Supplement of "Errata on the Impact of Jumps in Volatility and Returns"

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Abstract

We correct four places in Table I of the paper "Eraker B, Johannes M, Polson N (2003). The impact of jumps in volatility and returns. Journal of Finance, 58, 1269–1300." by two methods. The first method is based on the derivatives of the moment generating functions. The second method is by elementary and straightforward calculations of the variance and the covariance.

Keywords: conditional moments, SV, SVJ, SVCJ, SVIJ.

1. Model Assumptions

We assume that the logarithm of asset's price, $Y_t = \log(S_t)$, solves

$$\begin{pmatrix} dY_t \\ dV_t \end{pmatrix} = \begin{pmatrix} \mu \\ \kappa(\theta - V_{t-}) \end{pmatrix} dt + \sqrt{V_{t-}} \begin{pmatrix} 1 & 0 \\ \rho\sigma_v & \sqrt{1-\rho^2}\sigma_v \end{pmatrix} dW_t + \begin{pmatrix} \xi^y dN_t^y \\ \xi^v dN_t^v \end{pmatrix}. \tag{1.1}$$

Write (1.1) to two equations, we obtain

$$\begin{cases} dY_t = \mu dt + \sqrt{V_{t-}} dW_t^y + \xi^y dN_t^y, \\ dV_t = \kappa(\theta - V_{t-}) dt + \sigma_v \sqrt{V_{t-}} \left(\rho dW_t^y + \sqrt{1-\rho^2} dW_t^v \right) + \xi^v dN_t^v, \end{cases}$$

where $V_{t-} = \lim_{s \uparrow t} V_s$, $W_t = \begin{pmatrix} W_t^y \\ W_t^v \end{pmatrix} \in \mathbb{R}^2$ is a standard Brownian motion,

$$\begin{cases} dW_t^y = W_{t+dt}^y - W_t^y \sim N(0, dt), \\ dW_t^v = W_{t+dt}^v - W_t^v \sim N(0, dt), \\ E(dW_t^y) = E(dW_t^v) = 0, \\ Var(dW_t^y) = Var(dW_t^v) = dt, \\ Cov(dW_t^y, dW_t^v) = 0, \end{cases} \tag{1.2}$$

N_t^y and N_t^v are Poisson processes with constant intensities λ_y and λ_v,

$$\begin{aligned} N_t^y &\sim P(\lambda_y t), \; N_t^v \sim P(\lambda_v t), \\ dN_t^y &= N_{t+dt}^y - N_t^y \sim P(\lambda_y dt), \\ dN_t^v &= N_{t+dt}^v - N_t^v \sim P(\lambda_v dt), \end{aligned}$$

and ξ^y and ξ^y are the jump sizes in returns and volatility, respectively. We assume that the parameters and initial conditions have sufficient regularity for the solution of (1.1) to be well defined.

This specification nests many of the popular models used for option pricing and portfolio allocation applications. Without jumps, $\lambda_y = \lambda_v = 0$, (1.1) reduces to the square-root stochastic volatility model, the SV model (Heston (1993)). The SVJ model (Bates (1996)) has normally distributed jumps in returns, $\xi^y \sim N\left(\mu_y, \sigma_y^2\right)$, but no jumps in volatility, $\lambda_v = 0$. Duffie *et al.* (2000) introduced the models with jumps in volatility.

The SVIJ model has independently arriving jumps in volatility, $\xi^v \sim \exp(\mu_v)$, and jumps in returns, $\xi^y \sim N\left(\mu_y, \sigma_y^2\right)$. Moreover, dW_t^y, ξ^y, dN_t^y, dW_t^v, ξ^v, dN_t^v are mutually independent.

The SVCJ model has contemporaneous arrivals, $N_t^y = N_t^v = N_t$,

$$dN_t^y = dN_t^v = dN_t = N_{t+dt}^y - N_t^y \sim P(\lambda_y dt) = P(\lambda_v dt),$$

and correlated jump sizes, $\xi^v \sim \exp(\mu_v)$, and $\xi^y|\xi^v \sim N\left(\mu_y + \rho_J \xi^v, \sigma_y^2\right)$. The five quantities dW_t^y, ξ^y, dW_t^v, ξ^v, dN_t are mutually independent except that ξ^y and ξ^v have some relationship, as depicted below.

$$\begin{array}{ccc} dW_t^y & \xi^y & \\ & | & dN_t \\ dW_t^v & \xi^v & \end{array}$$

Therefore,

$$\begin{cases} E\xi^v = \mu_v,\ Var(\xi^v) = \mu_v^2,\ E\left[(\xi^v)^2\right] = (E\xi^v)^2 + Var(\xi^v) = \mu_v^2 + \mu_v^2 = 2\mu_v^2, \\ E(\xi^y|\xi^v) = \mu_y + \rho_J \xi^v,\ Var(\xi^y|\xi^v) = \sigma_y^2,\ E\left[(\xi^y)^2|\xi^v\right] = [E(\xi^y|\xi^v)]^2 + Var(\xi^y|\xi^v) = (\mu_y + \rho_J\xi^v)^2 + \sigma_y^2, \\ E(dN_t^y) = \lambda_y dt,\ Var(dN_t^y) = \lambda_y dt,\ E\left[(dN_t^y)^2\right] = [E(dN_t^y)]^2 + Var(dN_t^y) = (\lambda_y dt)^2 + \lambda_y dt = \lambda_y dt, \\ E(dN_t^v) = \lambda_v dt,\ Var(dN_t^v) = \lambda_v dt,\ E\left[(dN_t^v)^2\right] = [E(dN_t^v)]^2 + Var(dN_t^v) = (\lambda_v dt)^2 + \lambda_v dt = \lambda_v dt, \\ E(dN_t) = E(dN_t^y) = \lambda_y dt,\ Var(dN_t) = Var(dN_t^y) = \lambda_y dt,\ E\left[(dN_t)^2\right] = E\left[(dN_t^y)^2\right] = \lambda_y dt. \end{cases} \tag{1.3}$$

2. Conditional Moments

Table 1 provides the instantaneous variance and covariance of Y_t and V_t for each of the models. In the table, there are four places which are different from those in Eraker *et al.* (2003).

Now we calculate the conditional moments of the four models in Table 1. We provide two alternative methods to calculate the conditional moments. The first method is based on the derivatives of the moment generating functions. See Casella and Berger (2002) for more about univariate moment generating functions. See also Sato (1999) for properties of the multivariate characteristic functions. When the moment generating function exists, it can be used to generate moments, just as the characteristic function can. The second method is by elementary and straightforward calculations of the variance and the covariance.

