A Macro-Finance Model of Capital Reallocation and Misallocation *

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Abstract

I quantitatively study the efficiency in capital allocation and financial markets jointly in a general equilibrium model with heterogeneous investors. Leveraged intermediaries allocate wealth across sectors that differ in productivity and risk exposure, and face state-dependent margin constraints. I find that efficiency in financial markets and allocative efficiency in capital are mutually reinforcing. Three forces drive this result. First, efficient capital allocation in the presence of sectoral dispersion requires strong financial balance sheets and efficient risk sharing, and reflects risk-adjusted costs of capital. Second, binding constraints during downturns distort risk sharing among investors and raise risk premia, setting the stage for capital misallocation. Third, the resulting inefficiency in capital composition further affects valuations and growth through risk sharing among investors in constrained states. The mechanisms thus create a feedback loop between efficiency in capital allocation and efficiency in financial markets.

Keywords: General equilibrium, capital misallocation, heterogeneous investors, financial frictions.
JEL Codes: E13, E32, E44, G11, G12, G21.

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1 Introduction

Capital misallocation is pervasive across industries and countries. A common proxy is dispersion in the marginal product of capital (MPK), as in the seminal work of Hsieh and Klenow (2009) and Restuccia and Rogerson (2008). Much of the literature interprets such dispersion as evidence of misallocation and explains it through firm-level frictions, with limited attention to the equilibrium role of financial markets (Buera et al., 2011; Moll, 2014; David and Venkateswaran, 2019).

Recent work brings financial markets into the analysis of capital allocation. David et al. (2022) show that firm risk exposure can rationalize part of measured misallocation, while Choi et al. (2025) emphasize investor demand. Yet because these papers study firm decisions with an exogenous stochastic discount factor (SDF) in partial equilibrium, they leave open a central question in general equilibrium: how does capital allocation efficiency feed back into financial markets? Answering this question requires endogenizing the risk compensation that shapes firms' cost of capital.

I study these questions in a macro-finance model. The model links allocative efficiency on the real side of the economy to risk-sharing efficiency in financial markets. The paper makes two contributions. First, it identifies a risk-sharing channel of capital allocation: the efficiency with which investors share aggregate risks determines the sectoral allocation of capital. Second, it demonstrates a feedback loop between allocative and financial efficiency: distortions in sectoral capital allocation affect asset prices and growth through the same imperfect risk sharing among investors.

To develop the mechanism formally, I build a DSGE model with two main ingredients: two productive sectors with different exposures to aggregate shocks, and heterogeneous investors who differ in risk-bearing capacity and face portfolio constraints. The multi-sector structure lets financial frictions affect not only aggregate investment over time, but also the allocation of capital across sectors. I show that the interaction between sectoral composition and intermediary capitalization jointly determines capital allocation, valuations, and growth. When financial constraints bind, this interaction distorts risk pricing and pushes the allocation away from its efficient benchmark. The model abstracts from firm-level wedges and other standard sources of misallocation.[1] This isolation shows that, even without these frictions, imperfect risk sharing can generate capital misallocation that feeds back into risk premia, valuations, and growth.

The production side builds on multi-sector models in which adjustment costs generate

[1] Section 4 compares simpler economies that isolate the model's ingredients and shows that they are necessary to generate misallocation through the channel of impaired risk sharing among investors.

capital illiquidity, as in Eberly and Wang (2025) and Hansen et al. (2024). Capital is costly to reallocate across sectors, so sectoral composition evolves gradually. The two technologies produce the same consumption good but differ in productivity and exposure to aggregate shocks. In the quantitative model, one sector is more productive and more exposed to aggregate risk, while the other is safer. The firms in these sectors are equity-financed and owned by investors. The two-sector structure is a parsimonious theoretical device for representing a continuum of capital with varying degrees of riskiness across firms. I assume orthogonal fundamental shocks across sectors to isolate the endogenous comovement between the two sectors.

The economy is populated by heterogeneous investors with recursive preferences, who differ in their level of risk aversion. Financial intermediaries (labeled A) are relatively risk-tolerant, while households (labeled B) are more risk-averse.[2] All investors can trade the two risky assets (equity claims) and a riskless bond, but face state-dependent margin constraints. In equilibrium, intermediaries are leveraged and therefore more likely to be constrained (He and Krishnamurthy, 2013; Brunnermeier and Sannikov, 2014). The key distinction is that intermediaries do not only absorb aggregate risk; through their portfolio choices, they also determine how sectoral risks are allocated across the economy.

The margin constraints limit investors' risk taking and tighten when asset return volatilities are high, in the spirit of an approximate value-at-risk (VaR) rule (Kargar, 2021). The model extends this framework to a multi-sector, multi-shock setting. In the calibration, I focus on equilibria in which these constraints occasionally bind only for intermediaries, consistent with evidence that leverage constraints are most relevant for financial intermediaries. Absent these constraints, investors allocate wealth across assets and share risks efficiently, and capital is allocated according to the risk-adjusted cost of capital across sectors. Thus, any capital misallocation in the model arises from impaired risk sharing rather than firm-level wedges.[3] The key departure from the misallocation literature is therefore that misallocation originates in constrained risk sharing in financial markets.

To clarify the notion of misallocation, start from the benchmark in standard investment models. Let $MPK_{i,t+1}$ denote the marginal product of capital for firm i and let $r+\delta$ denote the user cost of capital, with r denoting the risk-free rate and δ denoting depreciation. In the absence of firm-specific wedges, optimal investment equates the expected marginal product of capital to the common user cost:

$$\mathbb{E}_t[MPK_{i,t+1}] = r + \delta. \tag{1}$$

[2]Intermediaries should be interpreted as representing leveraged institutions, such as commercial banks, investment banks, and hedge funds. Households correspond generically to institutions that provide short-term funding.

[3]In the model as shown later, the constraints tend to bind when the economy is financially fragile.

Dispersion in MPK is therefore often interpreted as evidence that this condition fails to hold for some firms. A common approach is to introduce firm-level wedges,

$$\mathbb{E}_t[MPK_{i,t+1}] = r + \delta + \tau_{i,t}, \tag{2}$$

where $\tau_{i,t}$ denotes the wedge of firm i. Thus, MPK need not be equalized across firms due to firm-specific wedges. These models explain a large fraction of the observed MPK dispersion, but leave a significant portion unexplained (David and Venkateswaran, 2019).

Risk premia change this benchmark. Instead of inserting a wedge, condition (1) can be modified to

$$\mathbb{E}_t[MPK_{i,t+1}] = r + \delta + \beta_{i,t}\lambda_t, \tag{3}$$

where $\beta_{i,t}$ denotes firm i's exposure to aggregate risk and λ_t denotes the market price of that risk (David et al., 2022). The cost of capital is no longer common across firms: firms with higher risk exposure require higher expected MPK. Efficient allocation therefore equalizes risk-adjusted returns, not necessarily MPK.

Motivated by this distinction, my model asks when risk-adjusted capital allocation itself becomes inefficient. In my model, a continuous analog of condition (3) holds, where risk premia are endogenously determined in general equilibrium. Because sectoral technologies have constant returns to scale, MPK can remain dispersed even in efficient allocations. I therefore define capital misallocation as a deviation from the unconstrained allocation with efficient risk sharing. When margin constraints bind and intermediaries' risk-allocation role breaks down, risk premia are distorted relative to that benchmark. Sectoral Tobin's Q then moves away from its efficient value.[4] Thus, misallocation in the model is not dispersion in marginal products. It is distorted risk-adjusted capital allocation caused by impaired risk sharing.

The model's equilibrium dynamics are described by two endogenous state variables: sectoral composition, summarized by the capital share of the riskier sector, and intermediary capitalization, summarized by intermediaries' share of total investor wealth. The first state variable governs how capital, and therefore fundamental risk, is allocated on the production side of the economy; the second governs who bears that risk in financial markets.

In the unconstrained benchmark, investors trade freely and capital is allocated efficiently in risk-adjusted terms. Risk-averse investors evaluate sectoral productivity after adjusting for differences in risk exposure. Even without binding constraints, the model generates balance-sheet recessions: when intermediaries' wealth share is low, risk prices rise and valuations fall. The model also generates an endogenous flight-to-safety mechanism. When the economy

[4]Because sectoral investment and growth are monotone functions of Tobin's Q in the model, I use deviations in sectoral Q as the main measure of misallocation in the quantitative analysis.

is sufficiently diversified, lower intermediary capitalization raises the relative valuation and investment of the safer sector. Investors also have a diversification motive that pushes capital composition toward the well-diversified state, and this motive strengthens when intermediary capitalization is weak.

With financial constraints, the efficient allocation is disrupted along the intermediary-capitalization dimension. When intermediaries are poorly capitalized, the state-dependent constraint binds on their exposure to sectoral equity claims.[5] Intermediaries must reduce their exposure to the dominant sector, raising the price of risk for the associated shock.[6] The dominant sector's valuation and investment decline, while the other sector is allocated relatively more capital. Consistent with Eisfeldt and Rampini (2006), the model generates countercyclical misallocation, but through a different channel: financial constraints impair risk sharing among investors.

The feedback operates through the capital-composition dimension. In constrained states, inefficient capital composition changes the sectoral risks that investors must share. Because risk sharing is impaired, these exposures command additional compensation, raising risk premia and depressing valuations and investment. These price effects move sectoral Q farther from its efficient benchmark and reinforce the misallocation. The result is a feedback loop between capital allocation and financial markets, with large fluctuations in sectoral Q and aggregate growth.

Because both state variables are endogenous equilibrium objects, their joint dynamics make the model strongly nonlinear. Since the equilibrium does not admit a closed-form solution, I solve it globally by casting the equilibrium conditions as a coupled system of second-order PDEs. I extend the projection approach in the spirit of Judd (1992, 2023) and develop an algorithm based on neural networks to recover policy rules and prices over the full state space. The calibration disciplines the sectoral characteristics, where constraints bind and how intermediary capitalization interacts with sectoral composition. Relative to local perturbation or log-linear methods, this global solution captures the model's nonlinear state dependence, the regions where constraints bind, and the misallocation dynamics that drive the quantitative results.

The remainder of the paper is organized as follows. Section 2 reviews the relevant literature. Section 3 lays out the model. Section 4 discusses the model's solution and the characterization of the equilibrium. In Section 5 I present the quantitative analysis and discuss the results. Finally, Section 6 concludes.

[5] Mathematically, the model's VaR-type margin constraints on portfolio weights are identical to restricting exposure to fundamental shocks, as will be shown formally in Section 3.

[6] The result applies whether the growth or value sector is dominant.

2 Related literature

This paper fits into several strands of literature. First, it contributes to the voluminous literature going back to the seminal contributions of Bernanke and Gertler (1989), Bernanke et al. (1999), and Kiyotaki and Moore (1997). I extend the framework of He and Krishnamurthy (2013) and Brunnermeier and Sannikov (2014, 2016), who study the role of financial sectors in amplifying systemic shocks into fluctuations of risk premia and asset prices. My paper blends these insights by incorporating heterogeneous investors with recursive preferences and state-dependent portfolio constraints into a multi-sector production economy. Notably, this framework allows not only for misallocation when frictions bind, but also for efficient capital allocation when frictions are loose.

Di Tella (2017), Gopalakrishna (2023), Maxted (2024), and Krishnamurthy and Li (2025) are recent examples that build on Brunnermeier and Sannikov (2014) to study additional amplification mechanisms. Key differences of my paper are: (i) investors share risks by trading equity in multiple productive sectors without physical capital transfers, so 'fire-sale' effects work through discount-rate changes rather than reductions in dividends; (ii) sectoral asymmetry in productivity endogenously generates time-varying aggregate productivity, and this yields a novel two-way feedback loop: negative shocks from capital production affect the balance sheets and risk-bearing capacity of intermediaries, leading to financial amplification; while rebalancing of capital across sectors alters aggregate productivity and the future distribution of risks, putting additional pressure on growth prospects. The interaction of capital reallocation and wealth redistribution in my model jointly drives asset prices and macro fluctuations.

There is a growing literature studying capital misallocation, early examples of which include Restuccia and Rogerson (2008) and Hsieh and Klenow (2009). Close to my work are papers exploring the role of financial frictions, e.g., Buera et al. (2011), Midrigan and Xu (2014), and Moll (2014). More recently, Ai et al. (2020) develop a contracting theory with intermediation to emphasize the amplifying role of exogenous financial shocks in capital misallocation. David et al. (2022) consider aggregate risk in a partial equilibrium (PE) setting, and show that risk premia can rationalize a large fraction of observed misallocation from the data. Dou et al. (2025) study the GE implications of the valuation channel and endogenous growth. I contribute to this literature by highlighting the risk sharing mechanism in a GE setting, and examining its interaction with capital misallocation in shaping aggregate growth.

On a technical note, this paper also contributes to the heterogeneous-agent asset pricing literature, for which Panageas (2020) provides an excellent survey. The seminal contributions of Dumas (1989, 1992), Wang (1996), Chan and Kogan (2002), Bhamra and Uppal

(2009, 2014), Longstaff and Wang (2012), Gârleanu and Panageas (2015) study equilibrium in economies with two heterogeneous agents and different preference assumptions. Basak and Cuoco (1998), Gârleanu and Pedersen (2011), He and Krishnamurthy (2012, 2013), Chabakauri (2013), and Rytchkov (2014) examine the asset pricing implications of exogenous and endogenous portfolio constraints in economies populated by two heterogeneous agents with one or many assets where constrained agents have logarithmic or constant relative risk aversion (CRRA) preferences. More recent examples include Drechsler et al. (2018), Silva (2020), Kargar (2021), Schneider (2022), who study the role of heterogeneous agents with recursive preferences in various settings.

This paper follows by allowing for investors with recursive preferences and endogenous portfolio constraints, and generalizes to a multi-sector production economy with real frictions. This insight draws on the pioneering works by Cochrane et al. (2008) and Martin (2013) study the rebalancing mechanism in general equilibrium and return comovement in economies with multiple Lucas 'trees'. These models assume infinite adjustment costs and do not allow capital movements. The other extreme of the spectrum, as in Cox et al. (1985), shows that capital adjusts immediately to ensure the balance of portfolios in the absence of adjustment costs. Eberly and Wang (2009, 2025) extend these models by endogenizing the size of 'trees' in a two-sector production economy, allowing for sluggish capital reallocation due to costly adjustment. Along with this line of literature, Duarte (2018) utilizes habit preference in a two-sector production setting to endogenize the risk aversion of a representative agent and shows this force prolongs economic downturns. Kozak (2022) examines bond-stock comovement in a similar economy with a representative agent, who has a recursive preference with an exogenously time-varying risk aversion. The multi-sector structure of my model builds on Eberly and Wang (2025) and is related to Duarte (2018) and Kozak (2022) by featuring asymmetric sectors with distinct cash-flow risks. I embed this structure in a macro-finance model, in which heterogeneous investors share risks subject to state-dependent portfolio constraints. As a result, misallocation emerges endogenously when the constraints bind. This framework brings financial amplification channel into the capital reallocation dynamics, which is missing in the representative-agent models.

3 Model

In this section, I outline a two-sector general equilibrium model in continuous time. A representative firm in each sector uses capital to produce consumption goods, subject to investment adjustment costs. The real technologies employed by firms are subject to aggregate shocks. The economy is populated by a continuum of investors with recursive preferences,

categorized into two classes: financial intermediaries and households. Financial intermediaries are less risk averse than households, although both types of investors have access to the financial market subject to portfolio constraints. The information structure of the economy is standard. Uncertainty is characterized in a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with a standard filtration $\mathscr{F}_t \subset \mathscr{F}$, $t \in [0, \infty)$.

In the remainder of this section, I first describe the economic setup and decision problems faced by firms and investors, and then define the competitive equilibrium.

3.1 Production

The productive side of the economy consists of two sectors, each using capital as the sole factor of production. Let K_n, I_n, and Y_n denote the capital stock, gross investment, and output, respectively, in sector n for $n \in \{1, 2\}$. I first introduce the firms' technology, followed by their optimization problems.

3.1.1 Technology

The representative firm in each sector employs a constant returns-to-scale production technology, specifically an "AK" type.[7] The stock of capital $K_{n,t}$ produces output at rate

$$Y_{n,t} = A_n K_{n,t} \tag{4}$$

per unit of time, where A_n is a constant governing the productivity of capital.

Capital stock in each sector is exposed to exogenous aggregate Brownian shocks. The evolution of capital stock in sector n follows

$$\frac{dK_{n,t}}{K_{n,t}} = \phi_n(\iota_{n,t})dt + \boldsymbol{\sigma}_n' d\boldsymbol{B}_t, \tag{5}$$

where $\iota_{n,t} \equiv I_{n,t}/K_{n,t}$ is the investment-capital ratio in sector n at time t. $\boldsymbol{B} = \{(B_{1,t}, B_{2,t})' \in \mathbb{R}^2; \mathscr{F}_t, t \geq 0\}$ is a two-dimensional Brownian motion in the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, representing aggregate shocks to the capital stock. Such shocks may reflect stochastic depreciation, investment-specific, or capital quality shocks.[8] The vector $\boldsymbol{\sigma}_n$ is the exposure of capital in Sector n to the aggregate shocks. Later in calibration, I assume no correlation in fundamentals between two sectors. Thus, capital in sector n is effectively only exposed to $B_{n,t}$. This allows

[7] See, for example, Kogan (2001, 2004), Eberly and Wang (2025), Pindyck and Wang (2013), and Brunnermeier and Sannikov (2014).

[8] See, for example, Brunnermeier and Sannikov (2014), and Maxted (2024). However, they do not focus on multi-sector setup, which is the focus of this paper. The vector notation is used to simplify the exposition.

the model to highlight the correlation between asset returns that emerge endogenously, rather than the exogenously specified correlation in fundamentals.

Following Hayashi (1982), Jermann (1998) and others, the investment in each sector incurs adjustment cost that is homogeneous of degree one in investment and capital stock.[9] Function $\phi(\cdot)$ satisfies $\phi'(\cdot) > 0$ and $\phi''(\cdot) \leq 0$ on the admissible domain. I assume a quadratic adjustment cost function given by

$$\phi(\iota) = \iota - \delta - \frac{\theta}{2}(\iota)^2, \tag{6}$$

where θ is the adjustment cost parameter and δ is the depreciation rate of capital.

3.1.2 Optimization problems

Firms pay output net of investment as dividends to shareholders. The representative firm in each sector chooses the investment rate to maximize the expected discounted value of dividends, given the *stochastic discount factor (SDF)*. The value of the firm in sector n is given by

$$S_{n,t} = \sup_{\iota_{n,s}} \mathbb{E}_t \left[\int_t^\infty \frac{\Lambda_s}{\Lambda_t} (A_n K_{n,s} - \iota_{n,s} K_{n,s}) ds \right]$$
$$\text{s.t. capital evolution (5) and adjustment cost (6).} \tag{7}$$

where Λ_t denotes the SDF that is determined in equilibrium. The evolution of SDF can be written as

$$\frac{d\Lambda_t}{\Lambda_t} = -r_t dt - \boldsymbol{\eta}_t' d\boldsymbol{B}_t \tag{8}$$

where r_t is the risk-free rate and $\boldsymbol{\eta}_t$ is the vector of market price of risk, both determined in equilibrium.

3.2 Investors

The economy is populated by a continuum of investors in two classes, A and B. I start from the demographic structure and preference of the investors in the economy, followed by their budget and financial constraints. Then I define the optimization problems faced by investors.

[9] Homogeneous adjustment cost functions have been widely used in the q theory of investment literature. Hayashi (1982) showed that with homogeneous adjustment costs and perfect capital markets, marginal and average q are equal. Jermann (1998) integrated this type of adjustment costs into a real business cycle model.

3.2.1 Demographics and preferences

The aggregate population remains constant and is normalized to one. To obtain stationarity in the model, I follow Gârleanu and Panageas (2015) and consider a simple overlapping-generations (OLG) framework. All investors, regardless of age, preference, or wealth, face an exogenous and constant mortality rate $\kappa > 0$, and a fraction κ of new investors are born per unit of time.[10] In aggregate, newborn investors inherit the wealth of their deceased parents on an equal per-capita basis. Of the newly born investors, a constant fraction $\bar{u}$ is of type A, while $1-\bar{u}$ is of type B. The OLG setup ensures the existence of a stationary distribution of wealth among investors.

All investors of two classes have recursive preferences with constant coefficient of relative risk aversion (RRA) and elasticity of intertemporal substitution (EIS).[11] Specifically, the preference is described by the stochastic differential utility (SDU) introduced by Duffie and Epstein (1992a,b). That is, an investor of type i for $i \in \{A, B\}$ has lifetime utility defined by

$$J_{i,t} = \mathbb{E}_t \left[\int_t^\infty f_i(C_{i,s}, J_{i,s}) ds \right], \tag{9}$$

where function $f(C,J)$ is known as the aggregator of the recursive utility for consumption C and the continuation value J. The aggregator function $f(C,J)$ is given by

$$f(C,J) = \frac{1-\gamma}{1-1/\psi} J \left[\left(\frac{C}{[(1-\gamma)J]^{1/(1-\gamma)}} \right)^{1-1/\psi} - (\hat{\rho}+\kappa) \right], \tag{10}$$

where the RRA is denoted as γ, and the EIS is denoted as ψ. For simplicity, assume $\gamma \neq 1$. The time-additive separable CRRA utility is a special case of the above recursive utility specification, when the RRA γ is equal to the reciprocal of the EIS ψ, i.e., $\gamma = 1/\psi$. Gârleanu and Panageas (2015) show that, due to mortality risk, the (effective) subjective discount rate is defined as $\rho \equiv \hat{\rho} + \kappa$, where $\hat{\rho}$ denotes the time impatience.

The two classes of investors have different RRA γ. I use A to denote financial intermediaries and B to denote households. In particular, households are more risk averse than financial intermediaries, i.e., $\gamma_A < \gamma_B$. It follows that in equilibrium financial intermediaries, including banks, hedge funds, and broker-dealers, etc., raise funds from households via short-term debt financing and use these funds to make levered investments. To focus on the risk aversion heterogeneity, I assume the same EIS across all investors.

[10] The total population size is thus $\int_{-\infty}^{t} \kappa e^{-\kappa(t-s)} ds = 1$.

[11] See, for example, Kreps and Porteus (1978), Epstein and Zin (1989), and Weil (1989).

3.2.2 Budget and financial constraints

All investors in the economy can trade claims on dividends from the two sectors and a risk-free bond in zero net supply, subject to portfolio constraints. Investors dynamically choose their consumption and portfolio weights. Let s denote the birth date of an investor and $W_{i,t}(s)$ denote the wealth of an investor of type i born in date s at time t. Since investors of the same type face the same optimization problems (defined below), the decisions do not depend on the age of each investor. Thus, the notation can be simplified by dropping s in the wealth dynamics. Let $\boldsymbol{\omega}_{i,t}$ be the vector of her portfolio weights in two risky assets. Then the financial wealth of an investor of type i evolves according to the dynamic budget constraint

$$\frac{dW_{i,t}}{W_{i,t}} = [\boldsymbol{\omega}_{i,t}'\boldsymbol{\lambda}_{R,t} + r_t - c_{i,t}]dt + \boldsymbol{\omega}_{i,t}'\boldsymbol{\Sigma}_{R,t}d\boldsymbol{B}_t, \tag{11}$$

where $c_{i,t} \equiv C_{i,t}/W_{i,t}$ is investor's consumption-wealth ratio. r_t is the risk-free rate, $\boldsymbol{\lambda}_{R,t}$ is the vector of expected excess return of risky assets, and $\boldsymbol{\Sigma}_{R,t}$ is the matrix of loadings on the aggregate shocks of asset returns (all defined in equilibrium in Section 4).

The leverage of an investor is defined as the ratio of asset over equity. In equilibrium, intermediaries lever up their balance sheets by borrowing from households. That is, when the sum of portfolio weights in two risky assets exceed one, the intermediaries lever up by issuing instantaneous risk-free debt to households. I assume investors face occasionally binding margin constraints, which depend inversely on the volatilities of returns of risky assets. Because return volatilities also depend on the state of the economy, margin requirements are state dependent as well.

I consider a change of variables that transforms the characterization to simplify the exposition. Define the risk exposure of an investor of type i as $\boldsymbol{\sigma}_{i,t}' \equiv \boldsymbol{\omega}_{i,t}'\boldsymbol{\Sigma}_{R,t}$. Given the absence of arbitrage, the vector of expected excess returns of risky assets is given by $\boldsymbol{\lambda}_{R,t} = \boldsymbol{\Sigma}_{R,t}\boldsymbol{\eta}_t$. The dynamic budget constraint can be rewritten as

$$\frac{dW_{i,t}}{W_{i,t}} = [\boldsymbol{\sigma}_{i,t}'\boldsymbol{\eta}_t + r_t - c_{i,t}]dt + \boldsymbol{\sigma}_{i,t}'d\boldsymbol{B}_t. \tag{12}$$

Following the change of variables, I assume financial constraint in terms of the risk exposure as

$$\|\boldsymbol{\sigma}_{i,t}\| \leq \zeta, \tag{13}$$

where $\|\cdot\|$ denotes the norm operator and measures the total risk exposure of investors. ζ

denotes the tightness of the constraint on the total risk exposure.[12] Therefore, the investors effectively face a VaR rule, as in Danielsson et al. (2012), Adrian and Shin (2014), and Kargar (2021).[13] I extend the framework to multiple assets and multiple shocks, which allows for a general form of VaR constraints. As will be shown, this friction in the financial market is the key driver of allocative efficiency on the real side of the economy.

