behaves like a pure bubble in the sense that it is held purely as a store of value, this stationary equilibrium is the unique equilibrium. When the consumption value is stochastic and could be high, in the unique equilibrium, the possible high consumption value increases the wealth transferred by the asset and the bubble price.

Taken together, the model delivers two insights. From a theory perspective, when the agents have strong incentives to invest, introducing consumption value uniquely selects the equilibrium where the asset serves as a non-negligible bubble, even in the long run. Practically, although the price trajectories of bubbles with consumption values and pure bubbles might look similar, they are fundamentally different since the value of a pure bubble is more fragile to agents' shifts in collective beliefs.

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A Appendix

Proof of Proposition 1:

We prove by contradiction. Suppose that there is a history H_t such that $x_{t+1} = 0$. By (4), $b^h_{t+i} \geq \frac{\epsilon_h}{u'(e_y)}$ for all i and thus $b^l_{t+i} u'(e_y) > \beta p u'(e_o) \frac{\epsilon_h}{u'(e_y)}$ for all i.

Let i be a large integer. We have

$$b^l_{t+i-1} > \frac{\beta u'(e_o)}{u'(e_y)} [p \frac{\epsilon_h}{u'(e_y)} + (1-p) \frac{\beta p u'(e_o)}{u'(e_y)} \frac{\epsilon_h}{u'(e_y)}]$$

Iterate back to get

$$b^l_{t+i-2} > \frac{\beta u'(e_o)}{u'(e_y)}[p\frac{\epsilon_h}{u'(e_y)} + (1-p)b^l_{t+i-1}]$$

From the assumption, $\frac{\beta u'(e_o)}{u'(e_y)} > 1$. If $\frac{p\beta u'(e_o)}{u'(e_y)} > 1$, easy to see that $b^l_{t+i} > \frac{\epsilon_h}{u'(e_y)}$. If $\frac{p\beta u'(e_o)}{u'(e_y)} < 1 < p(\frac{\beta u'(e_o)}{u'(e_y)})^2$, $b^l_{t+i-1} > p(\frac{\beta u'(e_o)}{u'(e_y)})^2\frac{\epsilon_h}{u'(e_y)}$. Since $p(\frac{\beta u'(e_o)}{u'(e_y)})^k$ is an increasing sequence, starting from a large enough i, we can always conclude that $b^l_{t+1} > \frac{\epsilon_h}{u'(e_y)}$.

However, this contradicts the existence of the "fundamental equilibrium". Indeed, the other no-deviation condition (3) is

$$\epsilon_h \geq \beta E\left[\tilde{b}_{t+1}u'(e_o)\right].$$

If $b^h_{t+1} > \frac{\epsilon_h}{u'(e_y)}$ and $b^l_{t+1} > \frac{\epsilon_h}{u'(e_y)}$, it must be that $u'(e_y) > \beta u'(e_o)$. Contradiction.

■

Proof of Proposition 2:

We prove the following more general result. The original statement is a special case of this proposition when $x = 1$.

Proposition A.1. *If $bu'(e_y - B) \geq \epsilon_h > \epsilon_l \geq 0$, $b_t = \frac{B}{x}$ for all t is the unique equilibrium of the economy.*

Easy to check $b_t = \frac{B}{x}$ for all t is an equilibrium. The uniqueness of the equilibrium is proved via the following sequence of lemmas.

Lemma A.1. *If $b_t u'(e_y - b_t x_t) > \epsilon$, $x_{t+1} = x_t$.*

Proof. By the first-order condition, if $b_t u'(e_y - b_t x_t) > \epsilon$, it must be $b_t u'(e_y - b_t x_t) = \beta E[b_{t+1}u'(e_o + b_{t+1}x_{t+1})] > \epsilon$. If $x_{t+1} < x_t$, an individual can improve his welfare by consuming less. ■

Lemma A.2. *In any equilibrium, $b_t \leq \frac{B}{x_t}$ for all t.*

Proof. If not, there is a time t such that $b_t > \frac{B}{x_t}$. Since $x_t \leq x$, $b_t u'(e_y - b_t x_t) > \epsilon_h$. By Lemma A.1, $x_{t+1} = x_t$. Let b^h_{t+1} denote $b(H_t \bigcup h)$ (given the history at time t) and b^l_{t+1} denote $b(H_t \bigcup l)$. Define $\bar{b}_{t+i} = max(b^h_{t+i}, b^l_{t+i})$ and $\underline{b}_{t+i} = min(b^h_{t+i}, b^l_{t+i})$. Since $b_t u'(e_y - b_t x_t) = \beta E(b_{t+1}u'(e_o + b_{t+1}x_t))$, $b_t u'(e_y - b_t x_t) \leq \beta \bar{b}_{t+1}u'(e_o + \bar{b}_{t+1}x_t)$. Note that the RHS is smaller than the LHS when $\bar{b}_{t+1} = b_t$ and it must be $\bar{b}_{t+1} > b_t$. Iterate forward to see that this is impossible. ■