Table 1: **Conditional Moments.** This table summarizes the instananeous conditional moments for the four models under consideration. In the case of the SVCJ model, the second moment of the jump sizes is $E\left[(\xi^y)^2\right] = \mu_y^2 + 2\mu_y\mu_v\rho_J + 2\rho_J^2\mu_v^2 + \sigma_y^2$ (a).

	SV	SVJ	SVCJ	SVIJ
$\frac{1}{dt}Var\left(dY_t\right)$	V_t	$V_t + \lambda_y\left(\mu_y^2 + \sigma_y^2\right)$	$V_t + \lambda_y E\left[(\xi^y)^2\right]$	$V_t + \lambda_y\left(\mu_y^2 + \sigma_y^2\right)$
$\frac{1}{dt}Var\left(dV_t\right)$	$\sigma_v^2 V_t$	$\sigma_v^2 V_t$	$\sigma_v^2 V_t + 2\mu_v^2\lambda_v$ (b)	$\sigma_v^2 V_t + 2\mu_v^2\lambda_v$ (d)
$\frac{1}{dt}Cov\left(dY_t, dV_t\right)$	$\rho\sigma_v V_t$	$\rho\sigma_v V_t$	$\rho\sigma_v V_t + \lambda_y\left(\mu_y\mu_v + 2\rho_J\mu_v^2\right)$ (c)	$\rho\sigma_v V_t$

2.1. Derivatives of the moment generating functions

Since the SV model and the SVJ model are special cases of the SVCJ model or the SVIJ model, we only derive the moment generating functions of the SVCJ model and the SVIJ model.

- The SVCJ model

The moment generating function of $\begin{pmatrix} dY_t \\ dV_t \end{pmatrix}$ for the SVCJ model is given by

$$
\begin{aligned}
& M_{\begin{pmatrix} dY_t \\ dV_t \end{pmatrix}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\
&= E\exp\left[\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \cdot \begin{pmatrix} dY_t \\ dV_t \end{pmatrix}\right] \\
&= E\exp\left(c_1 dY_t + c_2 dV_t\right) \\
&= E\exp\left\{c_1\left(\mu dt + \sqrt{V_t}dW_t^y + \xi^y dN_t\right) + c_2\left[\kappa\left(\theta - V_t\right)dt + \sigma_v\sqrt{V_t}\left(\rho dW_t^y + \sqrt{1-\rho^2}dW_t^v\right) + \xi^v dN_t\right]\right\} \\
&= E\exp\left\{\begin{array}{c} \left[c_1\mu + c_2\kappa\left(\theta - V_t\right)\right]dt + \left(c_1\sqrt{V_t} + c_2\sigma_v\rho\sqrt{V_t}\right)dW_t^y \\ +c_2\sigma_v\sqrt{1-\rho^2}\sqrt{V_t}dW_t^v + \left(c_1\xi^y + c_2\xi^v\right)dN_t \end{array}\right\}.
\end{aligned}
$$

Let $A = \exp\left\{\left[c_1\mu + c_2\kappa\left(\theta - V_t\right)\right]dt\right\}$, then

$$
\begin{aligned}
& M_{\left(\begin{array}{c} dY_t \\ dV_t \end{array}\right)}\left(\begin{array}{c} c_1 \\ c_2 \end{array}\right) \\
& = A \cdot E\left\{\exp\left[\left(c_1\sqrt{V_t} + c_2\sigma_v\rho\sqrt{V_t}\right) dW_t^y\right] \cdot \exp\left[c_2\sigma_v\sqrt{1-\rho^2}\sqrt{V_t}dW_t^v\right] \cdot \exp\left[\left(c_1\xi^y + c_2\xi^v\right) dN_t\right]\right\} \\
& = A \cdot E\left\{\exp\left[\left(c_1\sqrt{V_t} + c_2\sigma_v\rho\sqrt{V_t}\right) dW_t^y\right]\right\} \\
& \quad \cdot E\left\{\exp\left(c_2\sigma_v\sqrt{1-\rho^2}\sqrt{V_t}dW_t^v\right)\right\} \cdot E\left\{\exp\left[\left(c_1\xi^y + c_2\xi^v\right) dN_t\right]\right\} \\
& = A \cdot M_{dW_t^y}\left(c_1\sqrt{V_t} + c_2\sigma_v\rho\sqrt{V_t}\right) \cdot M_{dW_t^v}\left(c_2\sigma_v\sqrt{1-\rho^2}\sqrt{V_t}\right) \cdot E\left\{\exp\left[\left(c_1\xi^y + c_2\xi^v\right) dN_t\right]\right\} \\
& = A \cdot M_1 \cdot M_2 \cdot M_3,
\end{aligned}
$$

the second equality above holds by independence, where

$$
\begin{aligned}
M_1 &= M_{dW_t^y}\left(c_1\sqrt{V_t} + c_2\sigma_v\rho\sqrt{V_t}\right), \\
M_2 &= M_{dW_t^v}\left(c_2\sigma_v\sqrt{1-\rho^2}\sqrt{V_t}\right), \\
M_3 &= E\left\{\exp\left[\left(c_1\xi^y + c_2\xi^v\right) dN_t\right]\right\}.
\end{aligned}
$$

We only need to compute A, M_1, M_2, and M_3. In the following, we will frequently use the Taylor expansion of e^x,

$$
e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots.
$$

First,

$$
\begin{aligned}
A &= \exp\left\{\left[c_1\mu + c_2\kappa\left(\theta - V_t\right)\right] dt\right\} \\
&= 1 + \left[c_1\mu + c_2\kappa\left(\theta - V_t\right)\right] dt + \frac{\left\{\left[c_1\mu + c_2\kappa\left(\theta - V_t\right)\right] dt\right\}^2}{2!} + \cdots \\
&= 1 + \left[c_1\mu + c_2\kappa\left(\theta - V_t\right)\right] dt. \; \left(\, (dt)^k = 0, k \geq 2\right)
\end{aligned}
$$

For a normal random variable $X \sim N\left(\mu, \sigma^2\right)$, its moment generating function is (Casella and Berger (2002))

$$
M_X(t) = Ee^{tX} = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right).
$$

By (1.2), we have $dW_t^y \sim N(0, dt)$, thus

$$
\begin{aligned}
M_1 &= M_{dW_t^y}\left(c_1\sqrt{V_t} + c_2\sigma_v\rho\sqrt{V_t}\right) \\
&= \exp\left\{0 \cdot \left[\left(c_1 + c_2\sigma_v\rho\right)\sqrt{V_t}\right] + \frac{1}{2}dt\left[\left(c_1 + c_2\sigma_v\rho\right)\sqrt{V_t}\right]^2\right\} \\
&= \exp\left[\frac{1}{2}\left(c_1 + c_2\sigma_v\rho\right)^2 V_t dt\right] \\
&= 1 + \frac{1}{2}\left(c_1 + c_2\sigma_v\rho\right)^2 V_t dt.
\end{aligned}
$$