3.2.3 Optimization problems

Investors within each type face identical optimization problems, as mentioned above. They start with initial wealth $W_{i,0} > 0$, decides on consumption as a fraction of wealth $c_{i,t}$, and risk exposure $\boldsymbol{\sigma}_{i,t}$ to maximize lifetime utility. The optimization problem of an investor of type i is given by

$$V_{i,t} = \max_{\{c_{i,s}\geq 0, \boldsymbol{\sigma}_{i,s}\}} J_{i,t}$$
$$\text{s.t. dynamic budget constraint (12) and financial constraint (13),} \tag{14}$$

and a solvency constraint $W_{i,t} \geq 0$.

3.3 Equilibrium

The following defines the competitive equilibrium.

Definition 1 *A competitive equilibrium is a set of adapted stochastic processes for interest rate r_t, market price of risk $\boldsymbol{\eta}_t$, stochastic discount factor Λ_t; firms' output $Y_{n,t}$, capital $K_{n,t}$, investment $I_{n,t}$, and value $S_{n,t}$ for $n \in \{1,2\}$; investors' wealth $W_{i,t}$, consumption $C_{i,t}$, risk exposure $\boldsymbol{\sigma}_{i,t}$ for $i \in \{A,B\}$ such that*

i. firms solve an optimal investment problem (3.1.2);

ii. investors maximize utility by choosing consumption and risk exposure (3.2.3);

[12] This is equivalent to the equations (6) and (7) of Kargar (2021), in which the portfolio weight is bounded by $\frac{1}{\alpha\sigma_{R,t}}$ and $\sigma_{R,t}$ denotes the volatility of the risky asset return. The risk exposure is thus bounded by a constant $\frac{1}{\alpha}$. Note that investors will not short the risky asset in equilibrium if there is one single asset.

[13] VaR constraints, common for financial institutions in the Basel II and Basel III regulatory frameworks, aim at limiting downside risk and maintaining an equity cushion large enough so that the default probability is kept below some benchmark level.

iii. markets clear:

$$\begin{aligned}
\bar{u}C_{A,t} + (1-\bar{u})C_{B,t} &= Y_{1,t} + Y_{2,t} - I_{1,t} - I_{2,t} \quad \textit{(goods market)}, \\
\bar{u}W_{A,t} + (1-\bar{u})W_{B,t} &= S_{1,t} + S_{2,t} \quad \textit{(bond market)}, \\
\begin{pmatrix} \boldsymbol{\sigma}_{A,t} & \boldsymbol{\sigma}_{B,t} \end{pmatrix} \begin{pmatrix} \bar{u}W_{A,t} \\ (1-\bar{u})W_{B,t} \end{pmatrix} &= \begin{pmatrix} \boldsymbol{\sigma}_{R_1,t} & \boldsymbol{\sigma}_{R_2,t} \end{pmatrix} \begin{pmatrix} S_{1,t} \\ S_{2,t} \end{pmatrix} \quad \textit{(risk exposures market)}.
\end{aligned} \tag{15}$$

4 Solution

The equilibrium can be characterized in a recursive formulation, where all equilibrium objects are functions of aggregate state variables. In this section, I first define endogenous state variables and derive their dynamics. Subsequently, I derive the Hamilton-Jacobi-Bellman (HJB) equations for the scenario without portfolio constraints, characterize the solutions to the optimization problems of firms and investors, and provide intuition regarding the optimal policy functions. Then I define a recursive Markov equilibrium and briefly discuss the asset pricing implications from the unconstrained economy. Lastly, I use asymptotic expansion methods to characterize alternative economies to isolate key mechanisms. The full model with occasionally-binding constraints, which is solved numerically, will be discussed in Section 5.

4.1 State variables

Due to the homogeneity of Epstein-Zin preferences and the capital accumulation processes, all variables can be rescaled by the total wealth or total capital. Therefore, it suffices to track only two aggregate state variables: the share of the aggregate wealth that belongs to the investors of type A and the share of total capital that belongs to the firm in sector 1. Thus, the equilibrium conditions can be expressed as functions of these two endogenous state variables:

$$w_t = \frac{\bar{u}W_{A,t}}{\bar{u}W_{A,t} + (1-\bar{u})W_{B,t}}, \qquad x_t = \frac{K_{1,t}}{K_{1,t} + K_{2,t}}. \tag{16}$$

This paper highlights the joint dynamics of w and x as the main drivers of the equilibrium, and the interaction between the two state variables as a key channel through which financial frictions affect the efficiency of capital allocation. The state variable w is one of the key variables in the macro-finance models (e.g., He and Krishnamurthy (2013) and Brunnermeier and Sannikov (2014)).[14] In these models, shocks to the aggregate capital stock effectively

[14]Note that the combination of heterogeneity and recursive preferences gives rise to the wealth share as a state variable even under perfect risk sharing.

generate fluctuations in the wealth share of intermediaries, which in turn drive the dynamics of market price of risk and asset prices (as shown in Appendix D). Thus, the existing macro-finance models do not explicitly model the capital reallocation across sectors, which is captured by the state variable x in this model. The variable x has been studied in the literature on production-based models with multiple sectors (e.g., Eberly and Wang (2009), Duarte (2018) and Kozak (2022)). In these models, the representative investor allocates capital across sectors in response to shocks to the sectoral capital stock (as shown in Appendix D).

The following proposition gives the dynamics of state variable w_t, the wealth share of investors who belong to type A.

Proposition 1

$$dw_t = \kappa(\bar{u} - w_t)dt + w_t(1 - w_t)[\mu_{w,t}dt + \boldsymbol{\sigma}'_{w,t}d\boldsymbol{B}_t], \tag{17}$$

where $\mu_{w,t}$ *and* $\boldsymbol{\sigma}_{w,t}$ *are given by*

$$\mu_{w,t} = (\boldsymbol{\sigma}'_{A,t} - \boldsymbol{\sigma}'_{B,t})\boldsymbol{\eta}_t + c_{B,t} - c_{A,t} - (\boldsymbol{\sigma}'_{A,t} - \boldsymbol{\sigma}'_{B,t})\boldsymbol{\Sigma}'_{R,t}\begin{pmatrix} z_t \\ 1 - z_t \end{pmatrix},$$

$$\boldsymbol{\sigma}_{w,t} = \boldsymbol{\sigma}_{A,t} - \boldsymbol{\sigma}_{B,t}.$$

Proof. See Appendix A. ■

Notice that the law of motion of the intermediaries' wealth share w_t has an exogenous component in the drift function due to the demographic structure assumed earlier. This term ensures a non-degenerate long-run distribution of w, i.e., $w \in (0, 1)$. The endogenous component of (17) is driven by the consumption and risk exposure choices of investors of both types (in brackets). Term $\mu_{w,t}$ depends on the consumption-wealth ratio of households relative to that of intermediaries, and on the differential aggregate risk exposure between intermediaries and households. Term $\boldsymbol{\sigma}_{w,t}$ only depends on this latter force. As will be shown in (31), the wealth share dynamics depend on asset prices, which themselves depend on wealth share dynamics. The two-way dependence amplifies the return volatility in the two-sector model, extending the result in Brunnermeier and Sannikov (2014).

The purpose of modeling two productive sectors is a theoretical device to showcase a continuum of capital with various degrees of riskiness across all firms in the economy. For exposition, I assume sector 1 is riskier and more productive, while sector 2 is safer but less productive.[15] Thus, sector 1 and sector 2 can be interpreted as the growth sector and the value sector, respectively. Additionally, the fundamental correlation between the capital stocks of the two sectors is assumed to be zero. This is not meant to capture empirical correlations, but allows to

[15]The assumption about productive sectors is a general extension on Duarte (2018) and Kozak (2022).

focus on capital reallocation between two distinct sectors and on the endogenous correlation in asset returns.

Assumption 1 *The risk exposure of sector 1,* $\boldsymbol{\sigma}_1 = (\sigma_1, 0)'$*, the risk exposure of sector 2,* $\boldsymbol{\sigma}_2 = (0, \sigma_2)'$*, and the overall risk exposure of sector 1 is also higher than that of sector 2,* $\sigma_1 > \sigma_2$*. The productivity of sector 1 is higher than that of sector 2,* $A_1 > A_2$*.*

The following proposition gives the dynamics of state variable x_t, the share of capital that belongs to the firm in sector 1.

Proposition 2

$$dx_t = \mu_{x,t}dt + x_t(1-x_t)\sigma_1 dB_{1,t} - x_t(1-x_t)\sigma_2 dB_{2,t}, \tag{18}$$

where the drift of x_t *is given by*

$$\mu_{x,t} = x_t(1-x_t)\left[g_{1,t} - g_{2,t} + (1-x_t)\sigma_2^2 - x_t\sigma_1^2\right].$$

$g_{n,t}$ *denotes the growth rate of capital in sector* n *at time* t*.*

Proof. See Appendix A. ■

Note that through endogenous capital reallocation, $x_t \in [0,1]$. The endogenous drift term of (18) depends on the relative growth rates of capital in two sectors, adjusted by the difference in sectoral exposure to shocks. The volatility term is a quadratic function in x which attains its highest value at $x = 1/2$. Meanwhile, $x = 0$ and $x = 1$ correspond to the states where all capital is held by the firm in sector 2 and sector 1, respectively.[16] Using the definition of x, the expected growth rate of aggregate capital can be written as

$$g_t = x_t g_{1,t} + (1-x_t)g_{2,t}. \tag{19}$$

Similarly, let ι_t denote the aggregate investment rate, i.e., $\iota_t \equiv (I_{1,t} + I_{2,t})/(K_{1,t} + K_{2,t})$. The aggregate investment rate can be written as

$$\iota_t = x_t\iota_{1,t} + (1-x_t)\iota_{2,t}. \tag{20}$$

The aggregate quantities g_t and ι_t, which are functions of the state variables (x, w), are particularly important for understanding the model's implications for the business cycle.

It is also useful throughout to define another variable, $z_t \equiv S_{1,t}/(S_{1,t} + S_{2,t})$, the ratio of the firm value in sector 1 to the total value of firms in two sectors. Variable z_t, not a state variable

[16] These two states are absorbing states, as the sector cannot be rebuilt once it vanishes, because the loadings on the drift and diffusion terms of the capital stock evolution processes are zero.

per se, captures the weight of sector 1 in the market portfolio, and is a function of x_t and the relative valuation of firms in two sectors, as shown later.

4.2 Hamilton-Jacobi-Bellman equations

The recursive formulation of the equilibrium is characterized by the value function of firms and investors. This section derives the value functions that satisfy the HJB equations.

4.2.1 Firms' problems

The firm's problem in each sector $n \in \{1,2\}$ is to maximize the expected discounted value of dividends $D_{n,t} \equiv A_n K_{n,t} - \iota_{n,t} K_{n,t}$, which can be written recursively

$$0 = \max_{\iota_{n,t}} \left\{ D_{n,t}\, dt + \mathbb{E}_t \left[\frac{d(S_{n,t}\Lambda_t)}{\Lambda_t} \right] \right\}. \tag{21}$$

The homogeneity property implies that marginal q equals Tobin's average q, following Hayashi (1982). The following proposition gives the firms' value functions.

Proposition 3 *The value of firm in sector n for $n \in \{1,2\}$ has the form*

$$S_{n,t}(K_{n,t}, x_t, w_t) = K_{n,t}\, q_{n,t}(x_t, w_t), \tag{22}$$

where $q_{n,t}$ captures future investment opportunities of the firm in sector n. The dynamics of $q_{n,t}$ is given by

$$\frac{dq_{n,t}}{q_{n,t}} = \mu_{q_n,t}\, dt + \boldsymbol{\sigma}'_{q_n,t}\, d\boldsymbol{B}_t, \tag{23}$$

where $(\mu_{q_n,t}, \boldsymbol{\sigma}_{q_n,t})$ is determined in equilibrium. The expressions for $\mu_{q_n,t}$ and $\boldsymbol{\sigma}_{q_n,t}$ are given in the Appendix A.

The HJB equation for the firms' problem is given by the following partial differential equation (PDE):

$$0 = \max_{\iota_{n,t}} \left\{ \frac{A_n - \iota_{n,t}}{q_{n,t}} + g_{n,t} + \mu_{q_n,t} + \boldsymbol{\sigma}'_{q_n,t}\boldsymbol{\sigma}_n - r_t - \boldsymbol{\eta}'_t(\boldsymbol{\sigma}_{q_n,t} + \boldsymbol{\sigma}_n) \right\}, \tag{24}$$

where $g_{n,t}$ denotes the expected growth rate of the capital stock in sector n.

Proof. See Appendix A. ■

The first-order condition (FOC) gives the optimal investment rate as follows

$$g'(\iota_{n,t}) = \frac{1}{q_{n,t}}. \tag{25}$$

Consistent with Q-theory of investment, the optimal investment rate is an increasing function of Tobin's Q, which summarizes the marginal benefit of increasing capital stock. In equilibrium, the representative firm in each sector optimally chooses to equate marginal cost to marginal benefit.

It is straightforward to derive derive aggregate Tobin's Q, which is given by

$$q_t = x_t q_{1,t} + (1 - x_t) q_{2,t}. \tag{26}$$

Note that the aforementioned variable z_t, which captures the weight of sector 1 firm value in the market portfolio, can be written as

$$z_t = \left(1 + \frac{q_{2,t}}{q_{1,t}} \frac{1 - x_t}{x_t}\right)^{-1}. \tag{27}$$

The return of holding claims on the firm is given by the dividend yield plus capital gains, as follows

$$dR_{n,t} = \frac{A_n - \iota_{n,t}}{q_{n,t}} dt + \frac{dS_{n,t}}{S_{n,t}}, \tag{28}$$

$$\equiv \mu_{R_n,t} dt + \boldsymbol{\sigma}'_{R_n,t} d\boldsymbol{B}_t, \tag{29}$$

where

$$\mu_{R_n,t} = \frac{A_n - \iota_{n,t}}{q_{n,t}} + g_{n,t} + \mu_{q_n,t} + \boldsymbol{\sigma}'_{q_n,t} \boldsymbol{\sigma}_n, \tag{30}$$

$$\boldsymbol{\sigma}_{R_n,t} = \boldsymbol{\sigma}_{q_n,t} + \boldsymbol{\sigma}_n. \tag{31}$$

$\mu_{R_n,t}$ and $\boldsymbol{\sigma}_{R_n,t}$ denote the expected return and the vector of volatility terms (loadings on the aggregate shocks) of asset n's return, respectively. Notice that the volatilities of return are driven by an exogenous component $\boldsymbol{\sigma}_{n,t}$ and an endogenous component $\boldsymbol{\sigma}_{q_n,t}$, which captures the firm's investment opportunities. Define the vector of expected excess return, $\boldsymbol{\lambda}_{R,t} \equiv \binom{\mu_{R_1,t} - r_t}{\mu_{R_2,t} - r_t}$. The row n of the matrix $\boldsymbol{\Sigma}_{R,t}$ equals $\boldsymbol{\sigma}'_{R_n,t}$. The variance-covariance matrix of the returns is $\boldsymbol{\Omega}_{R,t} = \boldsymbol{\Sigma}_{R,t} \boldsymbol{\Sigma}'_{R,t}$.

4.2.2 Investors' problems

The investor's problem for each type $i \in \{A, B\}$ is to maximize lifetime utility, which can be written recursively

$$0 = \max_{c_{i,t}, \boldsymbol{\sigma}_{i,t}} \left\{ f_i(c_{i,t} W_{i,t}, V_{i,t}) dt + \mathbb{E}_t[dV_{i,t}] \right\}. \tag{32}$$

With homothetic preferences, the following proposition gives investors' value functions.

Proposition 4 *The value function of investor $i \in \{A, B\}$ has the form*

$$V_{i,t}(W_{i,t}, x_t, w_t) = \frac{W_{i,t}^{1-\gamma_i}}{1-\gamma_i} \xi_{i,t}(x_t, w_t)^{\frac{1-\gamma_i}{1-\psi}}, \tag{33}$$

where $\xi_{i,t}$ captures the type i investor's future investment opportunities. The dynamics of $\xi_{i,t}$ is given by

$$\frac{d\xi_{i,t}}{\xi_{i,t}} = \mu_{\xi_i,t} dt + \boldsymbol{\sigma}'_{\xi_i,t} d\boldsymbol{B}_t, \tag{34}$$

where $(\mu_{\xi_i,t}, \boldsymbol{\sigma}_{\xi_i,t})$ is determined in equilibrium. The expressions for $\mu_{\xi_i,t}$ and $\boldsymbol{\sigma}_{\xi_i,t}$ are given in the Appendix A.

The HJB equation for the investors' problem is given by the following partial differential equation (PDE):

$$\begin{aligned} 0 = \max_{c_{i,t}, \boldsymbol{\sigma}_{i,t}} \Bigg\{ & \frac{1}{1-1/\psi} \left(c_{i,t}^{1-1/\psi} \xi_{i,t}^{1/\psi} - \rho \right) + \left(\boldsymbol{\sigma}'_{i,t} \boldsymbol{\eta}_t + r_t - c_{i,t} \right) \\ & + \frac{1}{1-\psi} \mu_{\xi_i,t} - \frac{\gamma_i}{2} \boldsymbol{\sigma}'_{i,t} \boldsymbol{\sigma}_{i,t} \\ & + \frac{1}{2} \frac{1}{1-\psi} \left(\frac{1-\gamma_i}{1-\psi} - 1 \right) \boldsymbol{\sigma}'_{\xi_i,t} \boldsymbol{\sigma}_{\xi_i,t} \\ & + \frac{1-\gamma_i}{1-\psi} \boldsymbol{\sigma}'_{i,t} \boldsymbol{\sigma}_{\xi_i,t} \Bigg\}. \end{aligned} \tag{35}$$

Proof. See Appendix A. ■

The first-order conditions give the optimal consumption and unconstrained risk exposure as follows

$$c_{i,t} = \xi_{i,t}, \tag{36}$$

$$\boldsymbol{\sigma}^*_{i,t} = \frac{1}{\gamma_i} \boldsymbol{\eta}_t + \frac{1}{\gamma_i} \frac{1-\gamma_i}{1-\psi} \boldsymbol{\sigma}_{\xi_i,t}. \tag{37}$$

In equilibrium, the consumption-wealth ratio, which equals $\xi_{i,t}$, responds to changes in in-

vestment opportunities. The optimal unconstrained risk exposure has two components, as in the intertemporal capital asset pricing model (ICAPM) result of Merton (1973). The first term in (37) is the myopic demand of a one-period mean-variance investor, and is driven by the market price of risk. Since intermediaries are assumed to be less risk-averse, they tend to be more sensitive to changes in the market price of risk. The second term is the hedging demand that captures variation in the investor's investment opportunity set.

Combining the optimal unconstrained risk exposure (37) and the financial constraint (13) yields the optimal risk exposure

$$\boldsymbol{\sigma}_{i,t} = \min\left(\boldsymbol{\sigma}^*_{i,t}, \bar{\boldsymbol{\zeta}}\right). \tag{38}$$

4.3 Markov equilibrium

I define a stationary recursive Markov equilibrium with state variables x_t and w_t, in which all equilibrium objects are expressed as functions of (x, w).

Definition 2 *A Markov equilibrium in state variables (x, w) is a set of functions: equilibrium prices $(r, \boldsymbol{\eta})$, firms' valuation ratio q_n and policy function g_n for $n \in \{1, 2\}$, investors' marginal value of wealth ξ_i and policy functions c_i and $\boldsymbol{\sigma}_i$ for $i \in \{A, B\}$ such that*

i. valuation ratio q_n solves the HJB equation for the firm in sector n, and g_n is the corresponding policy function, taking $(r, \boldsymbol{\eta})$ and laws of motion for x and w as given;

ii. marginal value of wealth ξ_i solves the HJB equation for investor of type i, and c_i and $\boldsymbol{\sigma}_i$ are the corresponding policy functions, taking $(r, \boldsymbol{\eta})$, (q_1, q_2), and laws of motion for x and w as given;

iii. markets clear:

$$w c_A + (1-w) c_B = z\frac{A_1 - \iota_1}{q_1} + (1-z)\frac{A_2 - \iota_2}{q_2} \quad \textit{(goods market)},$$

$$\begin{pmatrix} \boldsymbol{\sigma}_A & \boldsymbol{\sigma}_B \end{pmatrix} \begin{pmatrix} w \\ 1-w \end{pmatrix} = \begin{pmatrix} \boldsymbol{\sigma}_{R_1} & \boldsymbol{\sigma}_{R_2} \end{pmatrix} \begin{pmatrix} z \\ 1-z \end{pmatrix} \quad \textit{(risk exposures market)};$$

iv. the laws of motion for x and w satisfy (18) and (17), respectively.

4.4 Implications from the unconstrained economy

While the equilibrium does not permit closed-form solutions even in the absence of financial constraints, this section presents propositions that characterize risk-taking decisions, asset pricing implications, and capital allocation efficiency from the unconstrained model.

4.4.1 Investors' risk-taking decisions

The following proposition characterizes the intermediaries' demand for risk.

Proposition 5 *In the absence of financial constraints, intermediaries' demand for risk from sector 1, i.e., the first element of their risk exposure* $\boldsymbol{\sigma}_A^*$*, is given by*

$$\sigma_A^{(1),*} = \frac{zR_w + R_w Q_x x(1-x) - R_x(Q_w + \frac{1}{w})x(1-x) - z\frac{\gamma_B}{w(1-w)} - \frac{\gamma_B}{w(1-w)}x(1-x)Q_x}{(\gamma_B - \gamma_A)Q_w - \frac{\gamma_B}{1-w} - \frac{\gamma_A}{w} + R_w}\sigma_1 \tag{39}$$

where

$$\begin{aligned} R_x &= \frac{1-\gamma_A}{1-\psi}\frac{\xi_{A,x}}{\xi_A} - \frac{1-\gamma_B}{1-\psi}\frac{\xi_{B,x}}{\xi_B}; \quad & R_w &= \frac{1-\gamma_A}{1-\psi}\frac{\xi_{A,w}}{\xi_A} - \frac{1-\gamma_B}{1-\psi}\frac{\xi_{B,w}}{\xi_B}; \\ Q_x &= z\frac{q_{1,x}}{q_1} + (1-z)\frac{q_{2,x}}{q_2}; & Q_w &= z\frac{q_{1,w}}{q_1} + (1-z)\frac{q_{2,w}}{q_2}. \end{aligned} \tag{40}$$

The intermediaries' demand for risk from sector 2, i.e., the second element of their risk exposure $\boldsymbol{\sigma}_A^*$ *can be found in the Appendix A.*

Proof. See Appendix A. ■

The above proposition warrants several remarks. First, the risk exposure of intermediaries depends on the aforementioned variables R_x and R_w, which capture the risk sharing mechanism. One of the main contributions of this paper is to highlight the role of R_x, beyond the wealth distribution channel captured by R_w. The variable R_x captures the investors' risk sharing motive that arises from the reallocation of capital across sectors. Mechanically, these terms can be written as the difference in the sensitivity of the investors' value functions with respect to state variables x and w, respectively, that is

$$R_x = \frac{\partial \log V_A}{\partial x} - \frac{\partial \log V_B}{\partial x}, \quad R_w = \frac{\partial \log V_A}{\partial w} - \frac{\partial \log V_B}{\partial w}. \tag{41}$$

A positive (negative) R implies that a marginal increase in state variable improves the utility of intermediaries (households) relatively more. Notice that R would be zero if there were no motives to share aggregate risks.[17]

Second, the risk exposure depends on the allocation of risks across sectors, captured by Q_x and Q_w. They can mechanically be written as the value-weighted average of the sensitivity of the firms' value functions with respect to state variables x and w, respectively, that is

$$Q_x = z\frac{\partial \log S_1}{\partial x} + (1-z)\frac{\partial \log S_2}{\partial x}, \quad Q_w = z\frac{\partial \log S_1}{\partial w} + (1-z)\frac{\partial \log S_2}{\partial w}. \tag{42}$$

[17] A slight abuse of notation is used here, as the value function V_i is negative when $\gamma_i > 1$. In this case, the absolute value of V_i is used.

Thus the variables Q_x and Q_w capture that sectoral valuations might respond differently to changes in the state variables, and that the composition of each sector's sensitivity matters for risk taking. To this end, the risk exposure of intermediaries depends on the interaction of these terms. The tractable characterization of the risk sharing and risk allocation mechanisms is a key contribution of this paper, which explicitly shows how the efficiency of financial markets and the efficiency of capital allocation are intertwined.