Lemma A.3. *In any equilibrium, if $b_t u'(e_y - b_t x_t) \geq \beta b^c_{t+1}u'(e_o + b^c_{t+1}x_{t+1})$, then $b_t \geq b^c_{t+1}$, $c \in \{h, l\}$. The inequality is tight if $b_t < \frac{B}{x_t}$.*

Proof. Since $x_t \geq x_{t+1}$, if $b_t u'(e_y - b_t x_t) \geq \beta b^c_{t+1} u'(e_o + b^c_{t+1} x_{t+1})$, $b_t u'(e_y - b_t x_t) \geq \beta b^c_{t+1} u'(e_o + b^c_{t+1} x_t)$. Since $b_t \leq \frac{B}{x_t}$, $u'(e_y - b_t x_t) \leq \beta u'(e_o + b_t x_t)$. Since $b^c_{t+1} u'(e_o + b^c_{t+1} x_t)$ is increasing in b^c_{t+1}, for the inequality to hold, it must be that $b_t \geq b_{t+1}$. All the follow-up inequalities are tight if $b_t < \frac{B}{x_t}$.

■

Lemma A.4. *If in the equilibrium, there exist a period t such that $b_t = \frac{B}{x_t}$, it must be $b_{t+i} = \frac{B}{x_t}$ for all $i > 0$.*

Proof. If the claim is not true, there is a period t (after a certain history H_t) such that $b^c_{t+1} \neq \frac{B}{x_t} = b_t$ for $c \in \{h, l\}$. Without loss of generality, we focus on the situation where the consumption value of generation t is ϵ_h. If $b_t u'(e_y - b_t x_t) > \epsilon_h$, $x_{t+1} = x_t$. Note that $b_t u'(e_y - b_t x_t) \leq \beta \bar{b}_{t+1} u'(e_o + \bar{b}_{t+1} x_t)$. If $\bar{b}_{t+1} > \underline{b}_{t+1}$, the inequality is strict and $\bar{b}_{t+1} > \frac{B}{x_{t+1}}$. Since $b_t \leq \frac{B}{x_t}$ for all t, it must be that $b^h_{t+1} = b^l_{t+1} = \frac{B}{x_t} = \frac{B}{x_{t+1}}$.

If $b_t u'(e_y - b_t x_t) = \epsilon_h$, given the previous argument, we only need to consider the situation where $x_{t+1} < x_t$. If so, it must be

$$b_t u'(e_y - b_t x_t) > \beta \underline{b}_{t+1} u'(e_o + \underline{b}_{t+1} x_t)$$

which by Lemma A.3 implies $b_t > \underline{b}_{t+1}$. Consider the equilibrium path where $b_{t+1} = \underline{b}_{t+1}$. Since we have already shown that there is no "fundamental equilibrium" where the agent consume everything, the first order condition implies that

$$b_{t+1} u'(e_y - b_{t+1} x_{t+1}) \geq \underline{b}_{t+2} \beta u'(e_o + \underline{b}_{t+2} x_{t+2}) \geq \underline{b}_{t+2} \beta u'(e_o + \underline{b}_{t+2} x_{t+1}).$$

Since $b_{t+1} < \frac{B}{x_t}$, $b_{t+1} x_{t+1} < B$ and it must be that $\underline{b}_{t+2} < b_{t+1}$. Consider the path where $b_{t+2} = \underline{b}_{t+2}$, we can iteratively construct a decreasing sequence b_{t+i} and it must be $lim_{i \to \infty} b_{t+i} \to 0$. However, the first-order condition then implies that b^h_{t+i} is bounded away from 0 since $b^h_{t+i} u'(e_y - b^h_{t+i} x_{t+i}) \geq \epsilon_h$. Contradiction. ■

Now we can prove the proposition. By the previous Lemmas, if in the equilibrium, $b_0 = \frac{B}{x}$, the only possibility is $b_t = \frac{B}{x}$ for all t. Moreover, it cannot be that $b_0 < \frac{B}{x}$ is not feasible since otherwise we have

$$b_0 u'(e_y - b_0 x_0) > \beta \underline{b}_1 u'(e_o + \underline{b}_1 x_0)$$

and a similar contradiction can be constructed.