By (1.2), we have $dW_t^v \sim N(0, dt)$, thus

$$\begin{aligned} M_2 &= M_{dW_t^v}\left(c_2\sigma_v\sqrt{1-\rho^2}\sqrt{V_t}\right) \\ &= \exp\left\{0\cdot\left(c_2\sigma_v\sqrt{1-\rho^2}\sqrt{V_t}\right) + \frac{1}{2}dt\left(c_2\sigma_v\sqrt{1-\rho^2}\sqrt{V_t}\right)^2\right\} \\ &= \exp\left\{\frac{1}{2}c_2^2\sigma_v^2\left(1-\rho^2\right)V_t dt\right\} \\ &= 1 + \frac{1}{2}c_2^2\sigma_v^2\left(1-\rho^2\right)V_t dt. \end{aligned}$$

Now turns to the hard part, the calculation of M_3.

$$\begin{aligned} M_3 &= E\left\{\exp\left[\left(c_1\xi^y + c_2\xi^v\right)dN_t\right]\right\} \\ &= E\left[\exp\left(c_1 dN_t\xi^y + c_2 dN_t\xi^v\right)\right] \\ &= E\left\{E\left[\exp\left(c_1 dN_t\xi^y + c_2 dN_t\xi^v\right)|dN_t\right]\right\} \\ &= E\left\{E\left[E\left(\exp\left(c_1 dN_t\xi^y + c_2 dN_t\xi^v\right)|\xi^v\right)|dN_t\right]\right\}. \end{aligned}$$

We first compute the innerest expectation. Given dN_t, then

$$\begin{aligned} E\left(\exp\left(c_1 dN_t\xi^y + c_2 dN_t\xi^v\right)|\xi^v\right) &= \exp\left(c_2 dN_t\xi^v\right)\cdot E\left\{\exp\left[c_1 dN_t\left(\xi^y|\xi^v\right)\right]\right\} \\ &= \exp\left(c_2 dN_t\xi^v\right)\cdot M_{\xi^y|\xi^v}\left(c_1 dN_t\right). \end{aligned}$$

Since $\xi^y|\xi^v \sim N\left(\mu_y + \rho_J\xi^v, \sigma_y^2\right)$, then

$$\begin{aligned} &E\left(\exp\left(c_1 dN_t\xi^y + c_2 dN_t\xi^v\right)|\xi^v\right) \\ &= \exp\left(c_2 dN_t\xi^v\right)\cdot\exp\left[\left(\mu_y + \rho_J\xi^v\right)c_1 dN_t + \frac{1}{2}\sigma_y^2 c_1^2\left(dN_t\right)^2\right] \\ &= \exp\left[\left(c_2 + c_1\rho_J\right)dN_t\xi^v + c_1\mu_y dN_t + \frac{1}{2}\sigma_y^2 c_1^2\left(dN_t\right)^2\right]. \end{aligned}$$

Note the exponent in the last equality of the above expression is a function of ξ^v. Substitute this expectation into M_3, we obtain

$$\begin{aligned} M_3 &= E\left\{E\left[\exp\left[\left(c_2 + c_1\rho_J\right)dN_t\xi^v + c_1\mu_y dN_t + \frac{1}{2}\sigma_y^2 c_1^2\left(dN_t\right)^2\right]|dN_t\right]\right\} \\ &= E\left\{\exp\left[c_1\mu_y dN_t + \frac{1}{2}\sigma_y^2 c_1^2\left(dN_t\right)^2\right]\cdot E\left\{\exp\left[\left(c_2 + c_1\rho_J\right)dN_t\xi^v\right]|dN_t\right\}\right\} \\ &= E\left\{\exp\left[c_1\mu_y dN_t + \frac{1}{2}\sigma_y^2 c_1^2\left(dN_t\right)^2\right]\cdot M_{\xi^v}\left(\left(c_2 + c_1\rho_J\right)dN_t\right)\right\}. \end{aligned}$$

For an exponential random variable $X \sim \exp(\beta)$, its moment generating function is (Casella and Berger (2002))

$$M_X(t) = Ee^{tX} = \frac{1}{1-\beta t}.$$

Since $\xi^v \sim \exp(\mu_v)$, then

$$M_{\xi^v}\left(\left(c_2 + c_1\rho_J\right)dN_t\right) = \frac{1}{1-\mu_v\left(c_2 + c_1\rho_J\right)dN_t}.$$

Thus,

$$M_3 = E\left\{ \frac{\exp\left[c_1\mu_y dN_t + \frac{1}{2}\sigma_y^2 c_1^2 \left(dN_t\right)^2\right]}{1-\mu_v\left(c_2+c_1\rho_J\right)dN_t} \right\}.$$

We know that $dN_t \sim P\left(\lambda_y dt\right)$. When dt is small, we will show that $dN_t \sim \text{Bernoulli}\left(\lambda_y dt\right)$. Remember that when dt is small, $\left(dt\right)^k = 0$, $k \geq 2$. Thus,

$$P\left(dN_t = k\right) = \frac{\left(\lambda_y dt\right)^k}{k!} e^{-\lambda_y dt},\ k = 0, 1, \cdots,$$

$$\begin{aligned} P\left(dN_t = 0\right) &= \frac{\left(\lambda_y dt\right)^0}{0!} e^{-\lambda_y dt} \\ &= 1 \cdot \left(1 - \lambda_y dt\right) \\ &= 1 - \lambda_y dt, \end{aligned}$$

$$\begin{aligned} P\left(dN_t = 1\right) &= \frac{\left(\lambda_y dt\right)^1}{1!} e^{-\lambda_y dt} \\ &= \lambda_y dt \left(1 - \lambda_y dt\right) \\ &= \lambda_y dt, \end{aligned}$$

when $k \geq 2$,

$$P\left(dN_t = k\right) = \frac{\left(\lambda_y dt\right)^k}{k!} e^{-\lambda_y dt} = 0.$$

Consequently, $dN_t \sim \text{Bernoulli}\left(\lambda_y dt\right)$. Now let us compute M_3. Let

$$g\left(dN_t\right) = \frac{\exp\left[c_1\mu_y dN_t + \frac{1}{2}\sigma_y^2 c_1^2 \left(dN_t\right)^2\right]}{1-\mu_v\left(c_2+c_1\rho_J\right)dN_t}.$$

Then,

$$\begin{aligned} M_3 &= E\left[g\left(dN_t\right)\right] \\ &= g\left(1\right) \cdot P\left(dN_t = 1\right) + g\left(0\right) \cdot P\left(dN_t = 0\right) \\ &= \frac{\exp\left(c_1\mu_y + \frac{1}{2}\sigma_y^2 c_1^2\right)}{1-\mu_v\left(c_2+c_1\rho_J\right)} \cdot \lambda_y dt + \frac{\exp\left(0\right)}{1-0} \cdot \left(1 - \lambda_y dt\right) \\ &= 1 + \left(\frac{\exp\left(c_1\mu_y + \frac{1}{2}\sigma_y^2 c_1^2\right)}{1-\mu_v\left(c_2+c_1\rho_J\right)} - 1\right) \lambda_y dt. \end{aligned}$$