It is worth noting that Proposition 5 nests the case of a single productive sector studied in the literature. When there is only one sector (i.e., one Brownian shock), z equals unity, and Q_x and R_x disappear. In this case, there is no capital reallocation, as investors do not have an option to shift risk exposure across sectors, and the risk sharing is driven solely by the wealth distribution channel.[18]

Corollary 1 *Risk is concentrated in intermediaries (i.e., $\omega_A^{(n),*} > \omega_B^{(n),*}$ for $n \in \{1,2\}$), if and only if $\gamma_A < \gamma_B$.*

A corollary of Proposition 5 is that intermediaries invest more of their wealth in both risky assets, and therefore bear more risks than households. Effectively, intermediaries act not only as risk 'bearers', but also as risk 'allocators' in the economy. As will be shown below, the risk pricing in the financial markets determines the capital allocation across sectors. In the absence of financial frictions, i.e., intermediaries are freely sharing risks with households, efficiency in financial markets and capital allocation is achieved simultaneously.

4.4.2 Asset pricing implications

The following propositions present the expressions for the market price of risk $\boldsymbol{\eta}_t$ and the interest rate r_t, which highlight the key implications of asset pricing.

Proposition 6 *In the absence of financial constraints, the market price of risk $\boldsymbol{\eta}_t$ can be expressed as following*

$$\begin{aligned}\boldsymbol{\eta}_t = {} & \gamma_t \left[z_t \boldsymbol{\sigma}_{R_1,t} + (1-z_t)\boldsymbol{\sigma}_{R_2,t} \right] \\ & - \gamma_t \left[w_t \left(\frac{1}{\gamma_A}\right)\left(\frac{1-\gamma_A}{1-\psi}\right)\boldsymbol{\sigma}_{\xi_A,t} + (1-w_t)\left(\frac{1}{\gamma_B}\right)\left(\frac{1-\gamma_B}{1-\psi}\right)\boldsymbol{\sigma}_{\xi_B,t} \right], \end{aligned} \tag{43}$$

where γ_t denotes aggregate risk aversion, and is given by

$$\gamma_t = \left(\frac{w_t}{\gamma_A} + \frac{1-w_t}{\gamma_B} \right)^{-1}. \tag{44}$$

[18] The risk exposure of intermediaries reduces to $\sigma_A^* = \frac{R_w - \frac{\gamma_B}{w(1-w)}}{(\gamma_B-\gamma_A)\frac{q_w}{q} - \frac{\gamma_B}{1-w} - \frac{\gamma_A}{w} + R_w}\sigma$.

Proof. See Appendix A. ■

The market price of risk (43) is the product of aggregate risk aversion and the difference between the supply of risk and the average hedging demand.[19] Aggregate risk aversion (44) is the reciprocal of the wealth-weighted average of the risk tolerance of the two types of investors. Hence, periods during which intermediaries are relatively less capitalized, so aggregate risk aversion is high, tend to experience a high market price of risk. This is consistent with the heterogeneous agents asset pricing literature (Panageas, 2020). The second term, wealth-weighted average of marginal value of wealth, with the weight also accounting for differences in preferences, captures the idea that if the average hedging demand is high, then a lower market price of risk is required to induce investors to bear the risk.

The vector of risk premia of the two assets is given by

$$\boldsymbol{\lambda}_{R,t} = \boldsymbol{\Sigma}_{R,t}\boldsymbol{\eta}_t. \tag{45}$$

Following the expression for market price of risk, the risk premia consist of two terms as well. The first term is driven by the covariance between each risky asset and total wealth. Intuitively, an asset that comoves a lot with total wealth provides little diversification benefit, is therefore risky, and commands a high risk premium. The second term captures the fact that an asset whose payoffs are high when the average hedging demand is high provides a good hedge to investors and therefore requires a lower risk premium.

Proposition 7 *In the absence of financial constraints, the interest rate r_t can be expressed as following*

$$\psi\rho + \mu_t + (1-\psi)(r_t + \Phi_t) = z_t\frac{A_1 - \iota_1(g_{1,t})}{q_{1,t}} + (1-z_t)\frac{A_2 - \iota_2(g_{2,t})}{q_{2,t}}, \tag{46}$$

where $\mu_t = w_t\mu_{\xi_A,t} + (1-w_t)\mu_{\xi_B,t}$, and $\Phi_t = w_t\Phi_{A,t} + (1-w_t)\Phi_{B,t}$. The expressions for $\mu_{\xi_i,t}$ and $\Phi_{i,t}$ are given in the Appendix A.

Proof. See Appendix A. ■

The left-hand side of (46) represents the (normalized) aggregate demand for goods, while the right-hand side represents the (normalized) aggregate supply of goods. The effects of the interest rate on aggregate demand depends crucially on the EIS: if $\psi > 1$, then an increase in the interest rate decreases aggregate demand (the opposite is true when $\psi < 1$).

[19] The term $z_t\boldsymbol{\sigma}_{R_1,t} + (1-z_t)\boldsymbol{\sigma}_{R_2,t}$ is the value-weighted average of the diffusion of the two risky assets, which is also the diffusion of total wealth.

4.4.3 Efficient sectoral allocation

The following proposition shows that the unconstrained equilibrium allocation of capital is efficient, given the different sectoral productivities and exposures to risks.

Proposition 8 *In the absence of financial constraints, the equilibrium allocation of capital across sectors is efficient in a risk-adjusted sense. That is, the following condition holds for firms in both sectors in equilibrium:*

$$\frac{A_n - \iota_{n,t}}{q_{n,t}} + g_{n,t} + \mu_{q_n,t} + \boldsymbol{\sigma}'_{q_n,t}\boldsymbol{\sigma}_n = r_t + \boldsymbol{\eta}'_t(\boldsymbol{\sigma}_{q_n,t} + \boldsymbol{\sigma}_n), \tag{47}$$

which can also be written as

$$\mu_{R_n,t} = r_t + \boldsymbol{\eta}'_t\boldsymbol{\sigma}_{R_n,t}. \tag{48}$$

Proof. See Appendix A. ■

The discount rate determines the investment (and valuation) of firms in both sectors, and therefore the allocation of capital across sectors. The difference in the A_n (MPK) across sectors is appropriately discounted by the required return of holding claims on the firm, which is given by the risk-free rate plus the risk premium. In other words, the firms equate the marginal benefit of investing to the marginal cost, which is the required return. The efficient allocation ensures that capital is allocated to the most productive uses, taking into account the risks associated with each sector.

In discrete time, (48) has a form similar to the equation $\mathbb{E}_t[R_{n,t+1}] = r_t + f_t\beta_{n,t}$ that holds for both sectors, where $\mathbb{E}_t[R_{n,t+1}]$ denotes the expected return of asset n over the next period, r_t denotes the risk-free rate, f_t denotes the pricing factor in the SDF, and $\beta_{n,t}$ denotes the asset's beta with respect to the factor. This result is thus consistent with David et al. (2022), who show in a partial equilibrium setting that this equation holds across firms.

The novelty of Proposition 8 is that the SDF is endogenously determined in general equilibrium, and the market price of risk (Proposition 6) is time-varying and tied to the distribution of wealth and capital. In addition, due to the 'AK' technology, the MPK across sectors is permanently different, which allows to center around the relation between the pricing of risk and the allocation of capital. Thus the efficient allocation of capital in the model is achieved through efficient risk sharing among investors. It implies that the efficiency of financial markets is crucial for the efficiency of allocation of capital. Moreover, as will be shown later, the binding financial constraint distorts investors' risk sharing, which leads to an inefficient allocation of capital across sectors.

4.5 Alternative economies

Although the model is analytically tractable, one may wonder how each segment of the model contributes to the equilibrium and if the full setup is necessary to capture the key mechanisms. In this section, I consider alternative economies that can provide intuition in this regard. I rely on the method of asymptotic expansion, which yields semi-closed form solutions for the equilibrium objects of these economies. This allows to shed light on the roles of each segment of the model before proceeding to numerical solutions.

The first (frictionless) economy features one representative sector, and no heterogeneity in risk aversion among investors. The following lemma shows that up to scale, there are no fluctuations in the equilibrium objects, such as investment, consumption, and asset prices.

Lemma 1 *Consider an economy where $\gamma_A = \gamma_B = \gamma$ and two sectors are identical. Under these assumptions, the structure of aggregate shocks simplifies to only one Brownian shock, with volatility σ, i.e.,*

$$\frac{dK_t}{K_t} = g_t\,dt + \sigma\,dB_t. \tag{49}$$

The following equilibrium objects can be derived.

i. The consumption-wealth ratio and risk exposure are given by

$$c_A^* = c_B^* = \rho - (1-\psi^{-1})\left(g^* - \frac{\gamma\sigma^2}{2}\right), \qquad \sigma_A^* = \sigma_B^* = \sigma. \tag{50}$$

ii. The investment rate and price of capital satisfy

$$\frac{A-\iota^*}{q^*} = \rho - \left(1-\psi^{-1}\right)\left(g^* - \frac{\gamma\sigma^2}{2}\right), \qquad g'(\iota^*) = \frac{1}{q^*}. \tag{51}$$

iii. The market price of risk and interest rate are given by

$$\eta^* = \gamma\sigma, \qquad r^* = \rho + \psi^{-1}g^* - (1+\psi^{-1})\frac{\gamma\sigma^2}{2}. \tag{52}$$

Proof. See Appendix A. ■

The above lemma shows that in the absence of heterogeneity in risk aversion, the one-sector economy is essentially deterministic. Output, consumption, and investment all move in proportion to the capital stock, which is subject to aggregate shocks. Since investment opportunities are constant in this case, investors maintain a constant portfolio weight of one over time. The market price of risk and the interest rate are both constant as well.

Consider next the second (frictionless) economy featuring two productive sectors and no heterogeneity in risk aversion among investors, à la Eberly and Wang (2025). Even though a closed-form solution is not available in this case, following Silva (2020), the behavior of the economy can be characterized by considering an asymptotic expansion of the equilibrium around the economy in Lemma 1.[20]

Lemma 2 *Consider a parametric family of economies indexed by ε, where two sectors are not identical. Assume $\gamma_A = \gamma_B = \gamma$. In addition, assume*

$$\frac{dK_{1,t}}{K_{1,t}} = g_{1,t}dt + \sigma\varepsilon dB_{1,t} + \sigma(1-\varepsilon)dB_{2,t}, \tag{53}$$

$$\frac{dK_{2,t}}{K_{2,t}} = g_{2,t}dt + \sigma dB_{2,t}, \tag{54}$$

and productivity A, depreciation rate δ, and adjustment cost (ϕ,θ) are assumed to be the same across two sectors for expositional simplicity.

The equilibrium objects, as functions of x, can be solved up to the first order in ε.

i. The prices of capital in sector 1 is given by

$$q_1(x;\varepsilon) = q^* + \frac{q^*\gamma\sigma^2(-c^*x\theta + c^*\theta\psi - q^*x + q^*)}{c^*(c^*\theta\psi + q^*)}\varepsilon + O(\varepsilon^2). \tag{55}$$

ii. The market prices of risks are given by

$$\eta_1(x;\varepsilon) = \gamma\sigma z\varepsilon + O(\varepsilon^2), \qquad \eta_2(x;\varepsilon) = \gamma(\sigma - \sigma z\varepsilon) + O(\varepsilon^2). \tag{56}$$

where z is the weight of sector 1 in the market portfolio, i.e., $z = \left(1 + \frac{q_2}{q_1}\frac{1-x}{x}\right)^{-1}$.

All other equilibrium objects can be found in Appendix A.

Proof. See Appendix A. ■

The above lemma highlights that the introduction of a second sector leads to the equilibrium objects becoming functions of the sectoral composition of the economy, x. The sectoral allocation is always efficient in risk-adjusted terms, as the representative investor frictionlessly reallocates capital in response to shocks.[21] There is, however, no role of frictions in the financial markets, such as the one considered in the full model (i.e., margin constraints), in generating capital misallocation, because the representative investor absorbs all the risks and does not have the capacity to share risks.

[20] Note that sectoral investment rate is monotonically increasing in its sectoral Tobin's Q, hence they convey essentially the same information for the given adjustment cost function.

[21] Appendix D presents the global solution of this economy.

Consider next the third (frictionless) economy featuring one sector and heterogeneous investors. The behavior of the economy can again be characterized by considering an asymptotic expansion of the equilibrium around the economy in Lemma 1, as no closed-form solution is available in this case either.

Lemma 3 *Consider a parametric family of economies indexed by ϵ, where two sectors are identical. The structure of aggregate shocks boils down to only one Brownian shock, with volatility σ, as in Lemma 1. Let $\gamma_A = \gamma(1-\Delta\gamma)$, $\gamma_B = \gamma(1+\Delta\gamma)$, $\Delta\gamma \equiv \delta_\gamma \epsilon$, and $\delta_\gamma > 0$.*

The equilibrium objects, as functions of w, can be solved up to the first order in ϵ.

i. Risk exposures

$$\sigma_A(w;\epsilon) = \sigma + 2(1-w)\sigma\Delta\gamma + O(\epsilon^2), \tag{57}$$

$$\sigma_B(w;\epsilon) = \sigma - 2w\sigma\Delta\gamma + O(\epsilon^2). \tag{58}$$

ii. Market price of risk

$$\eta(w;\epsilon) = \gamma\sigma - \gamma\sigma(2w-1)\Delta\gamma + O(\epsilon^2). \tag{59}$$

iii. The growth rate of capital

$$g(w;\epsilon) = g^* + \frac{(\psi-1)q^*}{q^* + \phi_1\psi c^*}\frac{\gamma\sigma^2}{2}(2w-1)\Delta\gamma + O(\epsilon^2). \tag{60}$$

All other equilibrium objects can be found in Appendix A.

Proof. See Appendix A. ■

The above lemma, in comparison to Lemma 1, showcases the role of heterogeneity in risk aversion in driving fluctuations in the equilibrium. As in Proposition 5, intermediaries (type A) are more exposed to risk than households (type B), i.e., $\sigma_A(w) > \sigma_B(w)$ for $w \in (0,1)$. Financial intermediaries operate leveraged, and after a negative shock, their wealth share declines, leading to an increase in the market price of risk and, provided $\psi > 1$, a fall in investment and growth.[22] This mechanism, often referred to as the *balance-sheet channel* in the macro-finance literature (Di Tella, 2017), is a result of risk sharing among investors, which makes the intermediaries' wealth share, w_t, a state variable that drives time-variation in risk premia.[23] There is no capital reallocation or misallocation in this economy, as the investors can only trade one risky asset, and the sectoral composition of the economy is constant.

[22] The condition $\psi > 1$ guarantees that the price of the risky asset falls after a negative shock, as the increase in the risk premium dominates the decline in interest rates.

[23] Appendix D presents the global solution of this economy.

The fourth economy introduces the financial constraints in the form of (13) (applied to one risky asset) on top of the setup in Lemma 3.

Lemma 4 *Consider a parametric family of economies indexed by ϵ, where two sectors are identical. The structure of aggregate shocks boils down to only one Brownian shock, with volatility σ, as in Lemma 1 and 3. Let $\gamma_A = \gamma(1-\Delta\gamma)$, $\gamma_B = \gamma(1+\Delta\gamma)$, $\Delta\gamma \equiv \delta_\gamma \epsilon$, and $\delta_\gamma > 0$. Moreover, investors are subject to the financial constraints. Specifically, $\sigma_i \leq \bar{\zeta}$.*

In equilibrium, when the constraint does not bind, σ_A can be expressed as (57) from Lemma 3. Other equilibrium objects behave like the ones in Lemma 3.

When the constraint binds, σ_A equals to $\bar{\zeta}$, and the equilibrium objects can be solved up to the first order in ϵ as follows.

i. Risk exposure for investors of type B

$$\sigma_B(w;\epsilon) = \frac{\sigma - w\bar{\zeta}}{1-w} + O(\epsilon^2). \tag{61}$$

ii. Market price of risk

$$\eta(w;\epsilon) = \gamma(1+\Delta\gamma)\frac{\sigma - w\bar{\zeta}}{1-w} + O(\epsilon^2). \tag{62}$$

iii. The growth rate of capital

$$g(w;\epsilon) = g^* + q^*(\psi-1)\frac{2w^2\bar{\zeta}^2 - w\bar{\zeta}^2 - 2w\bar{\zeta}\sigma + \sigma^2}{(g^*\theta\psi - q^* - r^*\theta\psi)(1-w) + w\gamma\bar{\zeta}\theta\sigma\psi - \gamma\theta\psi\sigma^2}\frac{\gamma}{2}\Delta\gamma + O(\epsilon^2). \tag{63}$$

All other equilibrium objects can be found in Appendix A.

Proof. See Appendix A. ■

The first observation is that in equilibrium, financial constraints will bind for intermediaries, when their wealth share w is sufficiently low. This is because the unconstrained risk exposure of intermediaries (57) is higher than that of households, and it increases as w decreases. Second, when the constraint binds, more risk-averse households become marginal investors. The risk exposure of households (61) increases relative to the unconstrained economy, as intermediaries are forced to sell assets. Importantly, the market price of risk increases relative to the unconstrained economy from Lemma 3, due to limited risk sharing among investors. The inefficient financial market in this economy leads to intertemporally distorted investment, and no capital misallocation arises in this economy.

The four lemmas in this section characterize simplified economies that separate specific segments from the full model. More specifically, Lemmas 2, 3 and 4 highlight sectoral reallocation, investor heterogeneity, and financial constraints in shaping equilibrium outcomes, respectively. Therefore, to capture the impact of financial markets on capital allocation efficiency, it is necessary to consider a model that incorporates all these features simultaneously.

5 Quantitative analysis

In this section, I explore the quantitative properties of the model. To that end, I solve the model and the corresponding PDEs numerically. First, I describe the global solution technique based on neural networks and calibration. I then discuss the results of the unconstrained model. I conclude this section by studying the causes and consequences of capital misallocation.

5.1 Numerical solution

The model is analytically tractable: the equilibrium dynamics can be fully characterized by a system of PDEs. The computation of equilibrium requires solving the HJB equations of firms and investors simultaneously. The value functions $q_n(x,w)$ for $n \in \{1,2\}$ and $\xi_i(x,w)$ for $i \in \{A,B\}$ can be found by solving a system of second-order PDEs in (x,w). To do so, all equilibrium objects need to be expressed in terms of these value functions and their derivatives. Unfortunately, the system of nonlinear differential equations does not admit a closed-form solution, so I rely on numerical techniques. This is challenging in the presence of the model's two endogenous state variables and financial frictions that occasionally bind.

I extend the global projection methods (Judd, 1992, 2023) using neural networks to approximate the value functions. Unlike a log-linearized representation around the steady state, this global method provides a full characterization of the whole dynamic system. This is particularly important as the model features regions in which financial constraints occasionally bind and the dynamics are highly non-linear. This is crucial for understanding the joint dynamics of (imperfect) risk sharing and capital misallocation. In practice, I use feedforward neural networks and experiment with various architectures, such as the number of hidden layers and the choice of activation functions. For the purposes of this paper, the neural network can be seen as a function approximator, like Chebyshev polynomials. In Appendix B, I discuss the algorithm used to solve the system of PDEs.

5.2 Calibration

Table 1 reports the calibration of the model. Note that the values correspond to annual values as the model is cast in continuous time.

Table 1: Calibration

Parameters	Symbol	Values
Rate of time preference	$\hat{\rho}$	0.001
Mortality rate	κ	0.02
Elasticity of intertemporal substitution	ψ	1.5
RRA of type A investors	γ_A	3
RRA of type B investors	γ_B	18
Population share of type A investors	$\bar{u}$	0.1
Productivity of capital in sector 1	A_1	0.30
Productivity of capital in sector 2	A_2	0.25
Volatility of capital in sector 1	σ_1	0.036
Volatility of capital in sector 2	σ_2	0.03
Depreciation rate of capital in sector 1	δ_1	0.016
Depreciation rate of capital in sector 2	δ_2	0.014
Adjustment cost of capital in sector 1	θ_1	14
Adjustment cost of capital in sector 2	θ_2	12
Tightness of the margin constraint	ζ	0.12

This table reports the parameter values used in calibrating the model. The values correspond to annual values as the model is cast in continuous time.

The rate of time preference $\hat{\rho}$ and the mortality rate κ are set to 0.001 and 0.02, respectively, implying an effective discount rate of $\hat{\rho}+\kappa = 0.021$, and a real interest rate around 2% in annual terms.[24] I set EIS to $\psi = 1.5$ for all investors, a common value found in the asset pricing and macro-finance literature, to focus on the heterogeneity in risk bearing capacity. The RRA of intermediaries (type A) is $\gamma_A = 3$, while households (type B) have $\gamma_B = 18$, preserving higher risk tolerance among intermediaries and investors' preferences for early resolution of uncertainty ($\gamma_i > 1/psi$). Other values within this range generate similar results. To stabilize intermediaries' share of wealth at moderate levels, the population share of type A investors is $\bar{u} = 0.1$. The tightness of the margin constraint is set as $\zeta = 0.12$, which indicate that the bound on the total risk exposure is 0.12, consistent with Kargar (2021).

On the production side, sectoral productivities are set as $A_1 = 0.25$ and $A_2 = 0.20$ to match a capital-output ratio of around 4 to 5. The sectoral volatilities are set as $\sigma_1 = 0.036$ and $\sigma_2 =$

[24]For instance, the effective discount rate is close to that in Gârleanu and Panageas (2015).

0.028, consistent with Assumption 1 that sector 1 is riskier and more productive, representing the growth sector. The values of the volatilities are broadly consistent with the volatility of output. To focus on the risk-adjusted capital reallocation, I set the common depreciation rate δ to 0.02, and the adjustment cost parameter θ to 12.5. The investment-output ratio is around 25%, and the growth rate of the economy is around 2% in annual terms. These parameter choices are broadly consistent with the macro-finance literature (Brunnermeier and Sannikov, 2014; Silva, 2020; Hansen et al., 2024).

5.3 Unconstrained benchmark

I now discuss the equilibrium where financial constraints are effectively not present, that is I set ζ to infinity. Therefore, this section focuses on the efficient capital reallocation in the frictionless economy. It highlights risk-adjusted capital allocation in an economy where the productivity of each sector is permanently different ('AK' type technology). Due to the presence of different risk exposures, however, capital can be efficiently allocated across sectors, as risk-averse investors require compensation for higher risk in exchange for higher productivity.

Moreover, the following results from GE framework emphasize the origins of the risk compensation and interaction between the two state variables, x and w, which shapes the dynamics of sectoral investment, asset prices, and investors' risk-taking. The model generates rich dynamics around the effects of diversification and *balance-sheet* channel. The results also show that the flight-to-safety mechanism is at play, in a way consistent with the reallocation from high- to low-risk assets when the financial health is weak.

5.3.1 Sectoral dynamics

Without frictions, investors can reallocate wealth across sectors in a risk-adjusted manner, and the financial market determines the cost of capital for each sector, which is crucial for the efficient capital allocation across sectors given permanent dispersion in marginal product of capital. To see this, note that (47) implies that the firms in two sectors maximize value by equating investment return to the cost of capital, which is the sum of the risk-free rate and the sector-specific risk premium. The optimal investment rates and prices of capital are determined by the cost of capital.[25]

Figure 1 shows the sectoral investment rates and prices of capital (Tobin's Q) as functions of the state variables.[26] First observation is that the sectoral dynamics are non-linear in both

[25]This is consistent with the intuition from David et al. (2022), who derive a risk-adjusted Euler equation in a frictionless neoclassical model in similar fashion. I focus on the GE implications of such risk-adjusted capital allocation.

[26]Note that the sectoral growth rates preserve similar patterns due to the assumptions of technology.

state variables, x and w. The investment of the growth sector (sector 1) tends to be high when x is low, due to the force of preserving balance among sectors conditional on w: when growth sector is in low share (low x), its price of capital q_1 is high to encourage investment, as can be seen from bottom-left panel of Figure 1. Similarly, the price of capital in the value sector (sector 2) tends to be high when x is high. Investors try to sustain the stationary distribution of x for diversification benefits, which is essential for the survival of two technologies. The discussions that follow will focus on Tobin's Q as the patterns of investment rates are similar to those of prices of capital.