■

Proof of Proposition 3:

The result is established via the following sequence of lemmas.

Lemma A.5. *In any equilibrium, the agent consumes the inventory down to x_h when the state first reaches h. The bubble price in that period satisfies $\tilde{b}_h u'(e_y - \tilde{b}_h) = \epsilon_h$. The bubble price then stays the same at $b_h = \frac{B}{x_h}$.*

Proof. Let

$$\tau(\omega) = min\{t : x_{t+1}(H_t(\omega)) < 1\}.$$

First note that if $\tau < \infty$, it must be that $\epsilon_\tau = \epsilon_h$. If not, $b_\tau u'(e_y - b_\tau) = \epsilon_l$ and thus $b_\tau < B$. Then the first-order condition for period τ implies that $\underline{b}_{\tau+1} < b_\tau$ and $\underline{b}_{\tau+1} x_{\tau+1} < \underline{b}_{\tau+1} < B$. Iterate forward to see that $\underline{b}$ is a decreasing sequence converging to 0, which is impossible since the future state could always be h with probability p.

Moreover, τ is the first period where the consumption value reaches ϵ_h. If not, let $t < \tau$ be the first period where the consumption value reaches ϵ_h. We have $b_t u'(e_y - b_t) = E[b_{t+1} u'(e_o + b_{t+1} x_{t+1})] > \epsilon_h$ and $\bar{b}_{t+1} > b_t$. Iterate forward, $\bar{b}_{t+i} u'(e_y - \bar{b}_{t+i}) > \epsilon_h$ for any $i > 0$ and $\bar{b}_{t+i}$ is increasing. Thus, $\bar{b}$ diverges. Contradiction. This implies that $b_\tau u'(e_y - b_\tau) = \epsilon_h$.

By proposition A.1, if $x_{\tau+1} = x_h$ at time t, the continuation equilibrium is unique and satisfies $b_{t+i} = \frac{B}{x_{t+1}}$. Thus, to establish the claim, we only need to rule out the possibility that $x_{\tau+1} \neq x_h$. By the discussion regarding autarky, we know that $x_{\tau+1} > 0$. If $0 < x_{\tau+1} < x_h$, by proposition A.1, the continuation equilibrium satisfies $b_{\tau+i} = \frac{B}{x_{\tau+1}}$. However, since $b_{\tau+1} > \frac{B}{x_h}$, $\beta b_{\tau+1} u'(e_o + b_{\tau+1} x_{\tau+1}) > \epsilon_h$, violating the optimal consumption condition of the generation τ.

If $x_{\tau+1} > x_h$, by the first-order condition,

$$b_\tau u'(e_y - b_\tau) = \epsilon_h = \beta E(b_{\tau+1} u'(e_o + b_{\tau+1} x_{\tau+1})) < \beta E(b_{\tau+1} u'(e_o + b_{\tau+1} x_h)). \tag{7}$$

This implies that $\bar{b}_{\tau+1} > b_h$. By the first-order condition in period $t+1$, $\bar{b}_{\tau+1} u'(e_y - \bar{b}_{\tau+1} x_{\tau+1}) > \epsilon_h$. Iterate over $\bar{b}_{t+i}$ to see that it diverges to infinity. Contradiction. ■

Lemma A.6. *The asset price before the first h realization satisfies $b_l u'(e_y - b_l) = \beta[p\tilde{b}_h u'(e_o + \tilde{b}_h) + (1-p) b_l u'(e_o + b_l)]$.*

Proof. Note that

$$b_l \left[u'(e_y - b_l) - (1-p) u'(e_o + b_l)\right]$$

is increasing in b_l. Thus, given $\tilde{b}_h u'(e_y - \tilde{b}_h) = \epsilon_h$, Equation (6) uniquely determines b_l. Moreover, the first-order condition is

$$b_t u'(e_y - b_t) - (1-p) b_{t+1} u'(e_o + b_{t+1}) = \beta p \tilde{b}_h u'(e_o + \tilde{b}_h).$$

If there exists b_t and b_{t+1} satisfying the equation above with $b_t > b_l$, since

$$b_t \left[u'(e_y - b_t) - (1-p) u'(e_o + b_t)\right] > \beta p \tilde{b}_h u'(e_o + \tilde{b}_h)$$

it must be $b_{t+1} > b_t$. Consider the equilibrium path with a long enough l realization. Then either b_{t+i} diverges or there is no feasible b_{t+i} satisfying the first-order condition. Contradiction