Therefore,

$$
\begin{aligned}
& M_{\begin{pmatrix} dY_t \\ dV_t \end{pmatrix}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\
&= A \cdot M_1 \cdot M_2 \cdot M_3 \\
&= \{1 + [c_1\mu + c_2\kappa(\theta - V_t)]\, dt\} \cdot \left[1 + \frac{1}{2}(c_1 + c_2\sigma_v\rho)^2 V_t dt\right] \cdot \left[1 + \frac{1}{2}c_2^2\sigma_v^2\left(1-\rho^2\right) V_t dt\right] \\
&\quad \cdot \left[1 + \left(\frac{\exp\left(c_1\mu_y + \frac{1}{2}\sigma_y^2 c_1^2\right)}{1 - \mu_v(c_2 + c_1\rho_J)} - 1\right)\lambda_y dt\right] \\
&= 1 + \left\{ \begin{array}{c} [c_1\mu + c_2\kappa(\theta - V_t)] + \frac{1}{2}(c_1 + c_2\sigma_v\rho)^2 V_t + \frac{1}{2}c_2^2\sigma_v^2\left(1-\rho^2\right) V_t \\ + \left(\frac{\exp\left(c_1\mu_y + \frac{1}{2}\sigma_y^2 c_1^2\right)}{1-\mu_v(c_2 + c_1\rho_J)} - 1\right)\lambda_y \end{array} \right\} dt.
\end{aligned}
$$

Our goal is to calculate

$$
\begin{aligned}
Var\begin{pmatrix} dY_t \\ dV_t \end{pmatrix} &= \begin{pmatrix} Var(dY_t) & Cov(dY_t, dV_t) \\ Cov(dY_t, dV_t) & Var(dV_t) \end{pmatrix} \\
&= \begin{pmatrix} E\left[(dY_t)^2\right] - [E(dY_t)]^2 & E(dY_t \cdot dV_t) - [E(dY_t)] \cdot [E(dV_t)] \\ Cov(dY_t, dV_t) & E\left[(dV_t)^2\right] - [E(dV_t)]^2 \end{pmatrix}.
\end{aligned}
$$

Thus it suffices to calculate the moments $E(dY_t)$, $E\left[(dY_t)^2\right]$, $E(dV_t)$, $E\left[(dV_t)^2\right]$, and the cross moment $E(dY_t \cdot dV_t)$. The moments and the cross moment can be calculated by using the following property of the moment generating function (see Sato (1999) for properties of the multivariate characteristic functions):

$$
E\left[(dY_t)^{k_1}(dV_t)^{k_2}\right] = \left. \frac{\partial^{k_1+k_2} M_{\begin{pmatrix} dY_t \\ dV_t \end{pmatrix}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}{\partial c_1^{k_1}\partial c_2^{k_2}} \right|_{c_1=c_2=0},
$$

$$
k_1 = 0, 1, 2,\ k_2 = 0, 1, 2,\ 1 \le k_1 + k_2 \le 2.
$$

Thus,

$$
E(dY_t) = \left. \frac{\partial M}{\partial c_1} \right|_{c_1=c_2=0}.
$$

$$
\frac{\partial M}{\partial c_1} = \left[\mu + (c_1 + c_2\sigma_v\rho) V_t + \lambda_y \left(\frac{\exp\left(c_1\mu_y + \frac{1}{2}\sigma_y^2 c_1^2\right)}{1 - \mu_v(c_2 + c_1\rho_J)}\right)'\right] dt,
$$

$$\left(\frac{\exp\left(c_1\mu_y+\frac{1}{2}\sigma_y^2c_1^2\right)}{1-\mu_v\left(c_2+c_1\rho_J\right)}\right)'$$
$$=\frac{\exp\left(c_1\mu_y+\frac{1}{2}\sigma_y^2c_1^2\right)\cdot\left(\mu_y+\sigma_y^2c_1\right)\cdot\left[1-\mu_v\left(c_2+c_1\rho_J\right)\right]-\exp\left(c_1\mu_y+\frac{1}{2}\sigma_y^2c_1^2\right)\cdot\left(-\mu_v\rho_J\right)}{\left[1-\mu_v\left(c_2+c_1\rho_J\right)\right]^2}$$
$$=\frac{\exp\left(c_1\mu_y+\frac{1}{2}\sigma_y^2c_1^2\right)\left\{\left(\mu_y+\sigma_y^2c_1\right)\cdot\left[1-\mu_v\left(c_2+c_1\rho_J\right)\right]+\mu_v\rho_J\right\}}{\left[1-\mu_v\left(c_2+c_1\rho_J\right)\right]^2},$$

$$\left(\frac{\exp\left(c_1\mu_y+\frac{1}{2}\sigma_y^2c_1^2\right)}{1-\mu_v\left(c_2+c_1\rho_J\right)}\right)'\Bigg|_{c_1=c_2=0}=\frac{1\cdot\{\mu_y\cdot 1+\mu_v\rho_J\}}{1^2}=\mu_y+\mu_v\rho_J.$$

Therefore,

$$\begin{aligned}E\left(dY_t\right)&=\left.\frac{\partial M}{\partial c_1}\right|_{c_1=c_2=0}\\&=\left[\mu+\lambda_y\left(\mu_y+\mu_v\rho_J\right)\right]dt.\end{aligned}$$

The other moments and the cross moment can be similarly calculated. But the calculations will be more involved.