The balance-sheet channel is at play: price of capital of the dominant sector increase in w, the wealth share of intermediaries. When intermediaries' share of wealth falls, more risk-averse households are dominant in the wealth distribution, increasing aggregate risk aversion implies that the price of risk rises (as will be shown later). This depresses asset prices and reduces investment and growth. A flight-to-safety mechanism is at play too: when the economy is diversified, that is when x is at its steady state, q_1 increases in w while q_2 decreases in w. When w is low, increasing aggregate risk aversion results in the reallocation from high- to low-risk assets and drives up the prices in low-risk sector (sector 2).[27]

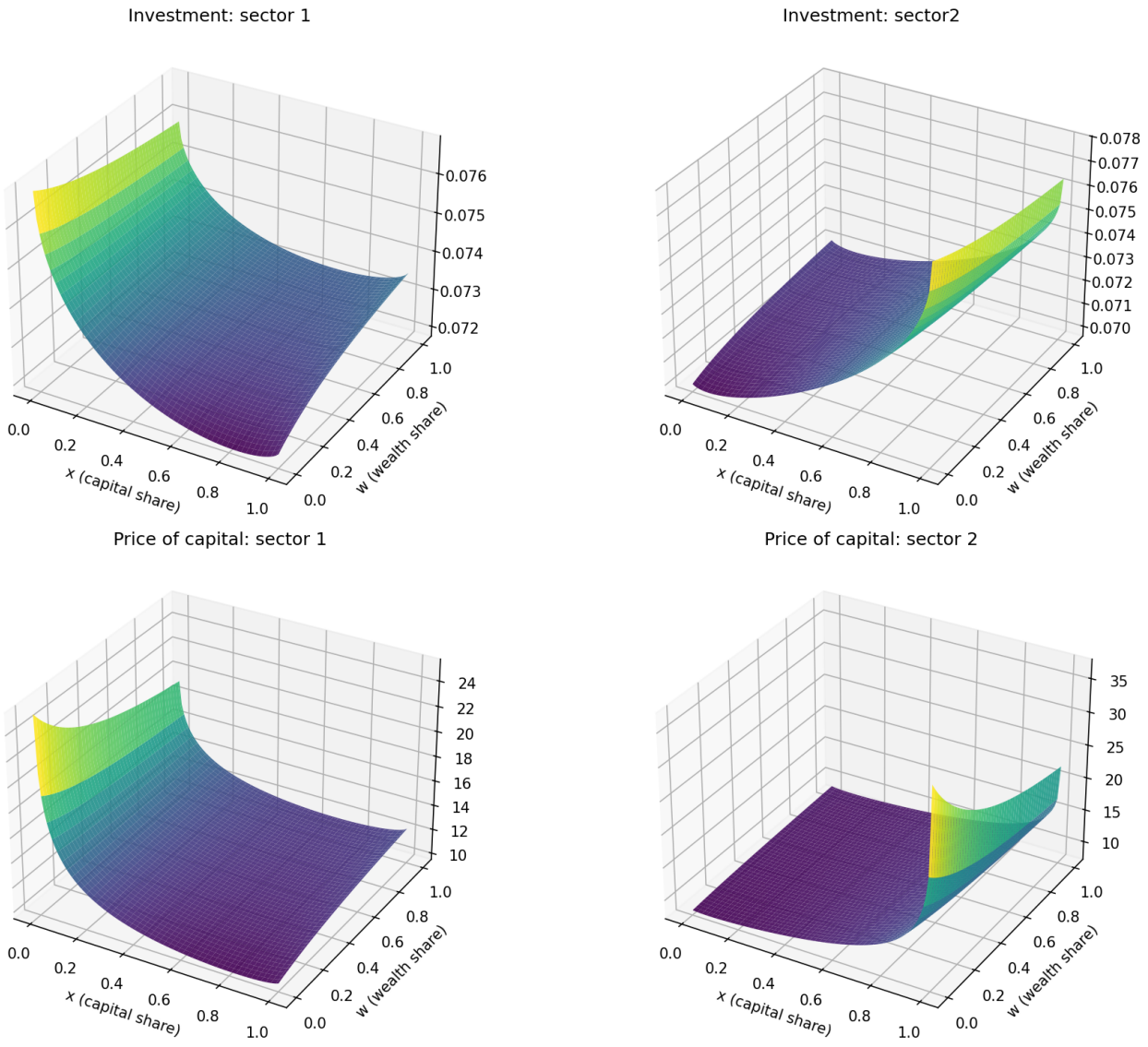
In addition, the motive for diversification is stronger when w is lower, as risk-averse households are dominant in the wealth distribution. This motive, for example, flips the relationship between price of capital in growth sector and w when growth sector is diminishing: q_1 decreases in w when x is low. The results highlight the contributions of my model, which characterizes full dynamics jointly and the interaction between various economic channels.

5.3.2 Asset pricing

Figure 2 presents the market prices of risk, η_1 and η_2 as functions of state variables under the unconstrained calibration. Both market prices of risk decrease in w. In periods when the balance sheet of intermediaries is relatively weak (low w), the return per unit of risk is high, as a result of countercyclical aggregate risk aversion. After a negative shock, intermediaries lose wealth and aggregate risk aversion increases as households become relatively more important in the market. An increase in aggregate risk aversion leads to an increase in the market price of risk. This mechanism, at the core of macro-finance literature, is often referred to as the *balance-sheet* channel. The increase in the market price of risk drives up the risk premia and leads to a decline in Tobin's Q and investment (Figure 1), which further weakens the balance sheet of intermediaries, a financial amplification documented in the literature (Brunnermeier

[27] In the model, the flight-to-safety mechanism comes from the endogenous risk-sharing among heterogeneous investors with recursive preferences, instead of exogenously time-varying risk aversion (Kozak, 2022) or habit utility (Duarte, 2018) in a representative investor framework.

Figure 1: Sectoral dynamics



This figure shows the sectoral investment rates and prices of capital (Tobin's Q) as functions of state variables x (capital share of sector 1) and w (wealth share of intermediaries) under the unconstrained calibration. The top panels present the investment rates of the two sectors, and the bottom panels present the prices of capital of the two sectors.

and Sannikov, 2014).

In the model presented in Section 3, I assume that capital of each sector is exposed only to one shock (Assumption 1). Nonetheless, sectoral prices and returns are exposed to both shocks, with different degrees of sensitivity. Thus, the market prices of risk further become drivers of the reallocation of capital across sectors. The relative dominate price of risk across shocks depends on the share of the capital x. As sectoral composition changes, the price of risk for each shock take turns to become dominant, which renders the *balance-sheet* channel stronger when the economy is less-diversified, that is when x is close to zero or one.

Moreover, due to investors' incentive for diversification, η_1 and η_2 tend to move in opposite directions with respect to x. For example, when w is at its steady state, η_1 increases with x while η_2 decreases. When x is high, growth sector (sector 1) is in high share, and investors want high compensation for bearing risks from growth sector, which drives up η_1. At the same time, investors want to reallocate to value sector (sector 2) for diversification, which drives down η_2. The opposite happens when x is low.

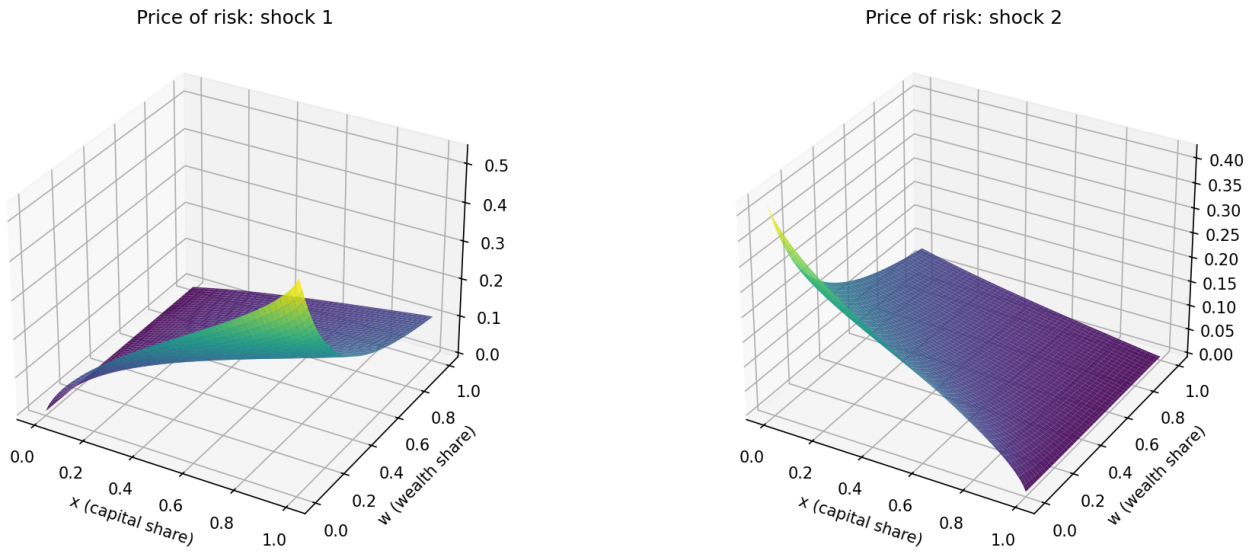
Consistent with the evidence from the real side (e.g., Tobin's Q), the motive for diversification is stronger when w is lower, as aggregate risk aversion increases. The stronger diversification motive creates highly nonlinear relation between market prices of risk and x when w is low, as the sensitivity of η_1 and η_2 to x is higher. Risk pricing is also consistent with the flight-to-safety mechanism on the real side (Figure 1), which puts a lifting force to η_1 and a pushing one to η_2 as w declines. The above mechanisms jointly showcase the risk channel of capital allocation efficiency, as the market prices of risk are the key components of the cost of capital and thus determine the sectoral capital allocation (Proposition 8).

Figure 3 shows the risk-free rate, the other component in (47), as functions of state variables under the unconstrained calibration. The risk-free rate is non-monotonic in both x and w. The risk-free rate tends to increase in w, consistent with the countercyclical aggregate risk aversion and the precautionary saving motive. It is inversely U-shaped in x. A high interest rate is necessary when diversification is achieved and precautionary saving motive is weak.

5.3.3 Portfolio choice, risk sharing, and risk allocation

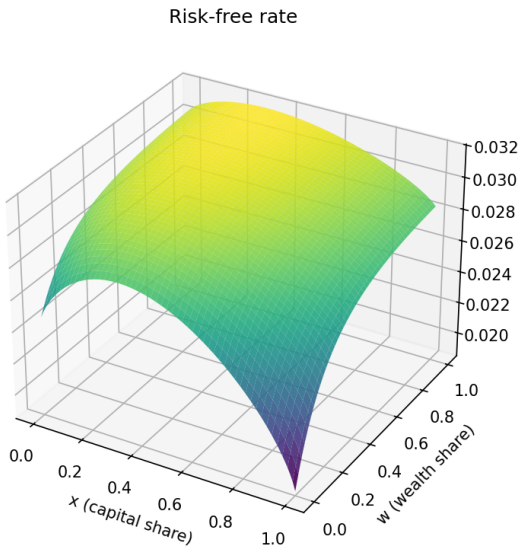
Investors' risk-taking considerations are directly linked to the pricing mechanisms as in Proposition 5. To further illustrate the risk channel of capital allocation efficiency, Figure 4 presents the portfolio choices of both types of investors. When w is close to zero, intermediaries' holding of growth sector is around 3 to 5 times net worth when growth sector is in high share. At the other extreme, when w is close to one, intermediaries take almost no leverage. At moderate levels of w between 0.2 and 0.4 (around its steady state), intermediaries' holding of growth sector decreases to around 2 to 3 times net worth. The holding of value sector is high when

Figure 2: Prices of risk



This figure shows the prices of risk as functions of state variables x (capital share of sector 1) and w (wealth share of intermediaries) under the unconstrained calibration. The left panel presents the price of risk for shock B_1, and the right panel presents the price of risk for shock B_2.

Figure 3: Risk-free rate



This figure shows the risk-free rate as functions of state variables x (capital share of sector 1) and w (wealth share of intermediaries) under the unconstrained calibration.

growth sector is in low share.

The countercyclical (with respect to w) leverage of intermediaries (top panels) imply that intermediaries are more exposed to the aggregate risk than households (bottom panels). This result holds when there are more than one risky asset and one aggregate shock. Given the countercyclical risk premium, returns for a leveraged investor increase in bad times (lower w), so intermediaries tend to reduce their risk exposure to shift resources to states with high returns when investment opportunities are more attractive (lower w).

Proposition 5 shows that the portfolio choices depend on the mechanism of risk sharing among investors, which is captured by R_x and R_w and varies across the state space. The sign of R_x depends on x being close to zero or one, and R_w is positive across the state space, as shown in Figure 5. The intuition, guided by (41), is that the changes in x improves intermediaries' utility relatively more when x is low. Therefore, when growth sector is in low share, an increase in x implies that intermediaries have to bear more risk from growth sector, and since they dislike risk relatively less, their utility increases relatively more than that of households. The same logic applies in the opposite direction when x is high. The changes in w always improve intermediaries' utility relatively more: an increase in w implies that they have to bear more aggregate risk overall, and since they dislike risk relatively less, their utility increases relatively more than that of households. If there were no gains from sharing risk, these two terms would be zero.

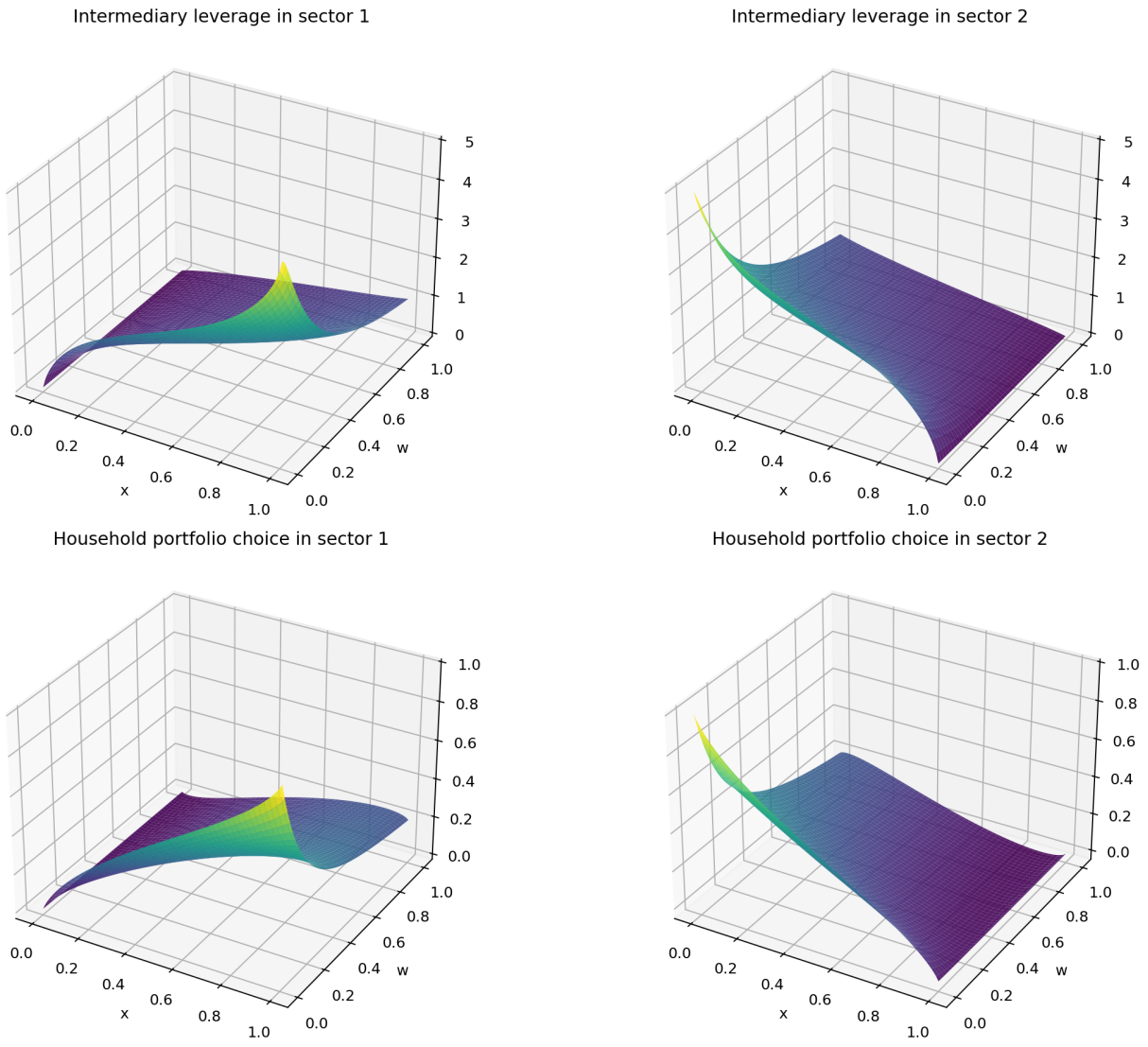
The portfolio choices also depend on risk allocation terms captured by Q_x and Q_w, as shown in Figure 6. The left panel shows that Q_x is positive when x is close to one and negative when x is close to zero, meaning that the average of valuation sensitivity to changes in x is higher when the economy is less-diversified and such sensitivity changes sign depending on which sector dominates. The right panel shows that Q_w is positive across the state space and becomes larger when w decreases, meaning that the average of valuation sensitivity to changes in w is high when w is low.

5.3.4 Volatility of returns

The second moment of returns, i.e., the total volatility of the two risky assets are shown in Figure 7. I calculate the total volatility of returns, which is the square root of the sum of squared diffusion loadings of returns. It is noteworthy that asset returns load on both shocks. This is so despite the fact that capital of each sector only loads on its Brownian shock (see Assumption 1). Economically, this highlights the strong contagion taking place through asset markets: a shock on the capital of a given sector has a large impact on the returns of the other sector.

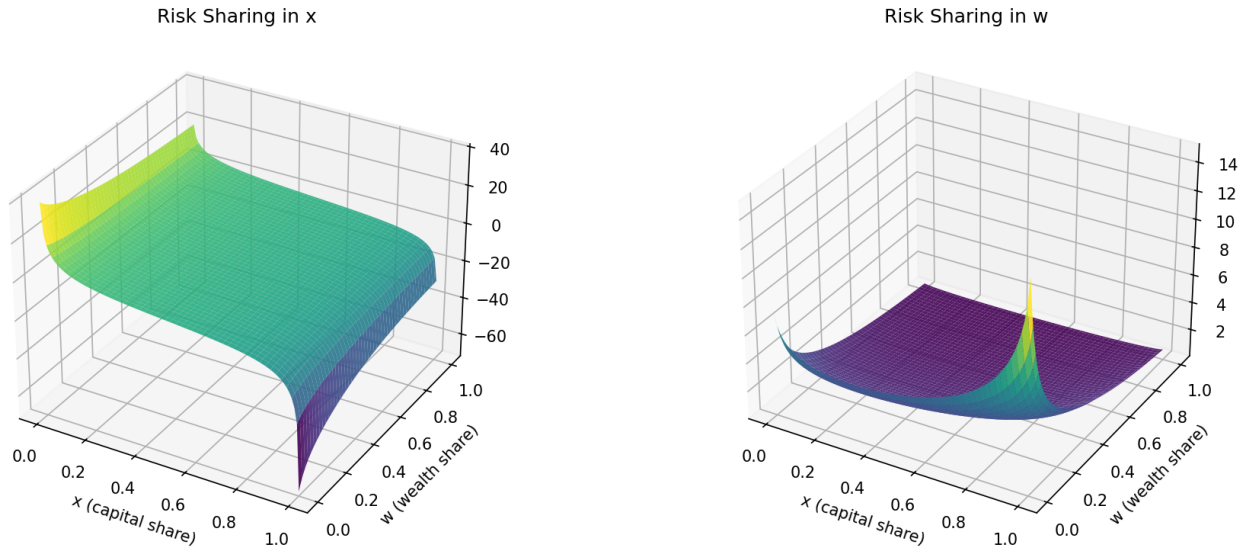
The diversification effect is particularly strong: the amplification of balance sheet channel is at play mostly when the economy is dominated by one sector, that is when x is close to zero

Figure 4: Portfolio choices



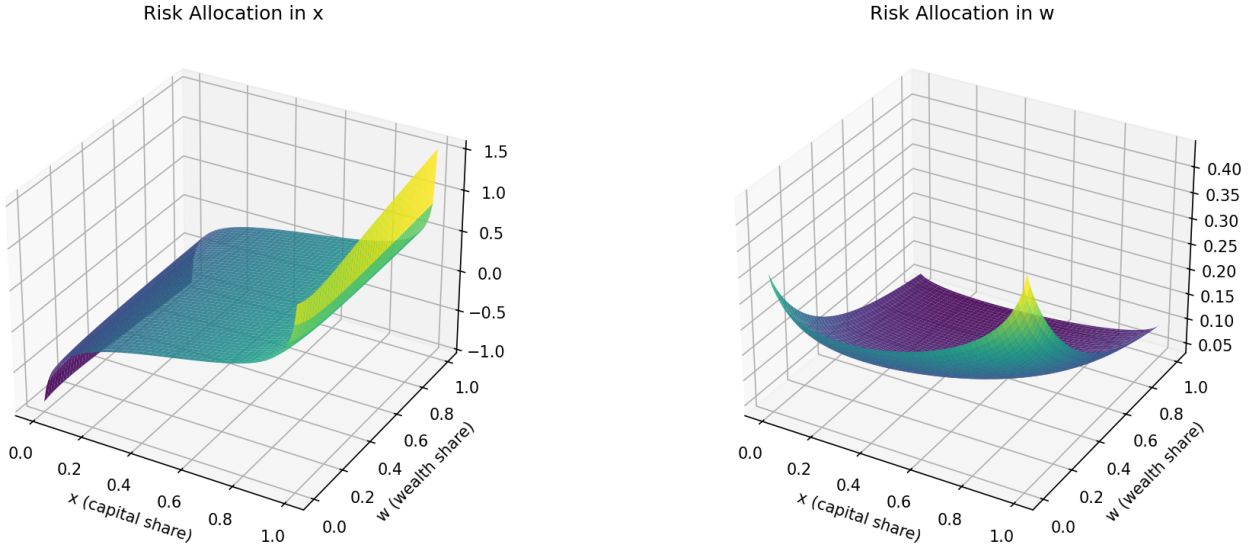
This figure shows the portfolio choices of intermediaries and households as functions of state variables x (capital share of sector 1) and w (wealth share of intermediaries) under the unconstrained calibration. The top-left panel presents the intermediaries' leverage in sector 1, the top-right panel presents the intermediaries' leverage in sector 2, the bottom-left panel presents the households' leverage in sector 1, and the bottom-right panel presents the households' leverage in sector 2.

Figure 5: Risk sharing R_x and R_w



This figure shows the risk-sharing terms from 5 as functions of state variables x (capital share of sector 1) and w (wealth share of intermediaries) under the unconstrained calibration. The left panel presents the risk-sharing term R_x, and the right panel presents the risk-sharing term R_w.

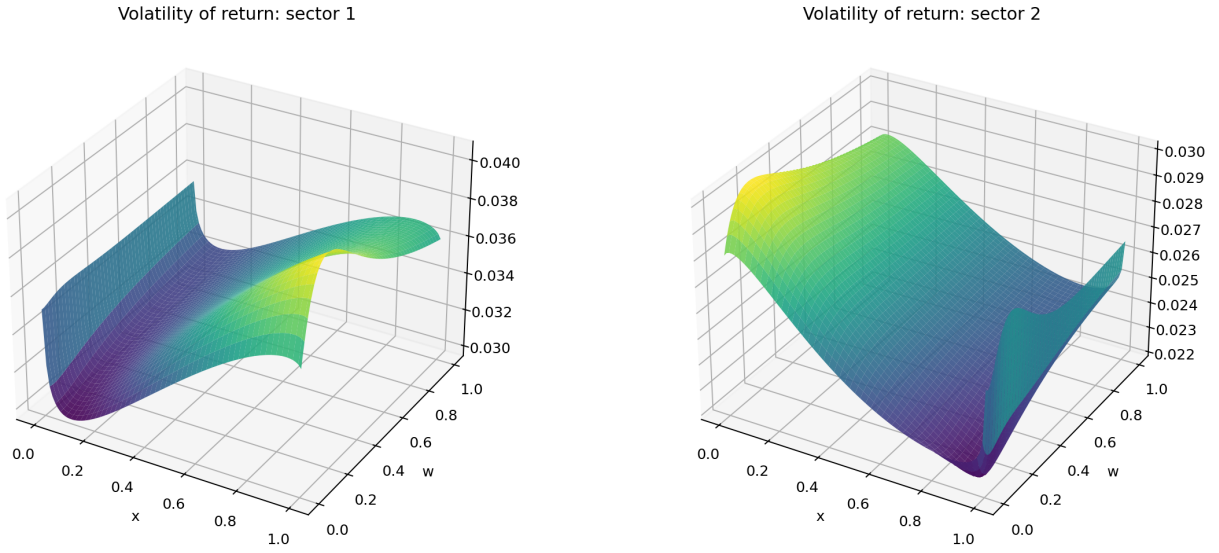
Figure 6: Risk allocation Q_x and Q_w



This figure shows the risk allocation terms from Proposition 5 as functions of state variables x (capital share of sector 1) and w (wealth share of intermediaries) under the unconstrained calibration. The left panel presents the risk allocation term Q_x, and the right panel presents the risk allocation term Q_w.

or one. In other states, the volatility of returns is relatively lower as investors can diversify across sectors. The joint forces of diversification and amplification lead to a non-monotonic behavior of the volatility of returns. Interestingly, return of either sector becomes more volatile when it is diminishing. This is because the price of capital in the diminishing sector becomes extremely sensitive to changes in x.

Figure 7: Volatility of returns



This figure shows the volatility of returns as functions of state variables x (capital share of sector 1) and w (wealth share of intermediaries) under the unconstrained calibration. The left panel presents the volatility of returns for the growth sector (sector 1), and the right panel presents the volatility of returns for the value sector (sector 2).