Conversely, if $b_t < b_l$ and there exists a b_{t+1} satisfying the first-order condition, it must be $b_{t+1} < b_t$. Consider the equilibrium path with a long enough l realization. Iterate forward to see that, regardless of whether the agent consumes or not, b_{t+i} must converge to either a positive number or 0. The uniqueness of b_l implies that b_{t+i} cannot converge to a positive number. $\tilde{b}_h > 0$ implies that b_{t+i} cannot converge to 0. ■

Easy to check that the equilibrium we propose satisfies all the first-order conditions. The Lemmas above together imply that the proposed equilibrium is the unique equilibrium.

■

Proof of Proposition 4:

By Lemma 1, there is a continuum of equilibrium paths. Now we characterize the maximum equilibrium path. Since $B > \underline{b}$, by the first-order condition, if $b_1 = B$, $b_2x_2 > b_2x_1 = B$. Alternatively, if $b_1 = \underline{b}$, $b_2x_2 < b_1x_1 < B$. By the continuity of b_2 and s with respect to b_1, there exists a $b_1^2 \in (\underline{b}, B)$ such that $b_2^2x_2^2 = B$. If $b_2^2 \leq \underline{b}$, we have obtained the maximum equilibrium path upon which $b_t = b_2^2$ and $x_t = x_2^2$ for all $t > 2$. If $b_2^2 > \underline{b}$, this path is not feasible since the first-order condition implies that $b_3^2x_3^2 > B$. By continuity, we can further lower the initial asset price b_1 such that $b_2 = \underline{b}$, which implies that $b_3 < \underline{b}$ and $b_3x_3 < B$. Since the composite of continuous functions is still continuous, there exists a $b_1^3 \in (\underline{b}, b_1^2)$ such that $b_1^3 > b_2^3 > b_3^3$ and $b_3^3x_3^3 = B$. This procedure thus generates the maximum equilibrium path if it stops in finite rounds of iteration. If not, note that $\{b_n^n\}$ is a decreasing sequence greater than $\underline{b}$. Since the aggregate bubble size is upper-bounded by B, it must be that $lim_{n\to\infty}b_n^n = \underline{b}$, otherwise the supply becomes unbounded. Moreover, $\{b_1^n\}$ is also a decreasing sequence converging to b_1^∞. By continuity, $(b_1^\infty, 1)$ is well-defined and induces an equilibrium path.

The discussion above establishes that the procedure described can induce an equilibrium path. Now we show that the procedure must stop in finite steps. That is, along this equilibrium path, the equilibrium reaches a steady state in finitely many periods. Consider any point (b_t, x_t) on the equilibrium path. By the first-order condition, the future prices are larger with a larger expected supply. Thus, the equilibrium price path is

If the equilibrium is not the maximum equilibrium, it cannot be that $lim_{t\to\infty}b_t = b \in (0, \underline{b}]$. This is because if the claim is true, since the equilibrium is not maximal, $lim_{t\to\infty}x_t < \frac{B}{b}$ and thus for n large, the first-order condition implies that $b_{n+1} < b < b_n$. Thus, $b_t \to 0$ and $x_tb_t \to 0$ as $t \to \infty$.

■

Proof of Proposition 5:

Consider the counterfactual where the asset does not have consumption value. By Proposition 4, there exists a maximal equilibrium path. Let b_n be the asset price when the asset price stabilizes. Clearly, $b_n < \underline{b}$ and $b_n x_n = B$.

When the asset has consumption value, let b_ϵ satisfies $b_\epsilon u'(e_y - B)$. If $b_n \geq b_\epsilon$, along the same maximal equilibrium trajectory, the agent never consumes the asset. Thus, the maximal equilibrium path also represents an equilibrium when the asset has consumption value. There cannot be any equilibrium where the initial price is higher. By Lemma 1, if there is an equilibrium with a higher initial asset price, on that path, it must be that $b_n x_n > B$.