We exploit the *Mathematica* software to calculate the moments and the cross moment. The *Mathematica* codes can be found in the supplemental file "SVCJ.nb". Since the calculations only involve differentiations and evaluations, the computing is very fast. We summarize the results calculated by the *Mathematica* software as follows.

$$\begin{aligned}E\left(dY_t\right)&=\left.\frac{\partial M}{\partial c_1}\right|_{c_1=c_2=0}=\left[\mu+\lambda_y\left(\mu_y+\mu_v\rho_J\right)\right]dt,\\E\left[\left(dY_t\right)^2\right]&=\left.\frac{\partial^2 M}{\partial c_1^2}\right|_{c_1=c_2=0}=\left[V_t+\lambda_y\left(\mu_y^2+2\mu_v\mu_y\rho_J+2\mu_v^2\rho_J^2+\sigma_y^2\right)\right]dt,\\E\left(dV_t\right)&=\left.\frac{\partial M}{\partial c_2}\right|_{c_1=c_2=0}=\left[\kappa\left(\theta-V_t\right)+\lambda_y\mu_v\right]dt,\\E\left[\left(dV_t\right)^2\right]&=\left.\frac{\partial^2 M}{\partial c_2^2}\right|_{c_1=c_2=0}=\left[2\lambda_y\mu_v^2+V_t\sigma_v^2\right]dt,\\E\left(dY_t\cdot dV_t\right)&=\left.\frac{\partial^2 M}{\partial c_1\partial c_2}\right|_{c_1=c_2=0}=\left[\lambda_y\left(\mu_v\mu_y+2\mu_v^2\rho_J\right)+\rho V_t\sigma_v\right]dt.\end{aligned}$$

$$\begin{aligned}Var\left(dY_t\right)&=E\left[\left(dY_t\right)^2\right]-\left[E\left(dY_t\right)\right]^2\\&=\left[V_t+\lambda_y\left(\mu_y^2+2\mu_y\mu_v\rho_J+2\rho_J^2\mu_v^2+\sigma_y^2\right)\right]dt,\end{aligned}$$

$$\begin{aligned}Var\left(dV_t\right)&=E\left[\left(dV_t\right)^2\right]-\left[E\left(dV_t\right)\right]^2\\&=\left[\sigma_v^2V_t+2\mu_v^2\lambda_v\right]dt,\ (\text{here } \lambda_v=\lambda_y)\end{aligned}$$

$$\begin{aligned}Cov\left(dY_t,dV_t\right)&=E\left(dY_t\cdot dV_t\right)-\left[E\left(dY_t\right)\right]\cdot\left[E\left(dV_t\right)\right]\\&=\left[\rho\sigma_vV_t+\lambda_y\left(\mu_y\mu_v+2\rho_J\mu_v^2\right)\right]dt.\end{aligned}$$

- The SVIJ model

To lighten notations, we use the same notations here as in those of the SVCJ model. The moment generating function of $\begin{pmatrix} dY_t \\ dV_t \end{pmatrix}$ for the SVIJ model is given by

$$
\begin{aligned}
& M_{\begin{pmatrix} dY_t \\ dV_t \end{pmatrix}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\
&= E \exp \left[\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \cdot \begin{pmatrix} dY_t \\ dV_t \end{pmatrix} \right] \\
&= E \exp \left(c_1 dY_t + c_2 dV_t \right) \\
&= E \exp \left\{ c_1 \left(\mu dt + \sqrt{V_t} dW_t^y + \xi^y dN_t^y \right) + c_2 \left[\kappa \left(\theta - V_t \right) dt + \sigma_v \sqrt{V_t} \left(\rho dW_t^y + \sqrt{1-\rho^2} dW_t^v \right) + \xi^v dN_t^v \right] \right\} \\
&= E \exp \left\{ \begin{array}{c} \left[c_1 \mu + c_2 \kappa \left(\theta - V_t \right) \right] dt + \left(c_1 \sqrt{V_t} + c_2 \sigma_v \rho \sqrt{V_t} \right) dW_t^y \\ + c_2 \sigma_v \sqrt{1-\rho^2} \sqrt{V_t} dW_t^v + c_1 \xi^y dN_t^y + c_2 \xi^v dN_t^v \end{array} \right\} \\
&= \exp \left\{ \left[c_1 \mu + c_2 \kappa \left(\theta - V_t \right) \right] dt \right\} \cdot \left\{ E \exp \left[\left(c_1 \sqrt{V_t} + c_2 \sigma_v \rho \sqrt{V_t} \right) dW_t^y \right] \right\} \cdot \left\{ E \exp \left[c_2 \sigma_v \sqrt{1-\rho^2} \sqrt{V_t} dW_t^v \right] \right\} \\
&\quad \cdot \left[E \exp \left(c_1 \xi^y dN_t^y \right) \right] \cdot \left[E \exp \left(c_2 \xi^v dN_t^v \right) \right] \\
&= \exp \left\{ \left[c_1 \mu + c_2 \kappa \left(\theta - V_t \right) \right] dt \right\} \cdot M_{dW_t^y} \left(c_1 \sqrt{V_t} + c_2 \sigma_v \rho \sqrt{V_t} \right) \cdot M_{dW_t^v} \left(c_2 \sigma_v \sqrt{1-\rho^2} \sqrt{V_t} \right) \\
&\quad \cdot \left[E \exp \left(c_1 \xi^y dN_t^y \right) \right] \cdot \left[E \exp \left(c_2 \xi^v dN_t^v \right) \right] \\
&= A \cdot M_1 \cdot M_2 \cdot M_3 \cdot M_4,
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \exp \left\{ \left[c_1 \mu + c_2 \kappa \left(\theta - V_t \right) \right] dt \right\}, \\
M_1 &= M_{dW_t^y} \left(c_1 \sqrt{V_t} + c_2 \sigma_v \rho \sqrt{V_t} \right), \\
M_2 &= M_{dW_t^v} \left(c_2 \sigma_v \sqrt{1-\rho^2} \sqrt{V_t} \right), \\
M_3 &= E \exp \left(c_1 \xi^y dN_t^y \right), \\
M_4 &= E \exp \left(c_2 \xi^v dN_t^v \right).
\end{aligned}
$$

We only need to compute A, M_1, M_2, M_3, and M_4. First,

$$
\begin{aligned}
A &= \exp \left\{ \left[c_1 \mu + c_2 \kappa \left(\theta - V_t \right) \right] dt \right\} \\
&= 1 + \left[c_1 \mu + c_2 \kappa \left(\theta - V_t \right) \right] dt.
\end{aligned}
$$

By (1.2), we have $dW_t^y \sim N(0, dt)$ and $dW_t^v \sim N(0, dt)$, thus

$$
\begin{aligned}
M_1 &= M_{dW_t^y} \left(c_1 \sqrt{V_t} + c_2 \sigma_v \rho \sqrt{V_t} \right) \\
&= \exp \left\{ 0 \cdot \left(c_1 + c_2 \sigma_v \rho \right) \sqrt{V_t} + \frac{1}{2} dt \left[\left(c_1 + c_2 \sigma_v \rho \right) \sqrt{V_t} \right]^2 \right\} \\
&= \exp \left\{ \frac{1}{2} \left(c_1 + c_2 \sigma_v \rho \right)^2 V_t dt \right\} \\
&= 1 + \frac{1}{2} \left(c_1 + c_2 \sigma_v \rho \right)^2 V_t dt,
\end{aligned}
$$

$$\begin{aligned} M_2 &= M_{dW_t^v}\left(c_2\sigma_v\sqrt{1-\rho^2}\sqrt{V_t}\right) \\ &= \exp\left\{0\cdot c_2\sigma_v\sqrt{1-\rho^2}\sqrt{V_t}+\frac{1}{2}dt\left[c_2\sigma_v\sqrt{1-\rho^2}\sqrt{V_t}\right]^2\right\} \\ &= \exp\left\{\frac{1}{2}c_2^2\sigma_v^2\left(1-\rho^2\right)V_t dt\right\} \\ &= 1+\frac{1}{2}c_2^2\sigma_v^2\left(1-\rho^2\right)V_t dt. \end{aligned}$$