5.4 Capital misallocation

In this section, I focus on the capital misallocation and its dynamics. Then I discuss the two-way feedback loop and the implications of misallocation for the economy.

To study the impact of financial frictions on capital allocation, I solve the model where investors face margin requirements in the form given in (13). As noted earlier, margin constraints are occasionally binding and state dependent. Although all investors face the constraints in their optimization problems, in my calibration I consider only those equilibria in which constraints occasionally bind only for the more risk-tolerant intermediaries, not for the

more risk-averse households.[28] For simplicity, in this section I focus on the regions of the state space where the constraint on risk exposures to shock B_1 tends to be binding in equilibrium (growth sector is in high share). I measure misallocation relative to the unconstrained benchmark at the same state (x, w). Let $q_n^u(x, w)$ denote the sector-n price of capital in the unconstrained economy and $q_n^c(x, w)$ the corresponding object when financial constraints are present. The sectoral allocation wedge $\Delta q_n(x, w)$ is defined as the difference between the prices of capital in the two scenarios. A negative value of Δq_n indicates under-allocation to sector n relative to the risk-adjusted efficient benchmark, while a positive value indicates over-allocation.[29]

As shown in the top-right panel of Figure 8, intermediaries invest significantly less of their wealth in the growth sector (solid blue line), when w becomes lower and the constraint binds. Consistent with Kargar (2021) and others, the binding constraint forces intermediaries to sell risky assets when the financial health becomes weaker. In this case, sector 1 is risky because it is dominant in the production side (x is close to one).

More importantly, unlike the model with a single risky asset, the presence of sector 2 (value sector) allows for a more complex reallocation of wealth, as shown in top-right panel of Figure 9. Financial intermediaries also inefficiently reallocate their portfolios towards the value sector (solid blue line), when they are forced to sell portfolios in sector 1 as w becomes lower. Thus the 'allocator' role of intermediaries is disrupted by the binding constraint when the economy is going into downturns. As will be shown later, the inefficiency in the financial market directly affects the allocation efficiency in the real side of the economy.

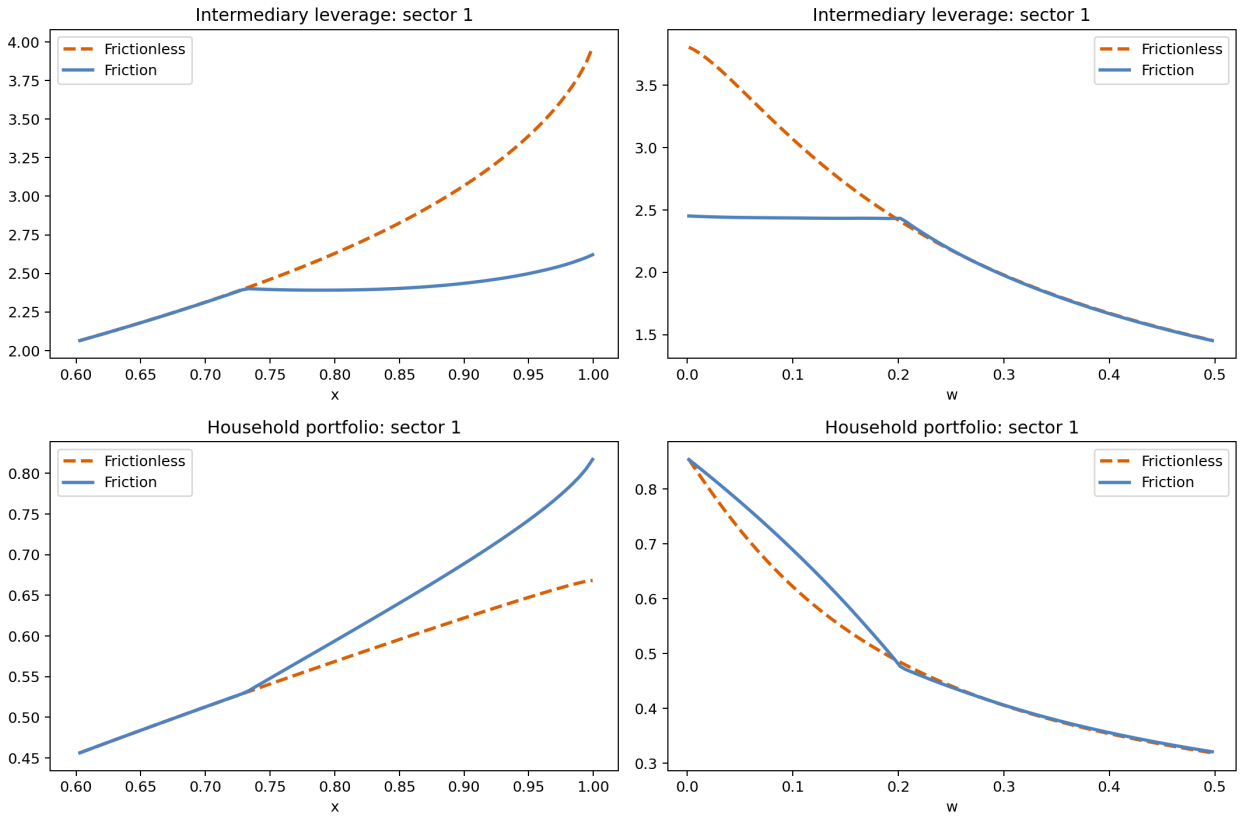
Due to market clearing, the households have to take on more risk from growth sector and reallocate their portfolios away from the value sector (solid blue lines in the bottom-right panels of Figure 8 and 9). The forced and inefficient reallocation of investors' wealth across assets, as a result, raises the price of risk for shock B_1 to entice more risk-averse households to take on the risk, as shown in the bottom-left panel of Figure 10 (solid blue line).[30] The market price of risk η_1 is inefficiently higher under the constrained economy compared to the unconstrained benchmark, which acts as a wedge between efficient and inefficient pricing and distorts the risk-return tradeoff (47).

[28]For example, it is conceivable that households may also be bound by margin constraints when the equilibrium properties of the risky assets are so different that these investors want to short one asset and invest more in the other one. However, such equilibria are not the focus of this paper. First, households are much less likely to be subject to regulation requirements in the form of margin constraints. Second, the constraint can bind households if one of their portfolio weights is highly negative. For this to happen, either a myopic component or a hedging component should be negative, which is unlikely in numerical terms.

[29]Since the firm's optimal investment rate is increasing in Tobin's Q, the sign of Δq_n also describes the direction of the investment and growth distortions.

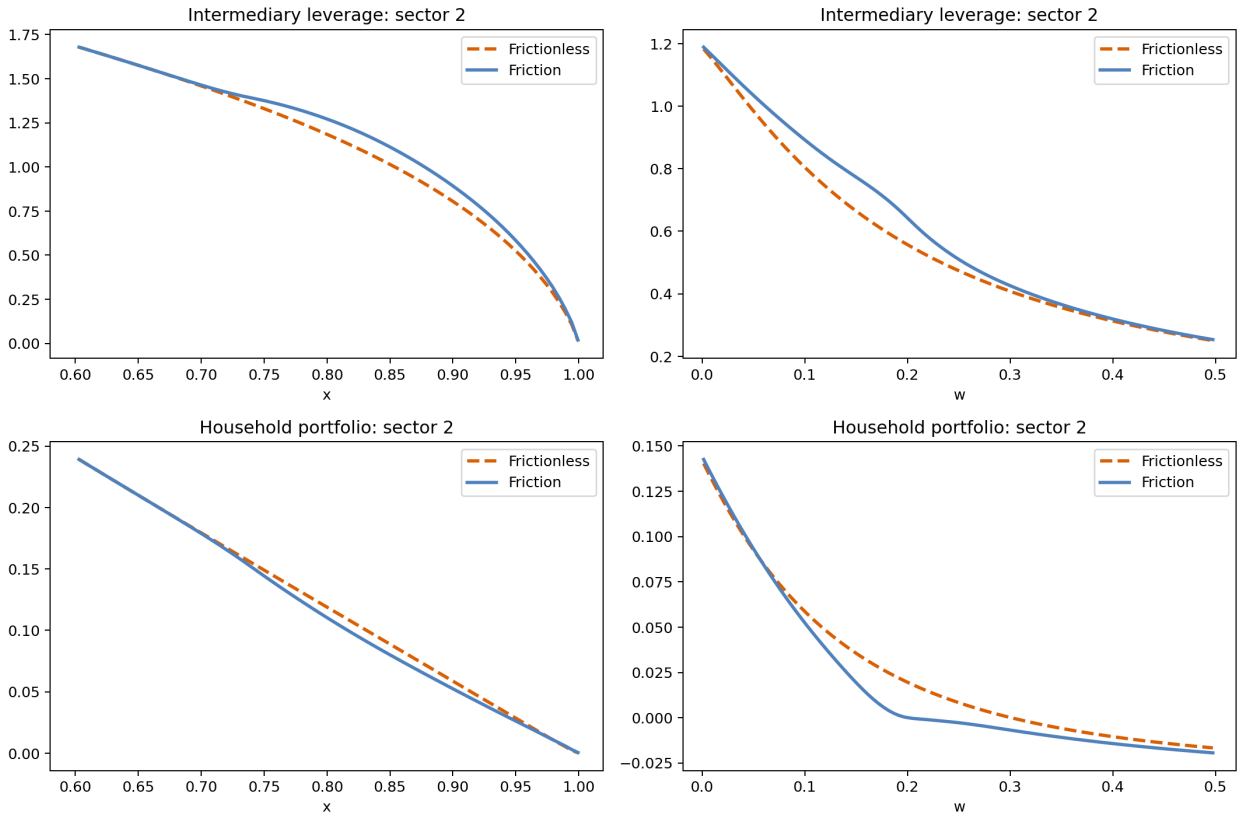
[30]The price of risk for shock B_2 (bottom-right panel) does not differ much with the portfolio rebalance since growth sector is less exposed to shock B_2.

Figure 8: Portfolio choice in sector 1



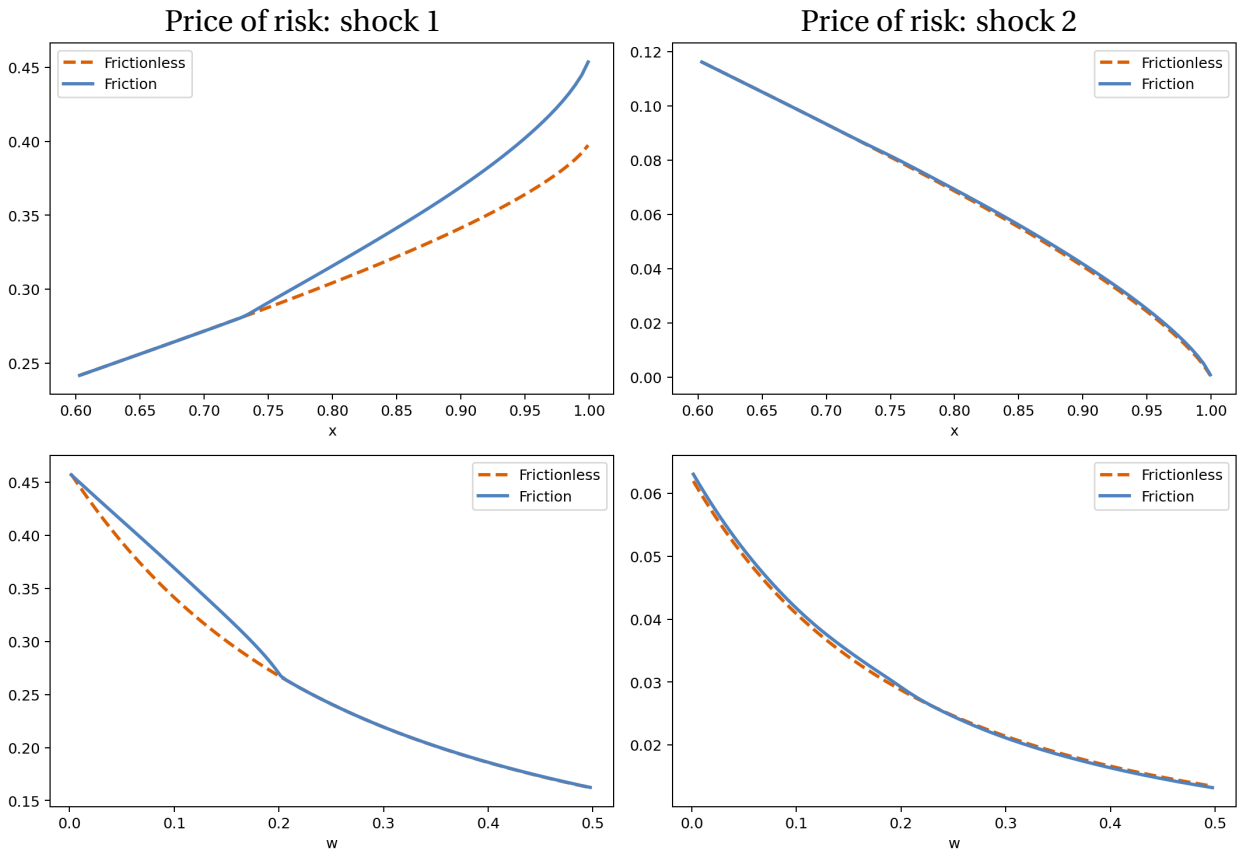
This figure shows the portfolio choice in sector 1 of investors at regions where the constraint on risk exposures to shock B_1 tends to be binding in equilibrium, as functions of state variables x (capital share of sector 1) or w (wealth share of intermediaries) under the constrained calibration. The solid blue lines represent the constrained equilibrium, while the dashed orangelines represent the unconstrained benchmark. The top panels present intermediary's portfolio choice in sector 1 as a function of x while the value of the state variable w is fixed at 0.1 (left) and intermediary's portfolio choice in sector 1 as a function of w while the value of the state variable x is fixed at 0.9 (right). The bottom panels present household's portfolio choice in sector 1 as a function of x while the value of the state variable w is fixed at 0.1 (left) and household's portfolio choice in sector 1 as a function of w while the value of the state variable x is fixed at 0.9 (right).

Figure 9: Portfolio choice in sector 2



This figure shows the portfolio choice in sector 2 of investors at regions where the constraint on risk exposures to shock B_1 tends to be binding in equilibrium, as functions of state variables x (capital share of sector 1) or w (wealth share of intermediaries) under the constrained calibration. The solid blue lines represent the constrained equilibrium, while the dashed orangelines represent the unconstrained benchmark. The top panels present intermediary's portfolio choice in sector 2 as a function of x while the value of the state variable w is fixed at 0.1 (left) and intermediary's portfolio choice in sector 2 as a function of w while the value of the state variable x is fixed at 0.9 (right). The bottom panels present household's portfolio choice in sector 2 as a function of x while the value of the state variable w is fixed at 0.1 (left) and household's portfolio choice in sector 2 as a function of w while the value of the state variable x is fixed at 0.9 (right).

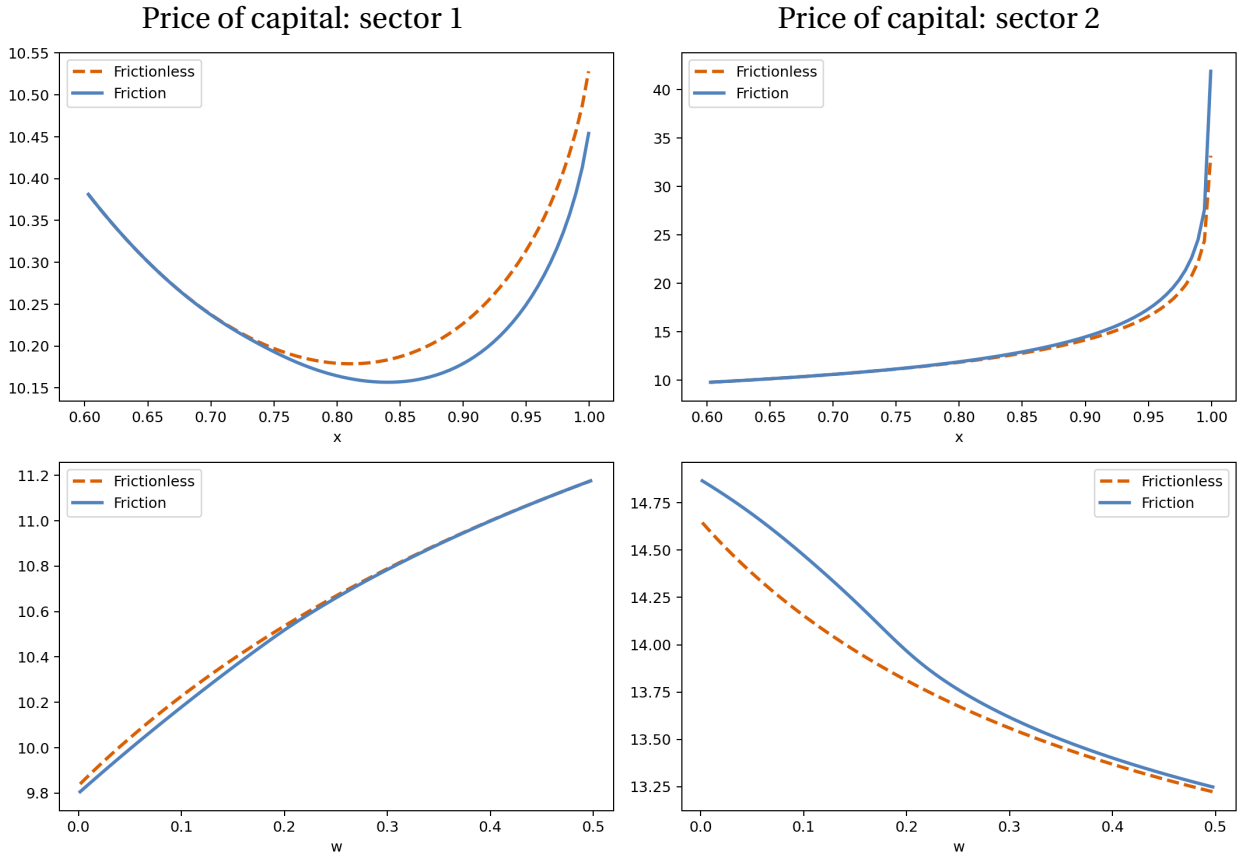
Figure 10: Financial constraint and price of risk



This figure shows the market prices of risks at regions where the constraint on risk exposures to shock B_1 tends to be binding in equilibrium, as functions of state variables x (capital share of sector 1) or w (wealth share of intermediaries) under the constrained calibration. The solid blue lines represent the constrained equilibrium, while the dashed orangelines represent the unconstrained benchmark. The top panels present the price of risk for shock B_1 (left) and the price of risk for shock B_2 (right) as a function of x while the value of the state variable w is fixed at 0.1. The bottom panels present the price of risk for shock B_1 (left) and the price of risk for shock B_2 (right) as a function of w while the value of the state variable x is fixed at 0.9.

As a result, the inefficient arrangement of financial market leads to the misallocation of capital, as intermediaries allocate less capital to the growth sector. The prices of capital of the two sectors as functions of w are displayed in the bottom panels of Figure 11. The risk premium (or discount rate) is inefficiently higher for growth sector and inefficiently lower for value sector. The price of capital q_1 (left) is thus lower than that in the unconstrained benchmark (dashed orange lines), while q_2 (right) is higher. This results in an allocation that is overly conservative. The results also imply that the misallocation is countercyclical with respect to w: when intermediaries are relatively weak in the economy, the constraints tend to be binding. Hence it is consistent with Eisfeldt and Rampini (2006), who find that empirically the misallocation is countercyclical.

Figure 11: Capital misallocation



This figure shows the sectoral prices of capital (Tobin's Q) at regions where the constraint on risk exposures to shock B_1 tends to be binding in equilibrium, as functions of state variables x (capital share of sector 1) or w (wealth share of intermediaries) under the constrained calibration. The solid blue lines represent the constrained equilibrium, while the dashed orange lines represent the unconstrained benchmark. The top panels present the Tobin's Q of sector 1 (left) and the Tobin's Q of sector 2 (right) as a function of x while the value of the state variable w is fixed at 0.1. The bottom panels present the Tobin's Q of sector 1 (left) and the Tobin's Q of sector 2 (right) as a function of w while the value of the state variable x is fixed at 0.9.

Since firms in each sector make investment decisions following Tobin's Q, the above distortions will result in inefficient dynamics of the state variable x. Proposition 2 then implies that, when x is close to one, the drift of x is more negative, as the constrained allocation of capital is overly conservative. The opposite happens when x is close to zero, that is the drift of x becomes more positive as the value sector becomes dominant. In other words, when financial health is weak and the constraint binds, the dynamics of capital composition x mean-reverts faster, as a result of misallocation.

The fluctuations act as extra source of risk, which affects the risk pricing in the financial market. The top-left panel of Figure 10 provides evidence of the effects on risk pricing from misallocation. The price of risk for shock B_1 is plotted as a function of x under frictions (solid blue line), that is when w is low and the constraint tends to be binding for intermediaries' portfolio holdings in the growth sector. The dashed orange line represents the same object under the unconstrained benchmark. It shows that across x (close to one), market price of risk η_1 is higher in the constrained economy. Such pricing effect is simultaneously reflected in the intermediaries' lower leverage in sector 1 across x, and higher leverage in sector 2 (top-left panel of Figures 8 and 9, respectively). Again, due to market clearing, households hold more of their wealth in sector 1, and less of their wealth in sector 2 (bottom-left panels of Figure 8 and 9, respectively).

Moreover, Figure 11 shows that q_1 (top-left) is lower, and q_2 (top-right) is higher in the constrained economy, as η_1 increases across x. Thus, sector 1 is under-invested, while sector 2 is over-invested. Note that sectoral Tobin's Q is determined by (24), and that the financial frictions apply directly to investors' portfolio decisions only. The resulting distortion in the capital composition comes from the inefficiency in the financial market. During periods of weak financial health, frictional financial system sets the stage for capital misallocation by elevating the price of risk through portfolio rebalance among investors. The misallocation, in turn, poses more risk to the financial market and increases the price of risk as well. Therefore, a feedback loop emerges: asset prices and growth is affected by inefficient allocation of capital, through its impact on risk sharing in the financial market. In short, the link between allocation efficiency, and asset prices and growth is amplified through the valuation channel (discount rates).

6 Conclusion

This paper develops a macro-finance model with two productive sectors and heterogeneous investors who differ in risk-bearing capacity and face state-dependent portfolio constraints. In the unconstrained benchmark, investors trade freely, the cost of capital is determined in

financial markets, and capital is allocated efficiently in a risk-adjusted way. With financial frictions, allocation departs from the efficient benchmark, and the misallocation is state dependent and cyclical. When intermediary balance sheets are weak and the dominant sector is large, constraints bind. Intermediaries deleverage and shift exposure away from the risky sector. Risk premia rise, valuations fall in the dominant sector, and investment shifts toward the other sector. The interaction between sectoral composition and intermediary wealth generates a feedback loop from balance-sheet stress to real allocation and asset prices.

Quantitatively, misallocation is most pronounced when the economy is less-diversified and financial health is weak. In these states, risk is reallocated to more risk-averse households only at higher compensation, which raises the price of risk. The valuation channel then amplifies the effects. Higher required returns depress valuations and investment, slowing the rebalancing of sectoral capital. Misallocation feeds back into risk pricing, and the loop tightens. The framework highlights a risk channel of capital allocation efficiency and shows how misallocation feeds back into asset prices and growth through balance-sheet and valuation channels.

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Appendices

A Proofs

Proof of the propositions and lemmas in the main text. To simplify notation, time subscripts are dropped when it is convenient, though it should be understood that they apply.

A.1 Proposition 1

Proof. Let $W_{i,t}(s)$ denote the time-t wealth of an individual investor of type-i born at time-s. The law of motion of $W_{i,t}(s)$ is given by the budget constraint (12)

$$\frac{dW_{i,t}(s)}{W_{i,t}(s)} = \mu_{i,t}(s)dt + \boldsymbol{\sigma}_{i,t}(s)'d\boldsymbol{B}_t. \tag{64}$$

As investors of the same type face the same optimization problem, regardless of their cohort/age, we have $\mu_{i,t}(s) = \mu_{i,t}$ and $\boldsymbol{\sigma}_{i,t}(s) = \boldsymbol{\sigma}_{i,t}$, i.e., the wealth dynamics do not depend on s (when they were born).