Suppose there is an equilibrium where the asset's initial price b_1 is lower. By Proposition 4, it cannot be that the agent never consumes the asset. Without consumption, the equilibrium path must be identical to the counterpart where the asset has no consumption value. However, other than the maximal equilibrium path, the asset prices along all other equilibrium paths converge to zero. Moreover, a similar analysis to the one in Section 3 shows that there is no equilibrium where the agent consumes the entire position. Thus, if there is such an equilibrium path, there must be a period t such that

$$\beta b_{t+1} u'(e_o + b_{t+1}(x_t - c_t)) = \epsilon.$$

However, the first-order condition and the market-clearing condition of the generation $t+1$ imply that

$$b_{t+1} u'(e_y - b_{t+1} x_{t+1}) \geq \epsilon$$

where $x_{t+1} = x_t - c_t + s(b_{t+1})$. Thus, $u'(e_y - b_{t+1}x_{t+1}) \geq \beta u'(e_o + b_{t+1}x_{t+1})$ and this implies that $b_{t+1}x_{t+1} > B$, which is impossible.

If $b_n < b_\epsilon$, clearly, the maximal equilibrium when the asset has zero consumption value fails to be an equilibrium. By continuity, there exists an initial value $\tilde{b}_1 > b_1$ such that $\tilde{b}_n = b_\epsilon$. The $n-th$ generation will consume $\tilde{x}_n - \frac{B}{b_\epsilon}$ and the price will stabilize at b_ϵ with bubble size B. We can show that this is the unique equilibrium following a similar procedure to the previous analysis.

∎

Proof of Proposition 6:

We first show that under the additional assumptions, the economy reaches the steady state in finitely many periods. If not, from the inductive procedure constructing the maximal equilibrium path, there must be a sequence of counterfactual prices and inventory (b_n, x_n) converging to $(\underline{b}, \frac{B}{\underline{b}})$ satisfying the following conditions: Within an arbitrarily small ball around

$(\underline{b}, \frac{B}{\underline{b}})$, there exists a pair (b_n, x_n), $b_n > \underline{b}$ and $x_n < \frac{B}{\underline{b}}$ such that there is a y_n with $b_n > y_n > \underline{b}$ satisfying both $y_n(x_n + s(b_n)) = B$ and $b_n u'(e_y - b_n x_n) = \beta y_n u'(e_o + y_n x_n)$.

Apply Taylor expansion on the right side of the equation. If $u'''(\cdot) > 0$ at $e_o + B$, when n is large, we have

$$b_n u'(e_y - b_n x_n) > \beta y_n u'(e_o + B) - \beta y_n^2 s(b_n) u''(e_o + B)$$

Note that $\beta u'(e_o + B) = u'(e_y - B)$. Thus, the inequality implies that

$$\frac{b_n}{\underline{b}} u'(e_y - b_n x_n) - u'(e_y - B) > C\underline{b}s(b_n)$$

where $C = -\beta u''(e_o + B) > 0$. Since $s(b_n) > m(b_n - \underline{b})$, and $b_n x_n < B$ it must be that

$$\frac{b_n - \underline{b}}{\underline{b}} u'(e_y - B) > Cm\underline{b}(b_n - \underline{b})$$

or equivalently

$$u'(e_y - B) > Cm\underline{b}^2.$$

When $m \geq -\frac{u'(e_y - B)}{\beta u''(e_o + B)\underline{b}^2}$, this inequality cannot hold.

For the second part of the proposition, suppose the claim is not true. Since $x_1 = \tilde{x}_1 = 1$, if $b_1 > \tilde{b}_1$, by the first-order condition, $b_2 > \tilde{b}_2$. Moreover, since $s > \tilde{s}$, $x_2 = \tilde{x}_2$. Thus, $b_2 x_2 > \tilde{b}_2 \tilde{x}_2$. Applying the same argument, we have $b_t x_t > \tilde{b}_t \tilde{x}_t$ for all t. However, this contradicts the fact that both equilibria eventually share the bubble size B. Specifically, if the price in both the s-economy and the $\tilde{s}$-economy stabilizes in finitely many periods, the contradiction is obvious. If the $\tilde{s}$-economy never stabilizes, the asset price in the $\tilde{s}$-economy must converge to $\underline{b}$. Then the no-crossing condition implies that the asset price in the s-economy must also stabilize at $b_\infty = \underline{b}$ in finitely many periods. This is impossible. Suppose the price in the s-economy stabilizes in period t. Then if $b_1 > \tilde{b}_1$, it must be $\tilde{b}_t < \underline{b}$.

A similar argument shows that if on the equilibrium paths for any indices of periods t and τ, if $\tilde{x}_\tau \leq x_t$, it must be $\tilde{b}_\tau > b_t$.

■

Proof of Proposition 7:

Since the s-economy stabilizes in finitely many periods, $b_t - b_{t+1}$ has a positive lower bound. When θ is small enough, $\tilde{x}_t < x_t$ for $t \geq 2$. By Proposition 6, $b_t < \tilde{b}_t$.

■