Since $\xi^y \sim N\left(\mu_y, \sigma_y^2\right)$ and $dN_t^y \sim P\left(\lambda_y dt\right) = \text{Bernoulli}\left(\lambda_y dt\right)$, then

$$\begin{aligned} M_3 &= E\left[\exp\left(c_1\xi^y dN_t^y\right)\right] \\ &= E\left\{E\left[\exp\left(c_1\xi^y dN_t^y\right)|dN_t^y\right]\right\} \\ &= E\left[M_{\xi^y}\left(c_1 dN_t^y\right)\right] \\ &= E\left\{\exp\left[\mu_y c_1 dN_t^y+\frac{1}{2}\sigma_y^2\left(c_1 dN_t^y\right)^2\right]\right\} \\ &= \exp\left(\mu_y c_1+\frac{1}{2}\sigma_y^2 c_1^2\right)\cdot\lambda_y dt+\exp\left(0\right)\cdot\left(1-\lambda_y dt\right) \\ &= 1+\left[\exp\left(\mu_y c_1+\frac{1}{2}\sigma_y^2 c_1^2\right)-1\right]\lambda_y dt. \end{aligned}$$

Since $\xi^v \sim \exp\left(\mu_v\right)$ and $dN_t^v \sim P\left(\lambda_v dt\right) = \text{Bernoulli}\left(\lambda_v dt\right)$, then

$$\begin{aligned} M_4 &= E\left[\exp\left(c_2\xi^v dN_t^v\right)\right] \\ &= E\left\{E\left[\exp\left(c_2\xi^v dN_t^v\right)|dN_t^v\right]\right\} \\ &= E\left[M_{\xi^v}\left(c_2 dN_t^v\right)\right] \\ &= E\left[\frac{1}{1-\mu_v c_2 dN_t^v}\right] \\ &= \frac{1}{1-\mu_v c_2}\cdot\lambda_v dt+\frac{1}{1}\cdot\left(1-\lambda_v dt\right) \\ &= 1+\left(\frac{1}{1-\mu_v c_2}-1\right)\lambda_v dt. \end{aligned}$$

Therefore,

$$\begin{aligned} & M_{\begin{pmatrix} dY_t \\ dV_t \end{pmatrix}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= A\cdot M_1\cdot M_2\cdot M_3\cdot M_4 \\ &= \left\{1+\left[c_1\mu+c_2\kappa\left(\theta-V_t\right)\right]dt\right\}\cdot\left[1+\frac{1}{2}\left(c_1+c_2\sigma_v\rho\right)^2 V_t dt\right]\cdot\left[1+\frac{1}{2}c_2^2\sigma_v^2\left(1-\rho^2\right)V_t dt\right] \\ &\quad\cdot\left\{1+\left[\exp\left(\mu_y c_1+\frac{1}{2}\sigma_y^2 c_1^2\right)-1\right]\lambda_y dt\right\}\cdot\left[1+\left(\frac{1}{1-\mu_v c_2}-1\right)\lambda_v dt\right] \\ &= 1+\left\{\begin{array}{c} \left[c_1\mu+c_2\kappa\left(\theta-V_t\right)\right]+\frac{1}{2}\left(c_1+c_2\sigma_v\rho\right)^2 V_t+\frac{1}{2}c_2^2\sigma_v^2\left(1-\rho^2\right)V_t \\ +\lambda_y\left[\exp\left(\mu_y c_1+\frac{1}{2}\sigma_y^2 c_1^2\right)-1\right]+\lambda_v\left(\frac{1}{1-\mu_v c_2}-1\right) \end{array}\right\}dt. \end{aligned}$$

We exploit the *Mathematica* software to calculate the moments and the cross moment. The *Mathematica* codes can be found in the supplemental file "SVIJ.nb". We summarize the results calculated by the *Mathematica* software as follows.

$$\begin{aligned}
E\left(dY_t\right) &= \left.\frac{\partial M}{\partial c_1}\right|_{c_1=c_2=0} = \left[\mu + \lambda_y \mu_y\right] dt, \\
E\left[\left(dY_t\right)^2\right] &= \left.\frac{\partial^2 M}{\partial c_1^2}\right|_{c_1=c_2=0} = \left[V_t + \lambda_y\left(\mu_y^2 + \sigma_y^2\right)\right] dt, \\
E\left(dV_t\right) &= \left.\frac{\partial M}{\partial c_2}\right|_{c_1=c_2=0} = \left[\kappa\left(\theta - V_t\right) + \lambda_v \mu_v\right] dt, \\
E\left[\left(dV_t\right)^2\right] &= \left.\frac{\partial^2 M}{\partial c_2^2}\right|_{c_1=c_2=0} = \left[2\lambda_v \mu_v^2 + V_t \sigma_v^2\right] dt, \\
E\left(dY_t \cdot dV_t\right) &= \left.\frac{\partial^2 M}{\partial c_1 \partial c_2}\right|_{c_1=c_2=0} = \rho V_t \sigma_v dt.
\end{aligned}$$

$$\begin{aligned}
Var\left(dY_t\right) &= E\left[\left(dY_t\right)^2\right] - \left[E\left(dY_t\right)\right]^2 \\
&= \left[V_t + \lambda_y\left(\mu_y^2 + \sigma_y^2\right)\right] dt,
\end{aligned}$$

$$\begin{aligned}
Var\left(dV_t\right) &= E\left[\left(dV_t\right)^2\right] - \left[E\left(dV_t\right)\right]^2 \\
&= \left[\sigma_v^2 V_t + 2\mu_v^2 \lambda_v\right] dt,
\end{aligned}$$

$$\begin{aligned}
Cov\left(dY_t, dV_t\right) &= E\left(dY_t \cdot dV_t\right) - \left[E\left(dY_t\right)\right] \cdot \left[E\left(dV_t\right)\right] \\
&= \rho \sigma_v V_t dt.
\end{aligned}$$

2.2. Elementary and straightforward calculations

In this section, we provide the elementary and straightforward calculations of the variance and the covariance for the four models. To save space, the calculations of the latter models may rely on the results of the former models.