Define the time-t average wealth of all investors of type i, including investors from different cohorts/ages s, as $W_{i,t} = \int_{-\infty}^{t} \kappa e^{-\kappa(t-s)} W_{i,t}(s)ds$. Applying Itô's lemma, the law of motion of $W_{i,t}$ is given by

$$dW_{i,t} = \frac{\partial W_{i,t}}{\partial t}dt + \frac{\partial W_{i,t}}{\partial W_{i,t}(s)}dW_{i,t}(s) \tag{65}$$

Applying Leibniz rule to the first term of (65), we have

$$\begin{aligned}\frac{\partial W_{i,t}}{\partial t} &= \kappa W_{i,t}(t) - \kappa \int_{-\infty}^{t} \kappa e^{-\kappa(t-s)} W_{i,t}(s)ds \\ &= \kappa W_{i,t}(t) - \kappa W_{i,t},\end{aligned} \tag{66}$$

using the definition of $W_{i,t}$. The last term of (65) can be written as

$$\begin{aligned}\frac{\partial W_{i,t}}{\partial W_{i,t}(s)}dW_{i,t}(s) &= \int_{-\infty}^{t} \kappa e^{-\kappa(t-s)} dW_{i,t}(s)ds \\ &= \int_{-\infty}^{t} \kappa e^{-\kappa(t-s)} W_{i,t}(s)ds(\mu_{i,t}dt + \boldsymbol{\sigma}_{i,t}'d\boldsymbol{B}_t) \\ &= W_{i,t}(\mu_{i,t}dt + \boldsymbol{\sigma}_{i,t}'d\boldsymbol{B}_t),\end{aligned} \tag{67}$$

using (64) and the definition of $W_{i,t}$ again.

Using the above two equations in (65), the law of motion of the average wealth of type-i

investors can be written as

$$dW_{i,t} = \kappa W_{i,t}(t)dt - \kappa W_{i,t}dt + W_{i,t}(\mu_{i,t}dt + \boldsymbol{\sigma}'_{i,t}d\boldsymbol{B}_t). \tag{68}$$

The first term in (68) corresponds to the wealth of newborn agents at time-t. Newborns of type-i inherit the wealth from their parents, which can be of type-A with probability $\bar{u}$ or type-B with probability $1-\bar{u}$. Hence, $W_{i,t}(t) = \bar{u}W_{A,t} + (1-\bar{u})W_{B,t}$.

Therefore, the law of motion of $W_{i,t}$ can be written as

$$dW_{i,t} = \kappa[\bar{u}W_{A,t} + (1-\bar{u})W_{B,t} - W_{i,t}]dt + W_{i,t}(\mu_{i,t}dt + \boldsymbol{\sigma}'_{i,t}d\boldsymbol{B}_t). \tag{69}$$

For $i = A$, the law of motion of $W_{A,t}$ is given by

$$\frac{dW_{A,t}}{W_{A,t}} = \kappa\left(\frac{\bar{u}}{w_t} - 1\right)dt + (\mu_{A,t}dt + \boldsymbol{\sigma}'_{A,t}d\boldsymbol{B}_t). \tag{70}$$

For $i = B$, the law of motion of $W_{B,t}$ is given by

$$\frac{dW_{B,t}}{W_{B,t}} = \kappa\left(\frac{1-\bar{u}}{1-w_t} - 1\right)dt + (\mu_{B,t}dt + \boldsymbol{\sigma}'_{B,t}d\boldsymbol{B}_t). \tag{71}$$

Denote $S_t = S_{1,t} + S_{2,t}$, and use the market clearing condition

$$\bar{u}W_{A,t} + (1-\bar{u})W_{B,t} = S_{1,t} + S_{2,t} = S_t \tag{72}$$

Apply Itô's lemma to the definition of w_t:

$$w_t = \frac{\bar{u}W_{A,t}}{\bar{u}W_{A,t} + (1-\bar{u})W_{B,t}} = \frac{\bar{u}W_{A,t}}{S_t}, \tag{73}$$

we have

$$dw_t = w_t\left[\frac{dW_{A,t}}{W_{A,t}} - \frac{dS_t}{S_t} + \left(\frac{dS_t}{S_t}\right)^2 - \frac{dS_t}{S_t}\frac{dW_{A,t}}{W_{A,t}}\right], \tag{74}$$

in which we can use

$$\frac{dS_t}{S_t} = w_t\frac{dW_{A,t}}{W_{A,t}} + (1-w_t)\frac{dW_{B,t}}{W_{B,t}}. \tag{75}$$

from the definition of S_t. The terms inside the brackets of (74) can be written as

$$\frac{dW_{A,t}}{W_{A,t}} - \frac{dS_t}{S_t} = (1-w_t)\left(\frac{dW_{A,t}}{W_{A,t}} - \frac{dW_{B,t}}{W_{B,t}}\right)$$
$$\left(\frac{dS_t}{S_t}\right)^2 - \frac{dS_t}{S_t}\frac{dW_{A,t}}{W_{A,t}} = -(1-w_t)\left(\frac{dW_{A,t}}{W_{A,t}} - \frac{dW_{B,t}}{W_{B,t}}\right)\left[w_t\frac{dW_{A,t}}{W_{A,t}} + (1-w_t)\frac{dW_{B,t}}{W_{B,t}}\right]$$

Substituting the above two equations into (74) and using (70) and (71), we have

$$\begin{aligned} dw_t = {} & \kappa[\bar{u} - w_t]dt \\ & + (1-w_t)w_t\left[(\boldsymbol{\sigma}'_{A,t} - \boldsymbol{\sigma}'_{B,t})\boldsymbol{\eta}_t + c_{B,t} - c_{A,t}\right]dt \\ & - (1-w_t)w_t\left[(\boldsymbol{\sigma}'_{A,t} - \boldsymbol{\sigma}'_{B,t})\boldsymbol{\Sigma}'_{R,t}\begin{pmatrix} z_t \\ 1-z_t \end{pmatrix}\right]dt \\ & + (1-w_t)w_t(\boldsymbol{\sigma}'_{A,t} - \boldsymbol{\sigma}'_{B,t})d\boldsymbol{B}_t, \end{aligned} \tag{76}$$

The above expression gives the law of motion for w, as shown in the main text. ■

A.2 Proposition 2

Proof. Define $K = K_1 + K_2$, $x = \frac{K_1}{K_1+K_2}$. Use the law of motion of K_1 and K_2

$$\frac{dK_1}{K_1} = g_1 dt + \sigma_1 dB_1,$$
$$\frac{dK_2}{K_2} = g_2 dt + \sigma_2 dB_2.$$

Applying Itô's lemma to K and substituting the definition of K and x, we have

$$\begin{aligned} d(K_1+K_2) &= (g_1K_1 + g_2K_2)dt + \sigma_1 K_1 dB_1 + \sigma_2 K_2 dB_2, \\ dK &= (g_1 xK + g_2(1-x)K)dt + \sigma_1 xK\, dB_1 + \sigma_2 K(1-x)dB_2. \end{aligned} \tag{77}$$

Applying Itô's lemma to x, we have

$$\begin{aligned} dx &= \frac{1}{K}dK_1 - \frac{K_1}{K^2}dK + \frac{K_1}{K^3}(dK)^2 - \frac{1}{K^2}dK_1 dK \\ &= x(g_1 dt + \sigma_1 dB_1) - x[(g_1 x + g_2(1-x))dt + \sigma_1 x dB_1 + \sigma_2(1-x)dB_2] \\ &\quad + x(\sigma_1^2 x^2 + \sigma_2^2(1-x)^2)dt - x\sigma_1^2 x dt \\ &= x(1-x)\left[g_1 - g_2 - \sigma_1^2 x + \sigma_2^2(1-x)\right]dt + \sigma_1 x(1-x)dB_1 - \sigma_2 x(1-x)dB_2. \end{aligned} \tag{78}$$

The above expression gives the law of motion for x, as shown in the main text. ■

A.3 Proposition 3

Proof. To simplify notation, time and sector n subscripts are dropped, as the proposition applies to both sectors at any time.

HJB equation for the firm's problem is given by

$$0 = \max_{\iota} \left\{ D\,dt + \mathbb{E}\left[\frac{d(\Lambda S)}{\Lambda} \right] \right\}, \tag{79}$$

where D is the dividend paid by the firm, Λ is the SDF.

We have the following law of motions

$$\begin{aligned}
\frac{dK}{K} &= g\,dt + \sigma_1 dB_1 + \sigma_2 dB_2, \\
\frac{dq}{q} &= \mu_q dt + \sigma_{q,1} dB_1 + \sigma_{q,2} dB_2, \\
\frac{d\Lambda}{\Lambda} &= -r\,dt - \eta_1 dB_1 - \eta_2 dB_2.
\end{aligned}$$

Applying Itô's lemma to $S = qK$, we have

$$\begin{aligned}
dS &= q\,dK + K\,dq + dq\,dK, \\
\frac{dS}{S} &= \left(g + \mu_q + \sigma_1\sigma_{q,1} + \sigma_2\sigma_{q,2}\right) dt + \left(\sigma_1 + \sigma_{q,1}\right) dB_1 + \left(\sigma_2 + \sigma_{q,2}\right) dB_2, \\
&= \mu_S dt + \sigma_{S,1} dB_1 + \sigma_{S,2} dB_2.
\end{aligned}$$

Applying Itô's lemma to ΛS, we have

$$\frac{d\Lambda S}{\Lambda S} = \left(\mu_S - r - \eta_1\sigma_{S,1} - \eta_2\sigma_{S,2}\right) dt + (\sigma_{S,1} - \eta_1) dB_1 + (\sigma_{S,2} - \eta_2) dB_2. \tag{80}$$

Equating the drift from (80) to D, the HJB equation for the firm's problem is thus given by

$$0 = \max_{\iota} \left\{ \frac{A - \iota}{q} + g + \mu_q + \boldsymbol{\sigma}_q' \boldsymbol{\sigma} - r - \boldsymbol{\eta}'(\boldsymbol{\sigma}_q + \boldsymbol{\sigma}) \right\} \tag{81}$$

where the bold symbols denote vectors for simplicity.

The optimal investment rate is given by the first order condition

$$g'(\iota) = 1/q. \tag{82}$$

Note that using Itô's lemma, the drift and diffusion μ_q, $\sigma_{q,1}$ and $\sigma_{q,2}$ can be expressed as follows

$$\begin{aligned}\mu_q = &\frac{q_x}{q}\mu_x + \frac{q_w}{q}\left[\kappa(\bar{\mu} - w) + w(1-w)\mu_w\right] \\ &+ \frac{1}{2}\frac{q_{xx}}{q}(\sigma_{x,1}^2 + \sigma_{x,2}^2) + \frac{1}{2}\frac{q_{ww}}{q}w^2(1-w)^2(\sigma_{w,1}^2 + \sigma_{w,2}^2) \\ &+ \frac{q_{xw}}{q}w(1-w)(\sigma_{x,1}\sigma_{w,1} + \sigma_{x,2}\sigma_{w,2}), \qquad (83)\\ \sigma_{q,1} = &\frac{q_x}{q}\sigma_{x,1} + \frac{q_w}{q}w(1-w)\sigma_{w,1}, \qquad (84)\\ \sigma_{q,2} = &\frac{q_x}{q}\sigma_{x,2} + \frac{q_w}{q}w(1-w)\sigma_{w,2}. \qquad (85)\end{aligned}$$

where $q_x = \partial q/\partial x$, $q_w = \partial q/\partial w$, $q_{xx} = \partial^2 q/\partial x^2$, $q_{ww} = \partial^2 q/\partial w^2$, $q_{xw} = \partial^2 q/\partial x \partial w$. $\mu_x, \mu_w, \sigma_{x,1}$, $\sigma_{x,2}, \sigma_{w,1}, \sigma_{w,2}$ are from the law of motion of state variables x and w, (18) and (17), respectively. ■

A.4 Proposition 4

Proof. To simplify notation, time and investor type i subscripts are dropped, as the proposition applies to investors of both types at any time.

HJB equation for the investor's problem is given by

$$0 = \max_{c,\boldsymbol{\sigma}}\{f(cW, V)dt + \mathbb{E}[dV]\}, \qquad (86)$$

where f is the aggregator of Epstein-Zin preferences.

We have the following law of motions

$$\begin{aligned}\frac{dW}{W} &= \mu_W dt + \sigma_{W,1}dB_1 + \sigma_{W,2}dB_2,\\ \frac{d\xi}{\xi} &= \mu_\xi dt + \sigma_{\xi,1}dB_1 + \sigma_{\xi,2}dB_2.\end{aligned}$$

Applying Itô's lemma to

$$V = \frac{W^{1-\gamma}}{1-\gamma}\xi^{\frac{1-\gamma}{1-\psi}},$$

we have

$$dV = \frac{\partial V}{\partial W}dW + \frac{\partial V}{\partial \xi}d\xi + \frac{1}{2}\frac{\partial^2 V}{\partial W^2}(dW)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \xi^2}(d\xi)^2 + \frac{\partial^2 V}{\partial W \partial \xi}dW d\xi,$$

in which

$$\begin{aligned}\frac{\mathbb{E}[dV]}{dt} &= (1-\gamma)V\mu_W + \frac{1-\gamma}{1-\psi}V\mu_\xi + \frac{1}{2}(1-\gamma)(-\gamma)V(\sigma_{W,1}^2+\sigma_{W,2}^2)\\ &+ \frac{1}{2}\frac{1-\gamma}{1-\psi}\left(\frac{1-\gamma}{1-\psi}-1\right)V(\sigma_{\xi,1}^2+\sigma_{\xi,2}^2)\\ &+ (1-\gamma)\frac{1-\gamma}{1-\psi}V(\sigma_{W,1}\sigma_{\xi,1}+\sigma_{W,2}\sigma_{\xi,2}).\end{aligned} \tag{87}$$

Meanwhile,

$$f(cW,V) = \frac{1-\gamma}{1-1/\psi}V\left[\left(\frac{cW}{((1-\gamma)V)^{\frac{1}{1-\gamma}}}\right)^{1-1/\psi}-\rho\right] \tag{88}$$

Equate the sum of (87) and (88) to zero and remove the term $(1-\gamma)V$, we have

$$\begin{aligned}0 = \max_{c,\boldsymbol{\sigma}}\Bigg\{&\frac{1}{1-1/\psi}\left(c^{1-1/\psi}\xi^{1/\psi}-\rho\right)+[\boldsymbol{\sigma}'\boldsymbol{\eta}_t + r_t - c]\\ &+\frac{1}{1-\psi}\mu_\xi - \frac{\gamma}{2}\boldsymbol{\sigma}'\boldsymbol{\sigma}\\ &+\frac{1}{2}\frac{1}{1-\psi}\left(\frac{1-\gamma}{1-\psi}-1\right)\boldsymbol{\sigma}_\xi'\boldsymbol{\sigma}_\xi\\ &+\frac{1-\gamma}{1-\psi}\boldsymbol{\sigma}'\boldsymbol{\sigma}_\xi\Bigg\},\end{aligned} \tag{89}$$

where matrix notation is used for simplicity. The optimal consumption-wealth ratio and risk exposure are given by the first order condition w.r.t. c and $\boldsymbol{\sigma}$,

$$c = \xi, \tag{90}$$

$$\boldsymbol{\sigma} = \frac{1}{\gamma}\boldsymbol{\eta} + \frac{1}{\gamma}\frac{1-\gamma}{1-\psi}\boldsymbol{\sigma}_\xi. \tag{91}$$

Note that using Itô's lemma, the drift and diffusion μ_ξ, $\sigma_{\xi,1}$ and $\sigma_{\xi,2}$ can be expressed as

follows

$$\mu_\xi = \frac{\xi_x}{\xi}\mu_x + \frac{\xi_w}{\xi}\left[\kappa(\bar{\mu} - w) + w(1-w)\mu_w\right] + \frac{1}{2}\frac{\xi_{xx}}{\xi}(\sigma_{x,1}^2 + \sigma_{x,2}^2) + \frac{1}{2}\frac{\xi_{ww}}{\xi}w^2(1-w)^2(\sigma_{w,1}^2 + \sigma_{w,2}^2) + \frac{\xi_{xw}}{\xi}w(1-w)(\sigma_{x,1}\sigma_{w,1} + \sigma_{x,2}\sigma_{w,2}), \tag{92}$$

$$\sigma_{\xi,1} = \frac{\xi_x}{\xi}\sigma_{x,1} + \frac{\xi_w}{\xi}w(1-w)\sigma_{w,1}, \tag{93}$$

$$\sigma_{\xi,2} = \frac{\xi_x}{\xi}\sigma_{x,2} + \frac{\xi_w}{\xi}w(1-w)\sigma_{w,2}. \tag{94}$$

where $\xi_x = \partial\xi/\partial x$, $\xi_w = \partial\xi/\partial w$, $\xi_{xx} = \partial^2\xi/\partial x^2$, $\xi_{ww} = \partial^2\xi/\partial w^2$, $\xi_{xw} = \partial^2\xi/\partial x\partial w$. μ_x, μ_w, $\sigma_{x,1}$, $\sigma_{x,2}$, $\sigma_{w,1}$, $\sigma_{w,2}$ are from the law of motion of state variables x and w, (18) and (17), respectively. ■

A.5 Proposition 5

Proof. Denote $\sigma_A^{(1),*}$ as type A investors' optimal risk exposure to risk from sector 1, i.e., the first element of their risk exposure $\boldsymbol{\sigma}_A^*$. Using the market clearing condition for the risk exposure,

$$w\sigma_A^{(1),*} + (1-w)\sigma_B^{(1)} = z\sigma_{R,1}^{(1)} + (1-z)\sigma_{R,2}^{(1)},$$

express $\sigma_B^{(1)}$ in terms of $\sigma_A^{(1),*}$. Then plug it into the expression for investor of type B's optimal risk exposure from (37),

$$\sigma_B^{(1)} = \frac{1}{\gamma_B}\eta^{(1)} + \frac{1}{\gamma_B}\frac{1-\gamma_B}{1-\psi}\sigma_{\xi_B}^{(1)},$$

substituting for $\eta^{(1)}$ using the expression for $\sigma_A^{(1),*}$,

$$\sigma_A^{(1),*} = \frac{1}{\gamma_A}\eta^{(1)} + \frac{1}{\gamma_A}\frac{1-\gamma_A}{1-\psi}\sigma_{\xi_A}^{(1)}.$$

Then I can compute $\sigma_A^{(1),*}$ as follows:

$$\sigma_A^{(1),*} = \frac{zR_w + R_wQ_xx(1-x) - R_x(Q_w + \frac{1}{w})x(1-x) - z\frac{\gamma_B}{w(1-w)} - \frac{\gamma_B}{w(1-w)}x(1-x)Q_x}{(\gamma_B - \gamma_A)Q_w - \frac{\gamma_B}{1-w} - \frac{\gamma_A}{w} + R_w}\sigma_1, \tag{95}$$

where

$$R_x = \frac{1-\gamma_A}{1-\psi}\frac{\xi_{A,x}}{\xi_A} - \frac{1-\gamma_B}{1-\psi}\frac{\xi_{B,x}}{\xi_B}; \quad R_w = \frac{1-\gamma_A}{1-\psi}\frac{\xi_{A,w}}{\xi_A} - \frac{1-\gamma_B}{1-\psi}\frac{\xi_{B,w}}{\xi_B};$$
$$Q_x = z\frac{q_{1,x}}{q_1} + (1-z)\frac{q_{2,x}}{q_2}; \qquad Q_w = z\frac{q_{1,w}}{q_1} + (1-z)\frac{q_{2,w}}{q_2}.$$

In a similar manner, I can compute $\sigma_A^{(2),*}$, the second element of $\boldsymbol{\sigma}_A^*$, as follows:

$$\sigma_A^{(2),*} = \frac{(1-z)R_w - R_w Q_x x(1-x) + R_x(Q_w + \frac{1}{w})x(1-x) - \frac{(1-z)\gamma_B}{w(1-w)} + \frac{\gamma_B}{w(1-w)}x(1-x)Q_x}{(\gamma_B-\gamma_A)Q_w - \frac{\gamma_B}{1-w} - \frac{\gamma_A}{w} + R_w}\sigma_2. \tag{96}$$

The risk exposure of type B investors can be computed using the market clearing condition for risk exposure. ■

A.6 Proposition 6

Proof. Plugging the (frictionless) optimal risk exposure (37) for type A and B investors into the market clearing conditions of risk exposure, I derive the expression for the vector of market price of risk

$$\begin{aligned}\boldsymbol{\eta}_t = {} & \gamma_t\left[z_t\boldsymbol{\sigma}_{R_1,t} + (1-z_t)\boldsymbol{\sigma}_{R_2,t}\right] \\ & - \gamma_t\left[w_t\left(\frac{1}{\gamma_A}\right)\left(\frac{1-\gamma_A}{1-\psi}\right)\boldsymbol{\sigma}_{\xi_A,t} + (1-w_t)\left(\frac{1}{\gamma_B}\right)\left(\frac{1-\gamma_B}{1-\psi}\right)\boldsymbol{\sigma}_{\xi_B,t}\right],\end{aligned} \tag{97}$$

where γ_t denotes aggregate risk aversion, and is given by

$$\gamma_t = \left(\frac{w_t}{\gamma_A} + \frac{1-w_t}{\gamma_B}\right)^{-1}. \tag{98}$$

■

A.7 Proposition 7

Proof. Plugging the optimal consumption-wealth ratio (36) into the HJB (35), I obtain

$$c_{i,t} = \psi\rho + \mu_{\xi_i,t} + (1-\psi)(r_t + \Phi_{i,t}), \tag{99}$$

where

$$\Phi_{i,t} = \boldsymbol{\sigma}_{i,t}'\boldsymbol{\eta}_t - \frac{\gamma_i}{2}\boldsymbol{\sigma}_{i,t}'\boldsymbol{\sigma}_{i,t} + \frac{1}{2}\frac{1}{1-\psi}\left(\frac{1-\gamma_i}{1-\psi}-1\right)\boldsymbol{\sigma}_{\xi_i,t}'\boldsymbol{\sigma}_{\xi_i,t} + \frac{1-\gamma_i}{1-\psi}\boldsymbol{\sigma}_{i,t}'\boldsymbol{\sigma}_{\xi_i,t}. \tag{100}$$

Plugging the above equation into the goods market clearing condition, I obtain the expression for the interest rate

$$\psi\rho + \mu_t + (1-\psi)(r_t + \Phi_t) = z_t\frac{A_1 - \iota_1(g_{1,t})}{q_{1,t}} + (1-z_t)\frac{A_2 - \iota_2(g_{2,t})}{q_{2,t}}, \tag{101}$$

where $\mu_t = w_t\mu_{\xi_A,t} + (1-w_t)\mu_{\xi_B,t}$, and $\Phi_t = w_t\Phi_{A,t} + (1-w_t)\Phi_{B,t}$. ■

A.8 Lemma 1

Proof. The benchmark economy is essentially non-stochastic, up to scale. The equilibrium conditions are:

$$\begin{gathered}
\frac{1}{1-1/\psi}(c_i-\rho) + (\sigma_i\eta + r - c_i) - \frac{\gamma_i}{2}\sigma_i^2 = 0, \\
\sigma_i = \frac{\eta}{\gamma_i}, \\
\frac{A-\iota(g)}{q} + g - r - \eta\sigma = 0, \\
\iota'(g) = q, \\
c = \frac{A-\iota(g)}{q}, \\
\sigma_i = \sigma.
\end{gathered}$$

The market price of risk is

$$\eta^* = \gamma\sigma.$$

The interest rate is given by:

$$r^* = \rho + \psi^{-1}g^* - \left(1+\psi^{-1}\right)\frac{\gamma\sigma^2}{2}.$$

The consumption-wealth ratios, i.e., the dividend yield on aggregate wealth are

$$c_A^* = c_B^* = \rho - \left(1-\psi^{-1}\right)\left(g^* - \frac{\gamma\sigma^2}{2}\right).$$

The growth rate g^* and price of capital q^* satisfy

$$\frac{A-\iota(g^*)}{q^*} = \rho - \left(1-\psi^{-1}\right)\left(g^* - \frac{\gamma\sigma^2}{2}\right),$$
$$\iota'(g^*) = q^*.$$

■

A.9 Lemma 2

This economy is the same as the one in Eberly and Wang (2025), except that it is solved in decentralized equilibrium. Eberly and Wang (2025) solve the social planner's problem, which is equivalent to the decentralized equilibrium in this case.

Assume $\gamma_A = \gamma_B = \gamma$ and $\kappa = 0$, i.e., there is a representative investor. In addition, assume

$$\frac{dK_{1,t}}{K_{1,t}} = g_{1,t}dt + \sigma\varepsilon dB_{1,t} + \sigma(1-\varepsilon)dB_{2,t},$$
$$\frac{dK_{2,t}}{K_{2,t}} = g_{2,t}dt + \sigma dB_{2,t}.$$

The two sectors are assumed to be symmetric for expositional simplicity: productivity A, depreciation rate δ, and adjustment cost (ϕ,θ) are assumed to be the same across two sectors.

Proof. We consider a first-order asymptotic expansion around the point $\varepsilon = 0$, i.e., for each variable z, we want to find the functions $(z_0(x), z_1(x))$ such that $z(x;\varepsilon) = z_0(x) + z_1(x)\varepsilon + O(\varepsilon^2)$, in which x, the capital share of sector 1, is state variable.

The zeroth-order coefficients are derived in the benchmark economy, meaning that the economy in Lemma 1 corresponds to the case of $\varepsilon = 0$, for which the solution can be solved explicitly. For the case where $\varepsilon > 0$, the equilibrium objects can be approximated by a Taylor expansion around $\varepsilon = 0$. For example, the price of capital in sector 1 can be approximated as

$$q_1(x;\varepsilon) = q^* + q_{1,\varepsilon}(x;0)\varepsilon + O(\varepsilon^2),$$

where q^* is the price of capital in the economy in Lemma 1. By computing the derivatives with respect to ε, $q_{1,\varepsilon}(x;0)$, we can obtain explicit expressions for the equilibrium objects when the extent of heterogeneity is small. The other equilibrium objects can be approximated similarly.