- The SV model

$$\begin{aligned}
Var\left(dY_t\right) &= Var\left(\mu dt + \sqrt{V_t} dW_t^y\right) \\
&= Var\left(\sqrt{V_t} dW_t^y\right) \\
&= V_t \cdot Var\left(dW_t^y\right) = V_t dt.
\end{aligned}$$

Thus,

$$\frac{Var\left(dY_t\right)}{dt} = V_t.$$

$$\begin{aligned}
Var\left(dV_t\right) &= Var\left(\kappa\left(\theta - V_t\right)dt + \sigma_v\sqrt{V_t}\left(\rho dW_t^y + \sqrt{1-\rho^2}dW_t^v\right)\right) \\
&= Var\left(\sigma_v\sqrt{V_t}\left(\rho dW_t^y + \sqrt{1-\rho^2}dW_t^v\right)\right) \\
&= \sigma_v^2 V_t \cdot Var\left(\rho dW_t^y + \sqrt{1-\rho^2}dW_t^v\right) \\
&= \sigma_v^2 V_t \cdot \left[Var\left(\rho dW_t^y\right) + Var\left(\sqrt{1-\rho^2}dW_t^v\right)\right] \text{ (by (1.2))} \\
&= \sigma_v^2 V_t\left[\rho^2 dt + \left(1-\rho^2\right)dt\right] \\
&= \sigma_v^2 V_t dt.
\end{aligned}$$

$$\begin{aligned}
Cov\left(dY_t, dV_t\right) &= Cov\left(\mu dt + \sqrt{V_t}dW_t^y, \kappa\left(\theta - V_t\right)dt + \sigma_v\sqrt{V_t}\left(\rho dW_t^y + \sqrt{1-\rho^2}dW_t^v\right)\right) \\
&= Cov\left(\sqrt{V_t}dW_t^y, \sigma_v\sqrt{V_t}\left(\rho dW_t^y + \sqrt{1-\rho^2}dW_t^v\right)\right) \\
&= Cov\left(\sqrt{V_t}dW_t^y, \sigma_v\sqrt{V_t}\rho dW_t^y\right) \text{ (by (1.2))} \\
&= \rho\sigma_v V_t dt. \text{ (by (1.2))}
\end{aligned}$$

- The SVJ model

  First, notice that dW_t^y, ξ^y, dN_t^y are mutually independent, we have

$$\begin{aligned}
Var\left(dY_t\right) &= Var\left(\mu dt + \sqrt{V_t}dW_t^y + \xi^y dN_t^y\right) \\
&= Var\left(\sqrt{V_t}dW_t^y + \xi^y dN_t^y\right) \\
&= Var\left(\sqrt{V_t}dW_t^y\right) + Var\left(\xi^y dN_t^y\right) \\
&= V_t dt + Var\left(\xi^y dN_t^y\right).
\end{aligned}$$

  Since $\xi^y \sim N\left(\mu_y, \sigma_y^2\right)$, $dN_t^y = N_{t+dt}^y - N_t^y \sim P\left(\lambda_y dt\right)$, and they are independent, we have

$$E\xi^y = \mu_y,\ Var\left(\xi^y\right) = \sigma_y^2,\ E\left[\left(\xi^y\right)^2\right] = \left(E\xi^y\right)^2 + Var\left(\xi^y\right) = \mu_y^2 + \sigma_y^2,$$

$$E\left(dN_t^y\right) = \lambda_y dt,\ Var\left(dN_t^y\right) = \lambda_y dt,\ E\left[\left(dN_t^y\right)^2\right] = \left[E\left(dN_t^y\right)\right]^2 + Var\left(dN_t^y\right) = \left(\lambda_y dt\right)^2 + \lambda_y dt = \lambda_y dt.$$

  Since dt is very small, $(dt)^2$ is negligibly small, which can be treated as 0. Thus,

$$\begin{aligned}
Var\left(\xi^y dN_t^y\right) &= E\left[\left(\xi^y dN_t^y\right)^2\right] - \left[E\left(\xi^y dN_t^y\right)\right]^2 \\
&= E\left[\left(\xi^y\right)^2\left(dN_t^y\right)^2\right] - \left[E\left(\xi^y dN_t^y\right)\right]^2 \\
&= E\left[\left(\xi^y\right)^2\right] \cdot E\left[\left(dN_t^y\right)^2\right] - \left[E\left(\xi^y\right) \cdot E\left(dN_t^y\right)\right]^2 \text{ (by independence)} \\
&= \left(\mu_y^2 + \sigma_y^2\right)\lambda_y dt - \left(\mu_y\lambda_y dt\right)^2 \\
&= \left(\mu_y^2 + \sigma_y^2\right)\lambda_y dt.
\end{aligned}$$

  Thus,

$$Var\left(dY_t\right) = \left[V_t + \lambda_y\left(\mu_y^2 + \sigma_y^2\right)\right]dt.$$

$$Var\left(dV_t\right) = \sigma_v^2 V_t dt,$$

which is the same as that of the SV model.

$$\begin{aligned}
&Cov\left(dY_t, dV_t\right) \\
&= Cov\left(\mu dt + \sqrt{V_t} dW_t^y + \xi^y dN_t^y, \kappa\left(\theta - V_t\right) dt + \sigma_v \sqrt{V_t}\left(\rho dW_t^y + \sqrt{1-\rho^2} dW_t^v\right)\right) \\
&= Cov\left(\sqrt{V_t} dW_t^y + \xi^y dN_t^y, \sigma_v \sqrt{V_t}\left(\rho dW_t^y + \sqrt{1-\rho^2} dW_t^v\right)\right) \\
&= \sigma_v V_t Cov\left(dW_t^y, \rho dW_t^y + \sqrt{1-\rho^2} dW_t^v\right) + \sigma_v \sqrt{V_t} Cov\left(\xi^y dN_t^y, \rho dW_t^y + \sqrt{1-\rho^2} dW_t^v\right).
\end{aligned}$$

Since dW_t^y, dW_t^v, ξ^y, dN_t^y are mutually independent, we obtain

$$\begin{aligned}
Cov\left(dY_t, dV_t\right) &= \sigma_v V_t \rho Cov\left(dW_t^y, dW_t^y\right) + 0 \\
&= \rho \sigma_v V_t dt.
\end{aligned}$$

- The SVCJ model

  Similar to the derivations for the SVJ model, we have

$$\begin{aligned}
Var\left(dY_t\right) &= V_t dt + Var\left(\xi^y dN_t^y\right), \\
Var\left(\xi^y dN_t^y\right) &= E\left[\left(\xi^y\right)^2\right] \cdot E\left[\left(dN_t^y\right)^2\right] - \left[E\left(\xi^y\right) \cdot E\left(dN_t^y\right)\right]^2.
\end{aligned}$$

  First,

$$\begin{aligned}
E\xi^y &= E\left[E\left(\xi^y | \xi^v\right)\right] \\
&= E\left(\mu_y + \rho_J \xi^v\right) \text{ (by (1.3))} \\
&= \mu_y + \rho_J \mu_v. \text{ (by (1.3))}
\end{aligned}$$