Using the above two expressions for laws of motion of capital, the diffusion of x to first shock from (18) can be written as

$$\sigma_{x,1} = x(1-x)\varepsilon\sigma,$$

and the diffusion of x to second shock can be written as

$$\sigma_{x,2} = -x(1-x)\varepsilon\sigma.$$

Since the derivative of $q(x;0)$ (zeroth-order coefficient) with respect to x is zero, the derivative of $q(x;\varepsilon)$ with respect to x, namely $q_x(x;\varepsilon)$, is first-order in ε. Similarly, the derivative of consumption-wealth ratio $\xi_i(x;\varepsilon)$ with respect to x, namely $\xi_{i,x}(x;\varepsilon)$, is also first-order in ε. Meanwhile, the diffusion terms of x are also first-order in ε as can be seen from above. Thus $\sigma_{q,1}, \sigma_{q,2}$ and $\sigma_{\xi_{i,1}}, \sigma_{\xi_{i,2}}$ are of order $O(\varepsilon^2)$ (see proofs of Proposition 3 and 4 for expressions).

Combining the above with the market prices of risks from (43), we can write the market prices of risks as

$$\begin{aligned}\eta_1(x;\varepsilon) &= \gamma\sigma z\varepsilon + O(\varepsilon^2),\\ \eta_2(x;\varepsilon) &= \gamma(\sigma - \sigma z\varepsilon) + O(\varepsilon^2).\end{aligned}$$

The risk exposure of investors from (37) can be written as

$$\begin{aligned}\sigma_{i,1} &= \sigma z\varepsilon + O(\varepsilon^2),\\ \sigma_{i,2} &= \sigma - \sigma z\varepsilon + O(\varepsilon^2).\end{aligned}$$

From the optimal consumption-wealth ratio (99), we can write it as

$$c_i(x;\varepsilon) = \psi\rho + (1-\psi)(r_t + \frac{\gamma_i}{2}\sigma^2 - \gamma_i\sigma^2 z\varepsilon) + O(\varepsilon^2).$$

where it uses $\mu_{\xi_i,t} = O(\varepsilon^2)$, and $\Phi_{i,t} = \frac{\gamma_i}{2}\sigma^2 - \gamma_i\sigma^2 z\varepsilon + O(\varepsilon^2)$.

Write the drift of x from (18) as

$$\mu_x = x(1-x)\left[g_1 - g_2\right] + 2x^3\varepsilon^2\sigma^2 - 2x^2\varepsilon^2\sigma^2 - x^2\varepsilon\sigma^2 + x\varepsilon\sigma^2$$

Since derivative $c_{i,x}(x;\varepsilon)$ is first-order in ε. The drift terms of c and q are second-order in ε.

Thus the firms' HJB equations can be written as

$$\frac{A - \iota(g_n)}{q_n} + g_n + - r_t - \eta_1\sigma_{n,1} - \eta_2\sigma_{n,2} = 0.$$

where it uses $\mu_{q,n} = O(\varepsilon^2)$, and $\sigma_{q_n,1}, \sigma_{q_n,2} = O(\varepsilon^2)$.

Thus we have

$$\frac{A-\iota(g_1)}{q_1} = r_t - g_1 + \gamma\sigma^2 - \gamma\sigma^2 z\varepsilon - \gamma\sigma^2\varepsilon + O(\varepsilon^2),$$
$$\frac{A-\iota(g_2)}{q_2} = r_t - g_2 + \gamma\sigma^2 - \gamma\sigma^2 z\varepsilon + O(\varepsilon^2).$$

Use the above HJBs for investors and firms, and market clearing condition for consumption goods, the interest rate can be solved as

$$\begin{aligned} r_t &= \rho - (1-\psi^{-1})(\frac{\gamma}{2}\sigma^2 - \gamma\sigma^2 z\varepsilon) - \psi^{-1}\left[\gamma\sigma^2 - 2\gamma\sigma^2 z\varepsilon - zg_1 - (1-z)g_2\right] \\ &= \rho + \psi^{-1}g^* - (1+\psi^{-1})\frac{\gamma}{2}\sigma^2 \\ &+ (1+\psi^{-1})\gamma\sigma^2 z\varepsilon + \psi^{-1}\left[z(g_1-g^*) + (1-z)(g_2-g^*)\right] + O(\varepsilon^2), \\ &= r^* \\ &+ (1+\psi^{-1})\gamma\sigma^2 z\varepsilon + \psi^{-1}\left[z(g_1-g^*) + (1-z)(g_2-g^*)\right] + O(\varepsilon^2). \end{aligned}$$

Following the first-order asymptotic expansion, write the price of capital as

$$q_1 = q^* + q_{1,1}(x)\varepsilon + O(\varepsilon^2),$$
$$q_2 = q^* + q_{2,1}(x)\varepsilon + O(\varepsilon^2),$$

where q^* is the equilibrium price of capital from Lemma 1, and $q_{1,1}$ denotes the first-order coefficient of q_1 with respect to ε. Similarly for $q_{2,1}$.

From (25), $g = \frac{q-\phi}{\theta} - \delta$. Then growth rate can be written as

$$g_1 = g^* + \frac{q_{1,1}}{\theta}\varepsilon + O(\varepsilon^2),$$
$$g_2 = g^* + \frac{q_{2,1}}{\theta}\varepsilon + O(\varepsilon^2),$$

where $g^* = \frac{q^*-\phi}{\theta} - \delta$ is the equilibrium growth rate from Lemma 1.

Similarly from (25), $\iota = \frac{q^2-\phi^2}{2\theta}$, the investment rate can be written as

$$\iota_1 = \frac{(q^*)^2 + 2q^* q_{1,1}\varepsilon - \phi^2}{2\theta} + O(\varepsilon^2),$$
$$\iota_2 = \frac{(q^*)^2 + 2q^* q_{2,1}\varepsilon - \phi^2}{2\theta} + O(\varepsilon^2).$$

The variable z is defined in (27), which can be written as

$$z = \left(1 + \frac{q_{2,t}}{q_{1,t}} \frac{1 - x_t}{x_t}\right)^{-1}.$$

Replace r_t, growth rate, price of capital, investment rate into firms' HJB equations, we can solve for the pair of $(q_{1,1}, q_{2,1})$ jointly, which gives the first-order asymptotic expansion of the equilibrium objects. The pair of equations to be solved are the following:

$$\begin{aligned}
0 &= c^* q_{1,1}\varepsilon + q^* \psi^{-1} \gamma \sigma^2 z \varepsilon + \psi^{-1} q^* \left[z \frac{q_{1,1}}{\theta} + (1 - z) \frac{q_{2,1}}{\theta} \right] \varepsilon - q^* \gamma \sigma^2 \varepsilon, \\
0 &= c^* q_{2,1}\varepsilon + q^* \psi^{-1} \gamma \sigma^2 z \varepsilon + \psi^{-1} q^* \left[z \frac{q_{1,1}}{\theta} + (1 - z) \frac{q_{2,1}}{\theta} \right] \varepsilon.
\end{aligned}$$

where $c^* = r^* - g^* + \gamma\sigma^2$ is the consumption-wealth ratio from Lemma 1.

The solution to the above pair of equations gives

$$\begin{aligned}
q_{1,1} &= \frac{q^* \gamma \sigma^2 (-c^* x \theta + c^* \theta \psi - q^* x + q^*)}{c^* (c^* \theta \psi + q^*)}, \\
q_{2,1} &= \frac{q^* x \gamma \theta \sigma^2 (-c^* \theta - q^*)}{c^* (c^* \theta \psi + q^*)}.
\end{aligned}$$

The derivative of price of capital with respect to x, same for both sectors, can be written as

$$q_x(x; \varepsilon) = \frac{q^* \gamma \sigma^2 (-c^* \theta - q^*)}{c^* (c^* \theta \psi + q^*)} \varepsilon,$$

which is negative in the reasonable calibration.

To complete the proof, the drift of x can be written as

$$\mu_x = x(1 - x) \frac{q^* \gamma \sigma^2}{c^* \theta} \varepsilon - x^2 \varepsilon \sigma^2 + x \varepsilon \sigma^2 + O(\varepsilon^2).$$

■

A.10 Lemma 3

This economy is a standard frictionless heterogeneous agent asset pricing model in a production economy.

Proof. We consider a first-order asymptotic expansion around the point $\epsilon = 0$, i.e., for each variable z, we want to find the functions $(z_0(w), z_1(w))$ such that $z(w; \epsilon) = z_0(w) + z_1(w)\epsilon + O(\epsilon^2)$. The zeroth-order coefficients are derived in the benchmark economy (see Lemma 1).

Let $\gamma_A = \gamma(1-\Delta\gamma)$, $\gamma_B = \gamma(1+\Delta\gamma)$, $\Delta\gamma > 0$, and

$$\Delta\gamma = \delta_\gamma \epsilon, \quad \kappa = \delta_\kappa \epsilon.$$

Aggregate risk aversion from (44) can be written as

$$\gamma(w;\epsilon) = \gamma\left(\frac{1-w}{1+\Delta\gamma} + \frac{w}{1-\Delta\gamma}\right)^{-1} = \gamma - (2w-1)\gamma\Delta\gamma + O(\epsilon^2).$$

In equilibrium, one of the FOCs gives $c_i = \xi_i$, thus the proof focuses on characterizing c_i.

Since the derivative of $q(w;0)$ (zeroth-order coefficient) with respect to w, is zero, the derivative of $q(w;\epsilon)$ with respect to w, namely $q_w(w;\epsilon)$, is first-order in ϵ. Similarly, the derivative of consumption-wealth ratio $c_i(w;\epsilon)$ with respect to w, namely $c_{i,w}(w;\epsilon)$ is also first-order in ϵ.

Meanwhile, the diffusion of w, σ_w is also first-order in ϵ because it is equal to zero at $\epsilon = 0$. Thus σ_q and σ_{ξ_i} are both of order $O(\epsilon^2)$ (see proof of Proposition 3 and 4 for expressions).

Then it follows that the market price of risk from (43) can be written as

$$\eta_t(w;\epsilon) = \gamma\sigma - \gamma\sigma(2w-1)\Delta\gamma + O(\epsilon^2).$$

Note that at $\epsilon = 0$, $\eta(w;0) = \gamma\sigma$, which is the market price of risk η^* from the benchmark economy in Lemma 1.

Then the risk exposure of investors from (37) can be written as

$$\begin{aligned}\sigma_A(w;\epsilon) &= \sigma + 2(1-w)\sigma\Delta\gamma + O(\epsilon^2),\\ \sigma_B(w;\epsilon) &= \sigma - 2w\sigma\Delta\gamma + O(\epsilon^2).\end{aligned}$$

Similarly, note that at $\epsilon = 0$, $\sigma_A(w;0) = \sigma_B(w;0) = \sigma$, which is the risk exposure from the benchmark economy in Lemma 1.

Using the above two expressions, the diffusion of w from (17) can be written as

$$\sigma_w(w;\epsilon) = w(1-w)2\sigma\Delta\gamma + O(\epsilon^2).$$

The interest rate, from firm's HJB, can be written as

$$r_t = \frac{A - \iota(g_t)}{q_t} + g_t + \mu_q + \sigma_q\sigma - \eta_t(\sigma_q + \sigma)$$

Use the market clearing condition for consumption goods, interest rate can be written as

$$\begin{aligned} r_t &= [wc_A + (1-w)c_B] + g_t + \mu_q + \sigma_q\sigma - \eta_t(\sigma_q + \sigma) \\ &= \psi\rho + \mu_t + (1-\psi)(r_t + \Phi_t) + g_t + \mu_q + \sigma_q\sigma - \eta_t(\sigma_q + \sigma) \end{aligned}$$

where the second line uses the optimal conditions for the consumption-wealth ratio (99) for investors A and B. Move the terms related to r_t to one side to solve for r_t, we have

$$r_t = \rho + \psi^{-1}[\mu_t + g_t + \mu_q + \sigma_q\sigma - \eta_t(\sigma_q + \sigma)] - (1-\psi^{-1})\Phi_t.$$

where

$$\begin{aligned} \mu_t &= w\mu_{\xi_A,t} + (1-w)\mu_{\xi_B,t} = O(\epsilon^2), \\ \Phi_{A,t} &= \frac{\gamma\sigma^2}{2} - \gamma\sigma^2(2w-1)\Delta\gamma + \frac{\gamma\sigma^2}{2}\Delta\gamma + O(\epsilon^2) \\ \Phi_{B,t} &= \frac{\gamma\sigma^2}{2} - \gamma\sigma^2(2w-1)\Delta\gamma - \frac{\gamma\sigma^2}{2}\Delta\gamma + O(\epsilon^2) \\ \Phi_t &= w\Phi_{A,t} + (1-w)\Phi_{B,t} \end{aligned}$$

The first equation above is because μ_w is zero at $\epsilon = 0$, which means $\mu_w(w;\epsilon)$ is first-order in ϵ; and derivative $c_{i,w}(w;\epsilon)$ is first-order in ϵ. The drift terms of c and q are second-order in ϵ. Thus,

$$\begin{aligned} r_t(w;\epsilon) = \rho &+ \psi^{-1}g^* - (1+\psi^{-1})\frac{\gamma\sigma^2}{2} \\ &+ \psi^{-1}(g_t - g^*) - \psi^{-1}\sigma(\eta_t - \eta^*) \\ &- (1-\psi^{-1})[w(\Phi_{A,t} - \Phi^*_{A,t}) + (1-w)(\Phi_{B,t} - \Phi^*_{B,t})] + O(\epsilon^2) \\ &= r^* \\ &+ \psi^{-1}(g_t - g^*) - \psi^{-1}\sigma(\eta_t - \eta^*) \\ &- (1-\psi^{-1})[w(\Phi_{A,t} - \Phi^*_{A,t}) + (1-w)(\Phi_{B,t} - \Phi^*_{B,t})] + O(\epsilon^2) \end{aligned}$$

Following the first-order asymptotic expansion, write the price of capital as

$$q = q^* + q_1(w)\epsilon + O(\epsilon^2).$$

where q^* is the equilibrium price of capital from Lemma 1, and q_1 denotes the first-order coefficient of q with respect to ϵ.

From (25), $g = \frac{q-\phi}{\theta} - \delta$. Then growth rate can be written as

$$g = g^* + \frac{q_1}{\theta}\epsilon + O(\epsilon^2)$$

where $g^* = \frac{q^*-\phi}{\theta} - \delta$ is the equilibrium growth rate from Lemma 1.

Thus, the interest rate can be written as

$$r_t = r^* + \psi^{-1}\frac{q_1}{\theta}\epsilon + \frac{\gamma}{2}\sigma^2(1+\psi^{-1})(2w-1)\Delta\gamma + O(\epsilon^2)$$

After substituting the expression for r_t, the consumption-wealth ratios again from (99) can be written as

$$c_A = c^* + (1-\psi)\psi^{-1}\frac{q_1}{\theta}\epsilon + (1-\psi)[\frac{\gamma}{2}\sigma^2(\psi^{-1}-1)(2w-1) + \frac{\gamma\sigma^2}{2}]\Delta\gamma + O(\epsilon^2)$$
$$c_B = c^* + (1-\psi)\psi^{-1}\frac{q_1}{\theta}\epsilon + (1-\psi)[\frac{\gamma}{2}\sigma^2(\psi^{-1}-1)(2w-1) - \frac{\gamma\sigma^2}{2}]\Delta\gamma + O(\epsilon^2)$$

Plug the above expressions into the market clearing condition for consumption goods again, we can obtain an expression for $q_1(w)$ from

$$c^* + (\psi^{-1}-1)\frac{q_1}{\theta}\epsilon + (\psi^{-1}-1)\frac{\gamma}{2}\sigma^2(2w-1)\Delta\gamma + O(\epsilon^2) = \frac{A-\iota(g(q))}{q}$$

Using Lemma 1, $c^* = \frac{A-\iota(g(q^*))}{q^*}$, and $\iota = \frac{q^2-\phi^2}{2\theta}$, the expression for $q_1(w)$ can be simplified to

$$q_1(w) = \frac{(\psi-1)\theta q^*}{q^* + \psi\theta c^*}(2w-1)\frac{\gamma}{2}\sigma^2\delta_\gamma$$

Now all equilibrium objects can be expressed as functions of w and ϵ. For example, the growth rate of the economy can be written as

$$g(w;\epsilon) = g^* + \frac{(\psi-1)q^*}{q^* + \psi\theta c^*}(2w-1)\frac{\gamma}{2}\sigma^2\Delta\gamma + O(\epsilon^2)$$

Note when $\psi > 1$, $q_1(w)$ is increasing in w, i.e., the growth rate is increasing in w.

To complete the derivation, we have expressions for the drift term of w from (17). The second component can be written as

$$\begin{aligned}\mu_w(w;\epsilon) &= [(\sigma_A - \sigma_B)\eta + c_B - c_A] - (\sigma_A - \sigma_B)\sigma_R \\ &= (1 + \psi - \frac{2}{\gamma})\gamma\sigma^2\Delta\gamma + O(\epsilon^2),\end{aligned}$$

and the first component can be written as $\kappa(\bar{u} - w)$. ■

A.11 Lemma 4

This economy is a heterogeneous agent asset pricing model in a production economy with financial frictions. Lemma 3 shows that the frictionless optimal risk exposure σ_A, given by

$$\sigma_A^*(w;\epsilon) = \sigma + 2(1-w)\sigma\Delta\gamma + O(\epsilon^2),$$

is linearly decreasing in w. This implies that as the wealth share of type A investors decreases, their optimal risk exposure increases. Thus, financial constraints will bind for type A investors in equilibrium when their wealth share is sufficiently low.

Thus, in this lemma, on top of Lemma 3, we consider the case where the financial constraint is binding, i.e., the investor's risk exposure is constrained. We derive the equilibrium under this binding constraint. Thus the equilibrium in the presence of heterogeneous agents and financial constraint can be characterized as the following: Lemma 3 is the equilibrium when constraints do not bind, and this lemma is the equilibrium when the constraints bind for type A investors.

Proof. Consider $\sigma_A = \bar{\zeta}$. Following the same arguments that σ_q and σ_{ξ_i} are both of order $O(\epsilon^2)$, as in Lemma 3, we can solve for the risk exposure of type B investors under the binding constraint. Using the market clearing condition for risk exposure, the risk exposure of type B investors can be written as

$$\sigma_B(w;\epsilon) = \frac{\sigma - w\bar{\zeta}}{1-w} + O(\epsilon^2).$$

The market price of risk can be written as

$$\eta_t(w;\epsilon) = \gamma_B \frac{\sigma - w\bar{\zeta}}{1-w} + O(\epsilon^2).$$

We can write the following objects as

$$\begin{aligned}
\Phi_{A,t} &= \gamma_B \frac{(\sigma - w\bar{\zeta})\bar{\zeta}}{1-w} - \frac{\gamma_A \bar{\zeta}^2}{2} + O(\epsilon^2) \\
\Phi_{B,t} &= \frac{\gamma_B}{2}\left(\frac{(\sigma - w\bar{\zeta})}{1-w}\right)^2 + O(\epsilon^2) \\
\Phi_t &= w\Phi_{A,t} + (1-w)\Phi_{B,t}
\end{aligned}$$

Using the market clearing condition for goods, the interest rate can be expressed as

$$r_t = \rho + \psi^{-1}[\mu_t + g_t + \mu_q + \sigma_q\sigma - \eta_t(\sigma_q + \sigma)] - (1 - \psi^{-1})\Phi_t.$$

Using same arguments from Lemma 3, we can write

$$\begin{aligned} r_t(w;\epsilon) &= \rho + \psi^{-1}(g_t - \eta_t\sigma) - (1-\psi^{-1})\Phi_t + O(\epsilon^2) \\ &= \rho + \psi^{-1}g^* - (1+\psi^{-1})\frac{\gamma\sigma^2}{2} \\ &+ \psi^{-1}\frac{q_1}{\theta}\epsilon - \psi^{-1}\eta_t\sigma - (1-\psi^{-1})\Phi_t + (1+\psi^{-1})\frac{\gamma\sigma^2}{2} + O(\epsilon^2) \\ &= r^* \\ &+ \psi^{-1}\frac{q_1}{\theta}\epsilon - \psi^{-1}\eta_t\sigma - (1-\psi^{-1})\Phi_t + (1+\psi^{-1})\frac{\gamma\sigma^2}{2} + O(\epsilon^2) \end{aligned}$$

Putting the above expression for the interest rate back in the firm's HJB, then we can solve for $q_1(w)$ as

$$q_1(w) = q^* \frac{2w^2\bar{\zeta}^2\psi - 2w^2\bar{\zeta}^2 - w\bar{\zeta}^2\psi + w\bar{\zeta}^2 - 2w\bar{\zeta}\sigma\psi + 2w\bar{\zeta}\sigma + \psi\sigma^2 - \sigma^2}{-g^*w\theta\psi + g^*\theta\psi + q^*w - q^* + r^*w\theta\psi - r^*\theta\psi + w\gamma\bar{\zeta}\theta\sigma\psi - \gamma\theta\psi\sigma^2}\theta\frac{\gamma}{2}\delta_\gamma.$$

Other equilibrium objects, including the drift term of w, can be expressed accordingly, as in Lemma 3. ■

B Numerical solution

The computation of equilibrium is reduced to the solution of a system of partial differential equations (PDEs) involving $(\xi_A(x,w), \xi_B(x,w), q_1(x,w), q_2(x,w))$.

The following procedure shows how to obtain a set of conditions involving (ξ_A, ξ_B, q_1, q_2) and its derivatives:

i. Using the optimal investment rate for firms in sector n from (25), I can express ι_n and g_n as functions of q_n.

ii. Using the expression for g_n from above, I can compute μ_x, drift of state variable x from (18). Note that the diffusion terms of x are exogenously given.

iii. Compute the variable z as a function of (q_1, q_2) from (27).

iv. Then I solve for the dynamics of state variable w, in particular the diffusion terms of w. Using the solutions for type A investors' risk exposures (39) from Proposition 5, and considering the margin constraints, I have

$$\sigma_A^{(1)} = \min(\sigma_A^{(1),*}, \bar{\zeta}^{(1)}), \quad \sigma_A^{(2)} = \min(\sigma_A^{(2),*}, \bar{\zeta}^{(2)}).$$

v. Using the market clearing conditions for the risk exposures again, I can compute $\sigma_B^{(1)}$ and $\sigma_B^{(2)}$ as follows:

$$\sigma_B^{(1)} = \frac{z\sigma_1 + Q_x x(1-x)\sigma_1 + [Q_w w(1-w) - w]\sigma_A^{(1)}}{Q_w w(1-w) + 1 - w},$$
$$\sigma_B^{(2)} = \frac{(1-z)\sigma_2 - Q_x x(1-x)\sigma_2 + [Q_w w(1-w) - w]\sigma_A^{(2)}}{Q_w w(1-w) + 1 - w}.$$

vi. Compute $\sigma_w^{(1)}$ and $\sigma_w^{(2)}$ using (17).

vii. Applying Itô's lemma, and using diffusion terms of x and w, compute the following:

$$\sigma_{q,n}^{(1)} = \frac{q_{n,x}}{q_n}\sigma_x^{(1)} + \frac{q_{n,w}}{q_n} w(1-w)\sigma_w^{(1)}; \quad \sigma_{q,n}^{(2)} = \frac{q_{n,x}}{q_n}\sigma_x^{(2)} + \frac{q_{n,w}}{q_n} w(1-w)\sigma_w^{(2)};$$
$$\sigma_{\xi,i}^{(1)} = \frac{\xi_{i,x}}{\xi_i}\sigma_x^{(1)} + \frac{\xi_{i,w}}{\xi_i} w(1-w)\sigma_w^{(1)}; \quad \sigma_{\xi,i}^{(2)} = \frac{\xi_{i,x}}{\xi_i}\sigma_x^{(2)} + \frac{\xi_{i,w}}{\xi_i} w(1-w)\sigma_w^{(2)}.$$

viii. Compute $(\eta^{(1)}, \eta^{(2)})$ using the expression for optimal risk exposures for type B investors from (37):

$$\eta^{(1)} = \gamma_B \sigma_B^{(1)} - \frac{1-\gamma_B}{1-\psi}\sigma_{\xi,B}^{(1)}, \quad \eta^{(2)} = \gamma_B \sigma_B^{(2)} - \frac{1-\gamma_B}{1-\psi}\sigma_{\xi,B}^{(2)}.$$

ix. Compute $(\boldsymbol{\sigma}_{R,1}, \boldsymbol{\sigma}_{R,2})$ by definition (31), which also defines $\boldsymbol{\Sigma}_R$.

x. Compute μ_w from (17), using optimal consumption-wealth ratios $c_i = \xi_i$ from (36) and above expressions.

xi. Applying Itô's lemma, compute $\mu_{q,n}$ for $n \in \{1,2\}$ and $\mu_{\xi,i}$ for $i \in \{A,B\}$ as follows:

$$\begin{aligned}
\mu_{q,n} &= \frac{q_{n,x}}{q_n}\mu_x + \frac{q_{n,w}}{q_n}\left[\kappa(\bar{\mu} - w) + w(1-w)\mu_w\right] \\
&\quad + \frac{1}{2}\frac{q_{n,xx}}{q_n}(\sigma_{x,1}^2 + \sigma_{x,2}^2) + \frac{1}{2}\frac{q_{n,ww}}{q_n}w^2(1-w)^2(\sigma_{w,1}^2 + \sigma_{w,2}^2) \\
&\quad + \frac{q_{n,xw}}{q_n}w(1-w)(\sigma_{x,1}\sigma_{w,1} + \sigma_{x,2}\sigma_{w,2}); \\
\mu_{\xi,i} &= \frac{\xi_{i,x}}{\xi_i}\mu_x + \frac{\xi_{i,w}}{\xi_i}\left[\kappa(\bar{\mu} - w) + w(1-w)\mu_w\right] \\
&\quad + \frac{1}{2}\frac{\xi_{i,xx}}{\xi_i}(\sigma_{x,1}^2 + \sigma_{x,2}^2) + \frac{1}{2}\frac{\xi_{i,ww}}{\xi_i}w^2(1-w)^2(\sigma_{w,1}^2 + \sigma_{w,2}^2) \\
&\quad + \frac{\xi_{i,xw}}{\xi_i}w(1-w)(\sigma_{x,1}\sigma_{w,1} + \sigma_{x,2}\sigma_{w,2}).
\end{aligned}$$

xii. Next I solve for interest rate r. Plugging the optimal consumption-wealth ratio (36) into the investors' HJB (35), I obtain

$$c_{i,t} - \psi\rho - \mu_{\xi_i,t} - (1-\psi)(r_t + \Phi_{i,t}) = 0,$$

where

$$\begin{aligned}
\Phi_{i,t} &= \boldsymbol{\sigma}_{i,t}'\boldsymbol{\eta}_t - \frac{\gamma_i}{2}\boldsymbol{\sigma}_{i,t}'\boldsymbol{\sigma}_{i,t} \\
&\quad + \frac{1}{2}\frac{1}{1-\psi}\left(\frac{1-\gamma_i}{1-\psi} - 1\right)\boldsymbol{\sigma}_{\xi_i,t}'\boldsymbol{\sigma}_{\xi_i,t} + \frac{1-\gamma_i}{1-\psi}\boldsymbol{\sigma}_{i,t}'\boldsymbol{\sigma}_{\xi_i,t}.
\end{aligned}$$

Plugging the optimal investment rate (25) into the firms' HJB (24), I obtain

$$\frac{A_n - \iota_{n,t}}{q_{n,t}} + \Omega_{n,t} - r_t = 0,$$

where

$$\Omega_{n,t} = g_{n,t} + \mu_{q_{n,t}} + \boldsymbol{\sigma}_{q_{n,t}}'\boldsymbol{\sigma}_{n,t} - \boldsymbol{\eta}_t'(\boldsymbol{\sigma}_{q_{n,t}} + \boldsymbol{\sigma}_{n,t}).$$

Using above equations in the goods market clearing condition,

$$wc_A + (1-w)c_B = z\frac{A_1 - \iota_1}{q_1} + (1-z)\frac{A_2 - \iota_2}{q_2}.$$

Rearranging the terms, I obtain the expression for the interest rate

$$r_t = \rho + \frac{1}{\psi}\mu_t + \left(\frac{1}{\psi} - 1\right)\Phi_t + \frac{1}{\psi}\Omega_t$$

where $\mu_t = w_t\mu_{\xi_A,t} + (1-w_t)\mu_{\xi_B,t}$, $\Phi_t = w_t\Phi_{A,t} + (1-w_t)\Phi_{B,t}$, and $\Omega_t = z\Omega_{1,t} + (1-z)\Omega_{2,t}$.

xiii. Plug all necessary equations into HJBs (24) and (35) to obtain the system of PDEs.

C Data

C.1 Calibration details

To construct empirical counterparts to the model sectors, I use industry-level real-side data. The model distinguishes the growth sector from the value sector along two primitive dimensions: productivity and fundamental risk. I therefore classify industries using these same objects rather than imposing a classification based solely on asset-market characteristics. For each industry j, I measure average productivity by value added per unit of capital and fundamental risk by the volatility of real value-added growth. Specifically, I define

$$A_j = \frac{\sum_t Y_{j,t}}{\sum_t K_{j,t}},$$

where $Y_{j,t}$ denotes nominal value added and $K_{j,t}$ denotes the current-cost net stock of fixed assets. I measure fundamental risk as

$$\sigma_j = \mathrm{sd}_t\left(\Delta \log Q_{j,t}\right),$$

where $Q_{j,t}$ is the chain-type quantity index for real value added. I then form a standardized industry score,

$$Score_j = \frac{A_j - \mu_A}{\sigma_A} + \frac{\sigma_j - \mu_\sigma}{\sigma_\sigma},$$

where μ_A and σ_A denote the cross-industry mean and standard deviation of A_j, and μ_σ and σ_σ denote the cross-industry mean and standard deviation of σ_j. Industries with high values of $Score_j$ are more growth-like in the sense of the model: they have both higher measured productivity and higher fundamental risk.

The baseline classification uses a smooth split rather than assigning each industry entirely to one sector. I define the growth-sector share of industry j as

$$p_j^G = \frac{1}{1 + \exp\left(-\lambda Score_j\right)},$$

and the value-sector share as

$$p_j^V = 1 - p_j^G.$$

This procedure allows each industry to contain both growth-like and value-like capital, while ensuring that the two constructed sectors exhaust aggregate capital. Sectoral aggregates are then obtained by weighting BEA industry-level variables by p_j^G and p_j^V. For example, sec-

toral capital is given by

$$K_t^G = \sum_j p_j^G K_{j,t}, \qquad K_t^V = \sum_j p_j^V K_{j,t}.$$

Similarly, sectoral investment, depreciation, and nominal value added are constructed as

$$I_t^G = \sum_j p_j^G I_{j,t}, \qquad I_t^V = \sum_j p_j^V I_{j,t}.$$

$$D_t^G = \sum_j p_j^G D_{j,t}, \qquad D_t^V = \sum_j p_j^V D_{j,t}.$$

$$Y_t^G = \sum_j p_j^G Y_{j,t}, \qquad Y_t^V = \sum_j p_j^V Y_{j,t}.$$

The empirical counterpart of the model state variable is the share of capital assigned to the growth sector,

$$x_t = \frac{K_t^G}{K_t^G + K_t^V}.$$

I use the resulting sectoral aggregates to compute the calibration targets. In particular, the investment-output ratio, capital-output ratio, average product of capital, and depreciation rate for sector $s \in \{G, V\}$ are

$$\frac{I_s}{Y_s} = \frac{\sum_t I_t^s}{\sum_t Y_t^s},$$

$$\frac{K_s}{Y_s} = \frac{\sum_t K_t^s}{\sum_t Y_t^s},$$

$$A_s = \frac{\sum_t Y_t^s}{\sum_t K_t^s},$$

and

$$\delta_s = \frac{\sum_t D_t^s}{\sum_t K_t^s}.$$

To measure sectoral real output growth, I aggregate industry-level real value-added growth using lagged nominal value-added weights. For sector $s \in \{G, V\}$,

$$g_{s,t}^Y = \sum_j \omega_{j,t-1}^s \Delta \log Q_{j,t},$$

where

$$\omega^{s}_{j,t-1} = \frac{p^{s}_{j} Y_{j,t-1}}{\sum_{k} p^{s}_{k} Y_{k,t-1}}.$$

I then compute the mean and volatility as moments used to discipline the sectoral growth and risk parameters.

I also consider alternative classifications to assess the robustness of the empirical mapping by Duarte (2018) and Kozak (2022). First, following the spirit of capital-type classifications, I construct a BEA asset-type measure that assigns equipment and intellectual-property capital to the high-risk technology sector and structures to the low-risk sector. This measure has the advantage of covering the aggregate economy, but it is better interpreted as a technology-capital share than as a literal growth-value split. Second, I construct a firm-level measure using CRSP/Compustat by sorting firms on book-to-market and aggregating firm-level capital within growth and value portfolios. This measure is closer to the conventional asset-pricing definition of growth and value, but it covers only publicly traded firms and depends on accounting definitions of firm capital. For these reasons, I use the model-implied BEA industry classification as the baseline and treat the capital-type and CRSP/Compustat classifications as robustness checks.

D Global solutions of alternative economies

D.1 Solutions of heterogeneous investors model with one sector

This section describes alternative economies that feature heterogeneous investors, and their solutions. They differ from the main model in that the investors have access to one risky asset: a claim to a Lucas 'tree' or a production firm. The demographics and preferences of investors are the same as in the main model.

Consumption-Based Model This section describes the consumption-based model with heterogeneous investors and one Lucas 'tree', and its solution.

Consumption is assumed to follow a standard geometric Brownian motion.

$$\frac{dC_t}{C_t} = \mu dt + \sigma dB_t, \tag{102}$$

where μ and σ are constants.

There is one state variable: the wealth share of intermediaries w. Denote the dividend yield of the claim to the Lucas 'tree' as F_t. The return of the risky asset can be written as

$$dR_t \equiv \frac{d(D/F)}{D/F} + F dt = \mu_R dt + \sigma_R dB_t, \tag{103}$$

where μ_R and σ_R denote the expected return and volatility of the risky asset, respectively. Applying Itô's lemma, we can express μ_R and σ_R as follows:

$$\mu_R = \mu + F - \mu_F - \sigma\sigma_F + \sigma_F^2, \tag{104}$$

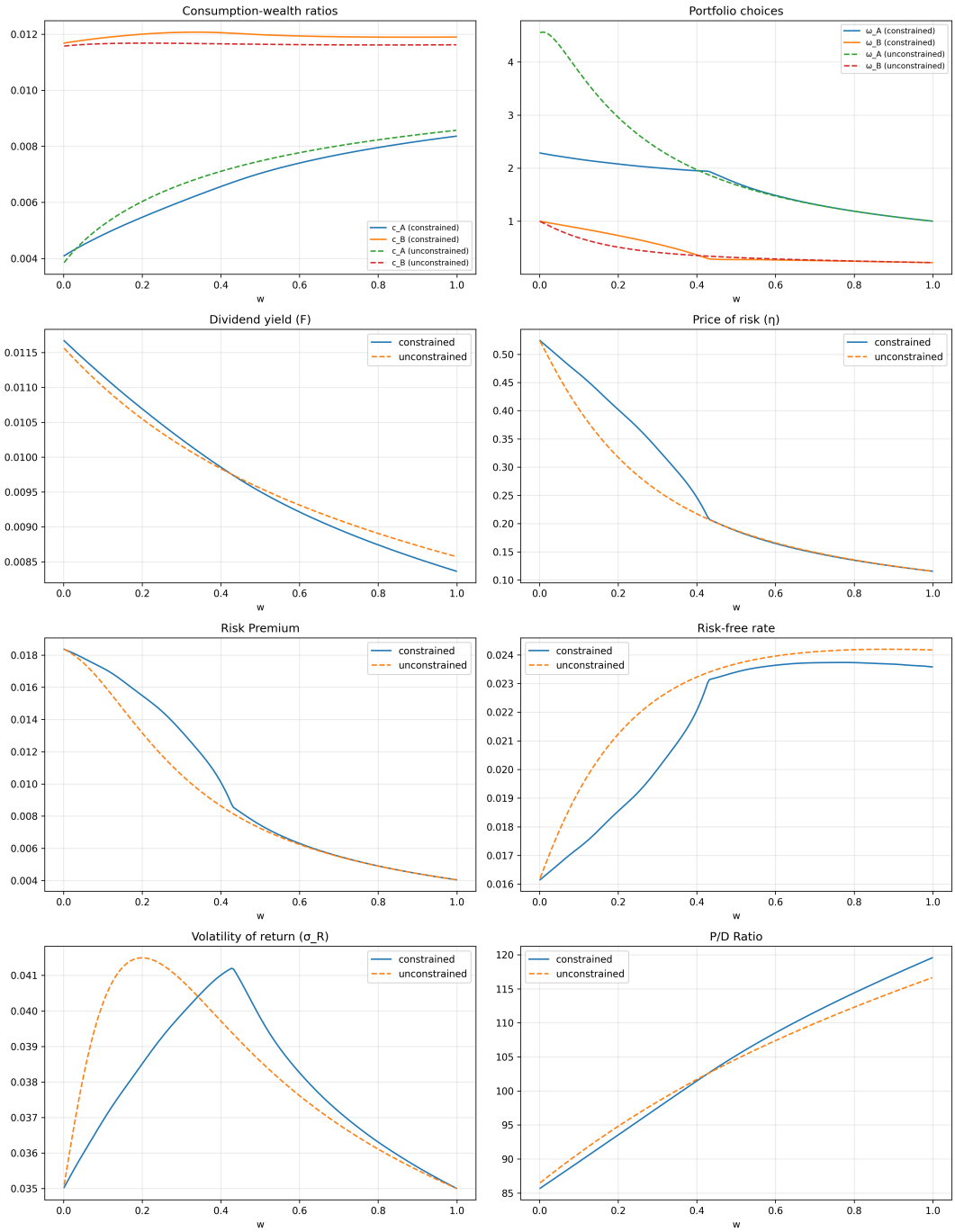
$$\sigma_R = \sigma - \sigma_F. \tag{105}$$

The equilibrium is characterized by two HJB equations for the two types of investors, the PDE that F_t satisfies, along with the market clearing conditions. The numerical solution is similar to that in the main model, except that there is only one state variable w. The calibration follows Kargar (2021) closely.

Figure D.1 shows the equilibrium results when the financial constraint is not binding for intermediaries, together with the results from the unconstrained economy for comparison. The solid lines represent the equilibrium with financial constraints, while the dashed lines represent the frictionless equilibrium. The panels on the first row display the consumption-wealth ratios $c_{i,t}$ and portfolio choices $\omega_{i,t}$ for investors of type i. The rest panels show the dividend yield F_t, price of risk η_t,risk premium $\lambda_{R,t}$, risk-free rate r_t, return volatility $\sigma_{R,t}$, and

price-dividend ratio $1/F_t$. Each quantity is plotted against w_t, the wealth share of intermediaries, which ranges from 0 to 1.

Figure D.1: Solutions of the consumption-based model



This figure shows the consumption-based model solution using the calibrated parameters from Kargar (2021). The solid lines represent the equilibrium with financial constraints, while the dashed lines represent the unconstrained equilibrium. The panels on the first row display the consumption-wealth ratios and portfolio choices for both types of investors. The rest panels show the dividend yield, price of risk,risk premium, risk-free rate, return volatility, and price-dividend ratio. Each quantity is plotted against w_t, the wealth share of intermediaries.

The frictionless economy is similar to Gârleanu and Panageas (2015).

The economy with financial constraints is similar to Kargar (2021), but with representative financial intermediaries.

Production-Based Model This section describes the production-based model with heterogeneous investors and one representative firm, and its solution.

The representative firm produces output using capital, and the same technology as the firms in the main model. The firm's capital stock evolves according to

$$\frac{dK_t}{K_t} = g_t dt + \sigma dB_t, \tag{106}$$

where σ is a constant, and g_t denotes the expected growth rate of capital at time t.

The equilibrium is characterized by three HJB equations for the firm and two types of investors, along with the market clearing conditions. The numerical solution is similar to that in the main model, except that there is only one state variable w. The calibration follows Silva (2020) closely.

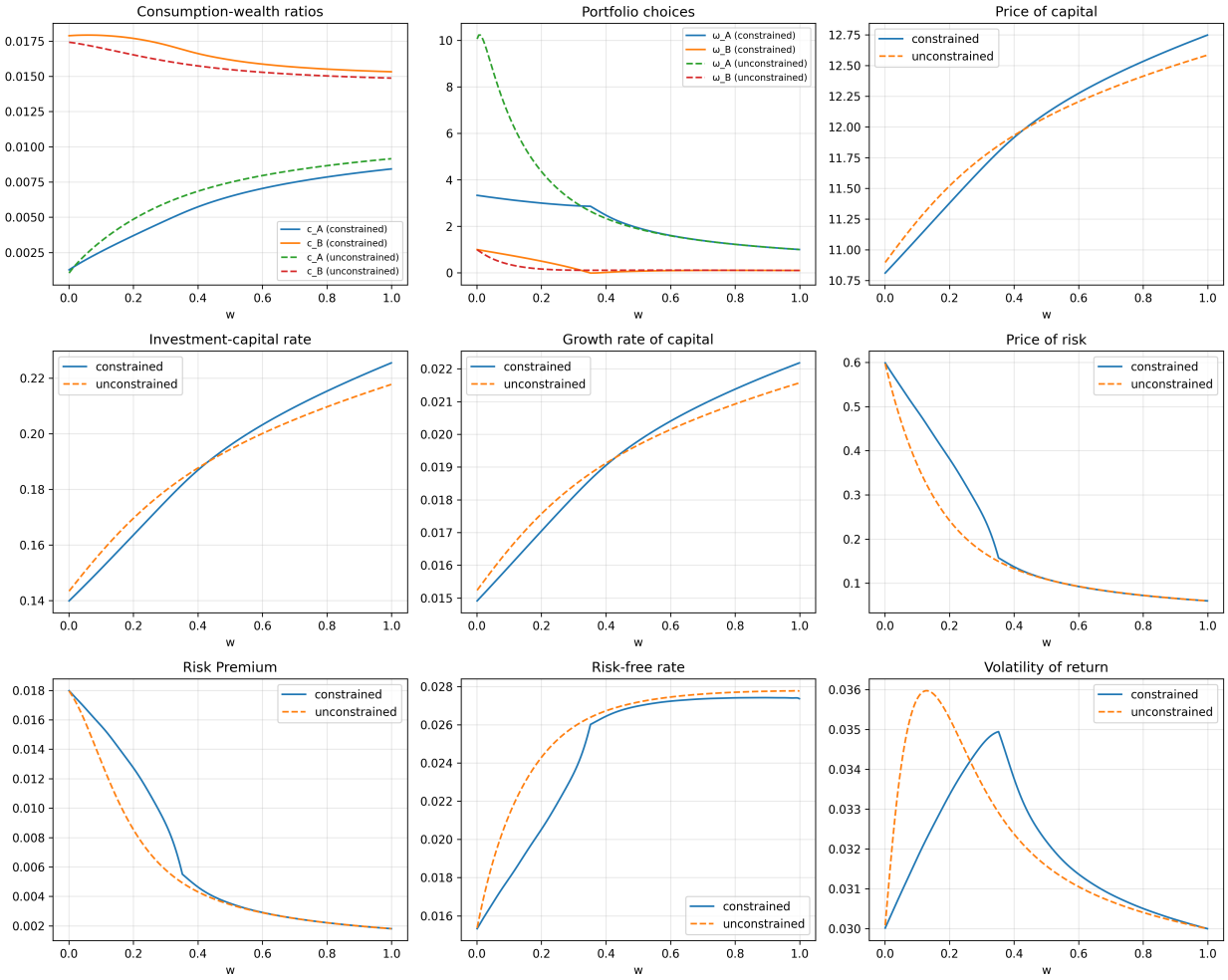
Figure D.2 shows the equilibrium results when the financial constraint is not binding for intermediaries, together with the results from the unconstrained economy for comparison.

D.2 Two-Sector Model

This section describes the economy from Eberly and Wang (2025) that features two sectors, and its solution. It differs from the main model in that there is a representative investor who has access to two risky assets: claims to dividends from firms in two sectors. The preference of investor is the same as that from Eberly and Wang (2025).

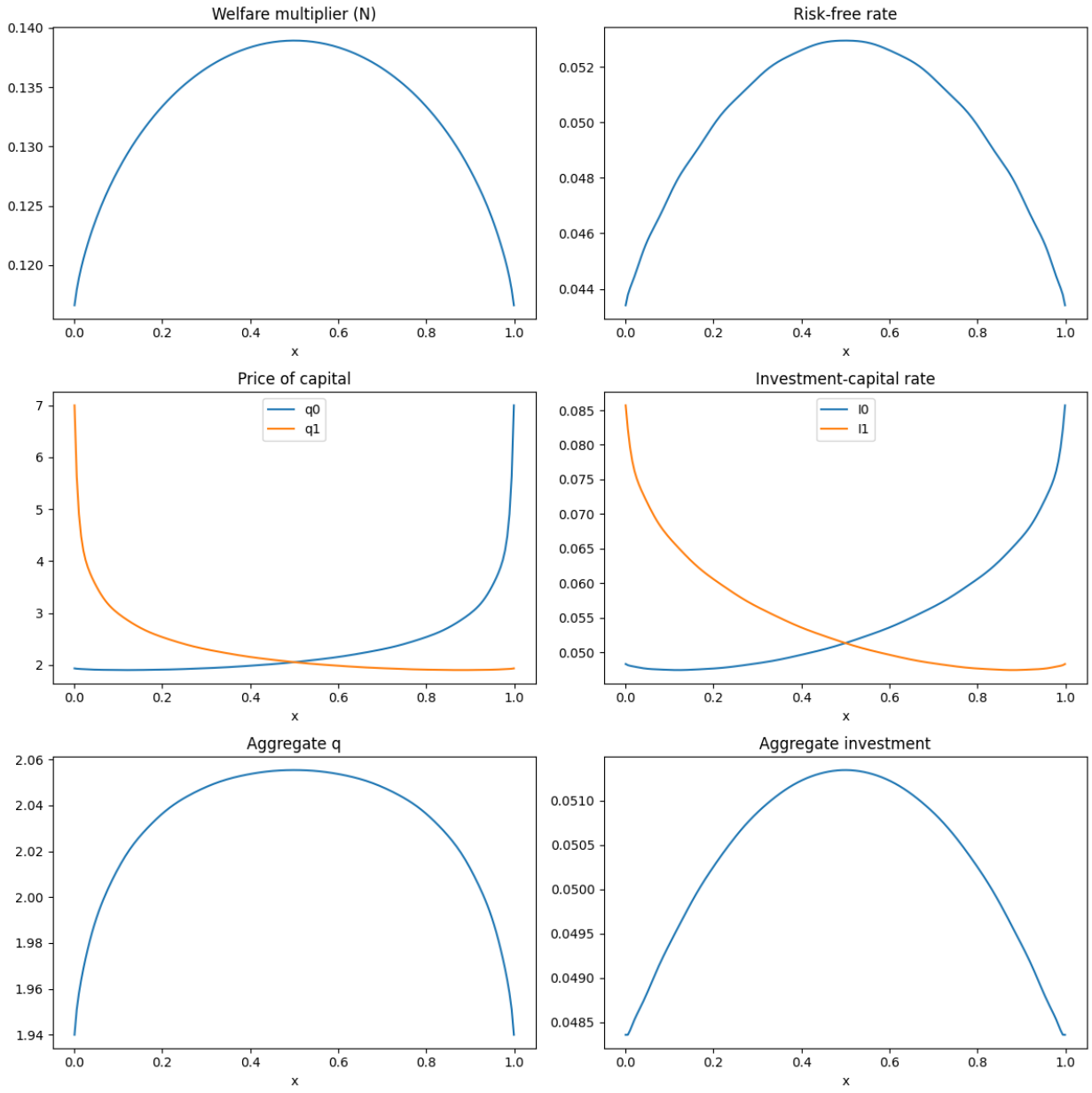
Due to the presence of two sectors and one representative investor, the model has one state variable: the capital share of sector 1, denoted as x. Here I follow Eberly and Wang (2025) and solve the model by invoking the welfare theorem. The social planner's problem can be characterized by a single HJB equation, and N denotes the welfare multiplier. Figure D.3 shows the equilibrium results when the two sectors are symmetric. The panels on the first row display the welfare multiplier of the representative investor and equilibrium risk-free rate r_t. The second row shows the prices of capital q and investment-capital rates for two sectors. The bottom row shows the aggregate price of capital and investment-capital rate. Each quantity is plotted against x_t, the capital share of sector 1, which ranges from 0 to 1.

Figure D.2: Solutions of the production-based model



This figure shows the production-based model solution using the calibrated parameters from Silva (2020). The solid lines represent the equilibrium with financial constraints, while the dashed lines represent the unconstrained equilibrium. The top and mid panels on the first row display the consumption-wealth ratios and portfolio choices for both types of investors. The rest panels show the price of capital, investment rate, growth rate of capital, price of risk,risk premium, risk-free rate, and return volatility. Each quantity is plotted against w_t, the wealth share of intermediaries.

Figure D.3: Solutions of the two-sector model



This figure shows the two-sector model solution using the calibrated parameters from Eberly and Wang (2025). The panels on the first row display the welfare multiplier of the representative investor and equilibrium risk-free rate r_t. The second row shows the prices of capital q and investment-capital rates for two sectors. The bottom row shows the aggregate price of capital and investment-capital rate. Each quantity is plotted against x_t, the capital share of sector 1.