  Thus,

$$\left[E\left(\xi^y\right) \cdot E\left(dN_t^y\right)\right]^2 = \left[\left(\mu_y + \rho_J \mu_v\right) \lambda_y dt\right]^2 = 0.$$

  Consequently,

$$\begin{aligned}
Var\left(dY_t\right) &= V_t dt + \lambda_y dt \cdot E\left[\left(\xi^y\right)^2\right] \text{ (by (1.3))} \\
&= \left[V_t + \lambda_y E\left[\left(\xi^y\right)^2\right]\right] dt.
\end{aligned}$$

  It suffices to compute $E\left[\left(\xi^y\right)^2\right]$.

$$\begin{aligned}
E\left[\left(\xi^y\right)^2\right] &= E\left\{E\left[\left(\xi^y\right)^2 | \xi^v\right]\right\} \\
&= E\left[\left(\mu_y + \rho_J \xi^v\right)^2 + \sigma_y^2\right] \text{ (by (1.3))} \\
&= E\left[\mu_y^2 + 2\mu_y \rho_J \xi^v + \rho_J^2 \left(\xi^v\right)^2 + \sigma_y^2\right] \\
&= \mu_y^2 + 2\mu_y \rho_J E\xi^v + \rho_J^2 E\left[\left(\xi^v\right)^2\right] + \sigma_y^2 \\
&= \mu_y^2 + 2\mu_y \rho_J \mu_v + \rho_J^2 2\mu_v^2 + \sigma_y^2 \text{ (by (1.3))} \\
&= \mu_y^2 + 2\mu_y \mu_v \rho_J + 2\rho_J^2 \mu_v^2 + \sigma_y^2.
\end{aligned}$$

Alternatively, we can compute $E\left[(\xi^y)^2\right]$ by the conditional variance formula. First,

$$\begin{aligned} Var(\xi^y) &= Var(E(\xi^y|\xi^v)) + E(Var(\xi^y|\xi^v)) \\ &= Var(\mu_y + \rho_J \xi^v) + E\left(\sigma_y^2\right) \text{ (by (1.3))} \\ &= \rho_J^2 \mu_v^2 + \sigma_y^2. \text{ (by (1.3))} \end{aligned}$$

Thus,

$$\begin{aligned} E\left[(\xi^y)^2\right] &= [E\xi^y]^2 + Var(\xi^y) \\ &= (\mu_y + \rho_J \mu_v)^2 + \rho_J^2 \mu_v^2 + \sigma_y^2 \\ &= \mu_y^2 + 2\mu_y \mu_v \rho_J + 2\rho_J^2 \mu_v^2 + \sigma_y^2. \end{aligned}$$

$$\begin{aligned} Var(dV_t) &= Var\left(\kappa(\theta - V_t)\,dt + \sigma_v \sqrt{V_t}\left(\rho dW_t^y + \sqrt{1-\rho^2} dW_t^v\right) + \xi^v dN_t^v\right) \\ &= Var\left(\sigma_v \sqrt{V_t}\left(\rho dW_t^y + \sqrt{1-\rho^2} dW_t^v\right) + \xi^v dN_t^v\right) \\ &= Var\left(\sigma_v \sqrt{V_t}\left(\rho dW_t^y + \sqrt{1-\rho^2} dW_t^v\right)\right) + Var(\xi^v dN_t^v) \text{ (by independence)} \\ &= \sigma_v^2 V_t dt + Var(\xi^v dN_t^v), \end{aligned}$$

$$\begin{aligned} Var(\xi^v dN_t^v) &= E\left[(\xi^v dN_t^v)^2\right] - [E(\xi^v dN_t^v)]^2 \\ &= E\left[(\xi^v)^2\right] \cdot E\left[(dN_t^v)^2\right] - [E(\xi^v) \cdot E(dN_t^v)]^2 \text{ (by independence)} \\ &= 2\mu_v^2 \cdot \lambda_v dt - (\mu_v \lambda_v dt)^2 \text{ (by (1.3))} \\ &= 2\mu_v^2 \lambda_v dt. \end{aligned}$$

Consequently,

$$Var(dV_t) = \sigma_v^2 V_t dt + 2\mu_v^2 \lambda_v dt = \left(\sigma_v^2 V_t + 2\mu_v^2 \lambda_v\right) dt.$$

$$\begin{aligned} &Cov(dY_t, dV_t) \\ &= Cov\left(\mu dt + \sqrt{V_t} dW_t^y + \xi^y dN_t^y, \kappa(\theta - V_t)\,dt + \sigma_v \sqrt{V_t}\left(\rho dW_t^y + \sqrt{1-\rho^2} dW_t^v\right) + \xi^v dN_t^v\right) \\ &= Cov\left(\sqrt{V_t} dW_t^y + \xi^y dN_t^y, \sigma_v \sqrt{V_t}\left(\rho dW_t^y + \sqrt{1-\rho^2} dW_t^v\right) + \xi^v dN_t^v\right) \\ &= Cov\left(\sqrt{V_t} dW_t^y, \sigma_v \sqrt{V_t}\left(\rho dW_t^y + \sqrt{1-\rho^2} dW_t^v\right)\right) + Cov\left(\sqrt{V_t} dW_t^y, \xi^v dN_t^v\right) \\ &\quad + Cov\left(\xi^y dN_t^y, \sigma_v \sqrt{V_t}\left(\rho dW_t^y + \sqrt{1-\rho^2} dW_t^v\right)\right) + Cov(\xi^y dN_t^y, \xi^v dN_t^v). \end{aligned}$$

Similar to the calculations in the SV model, we have

$$Cov\left(\sqrt{V_t} dW_t^y, \sigma_v \sqrt{V_t}\left(\rho dW_t^y + \sqrt{1-\rho^2} dW_t^v\right)\right) = \rho \sigma_v V_t dt.$$

By the dependence structure of the five quantities dW_t^y, ξ^y, dW_t^v, ξ^v, dN_t, we have

$$Cov\left(\sqrt{V_t} dW_t^y, \xi^v dN_t^v\right) = Cov\left(\xi^y dN_t^y, \sigma_v \sqrt{V_t}\left(\rho dW_t^y + \sqrt{1-\rho^2} dW_t^v\right)\right) = 0.$$

Thus,

$$Cov\left(dY_t, dV_t\right) = \rho\sigma_v V_t dt + Cov\left(\xi^y dN_t^y, \xi^v dN_t^v\right).$$

$$\begin{aligned} Cov\left(\xi^y dN_t^y, \xi^v dN_t^v\right) &= E\left(\xi^y dN_t^y \xi^v dN_t^v\right) - E\left(\xi^y dN_t^y\right) \cdot E\left(\xi^v dN_t^v\right) \\ &= E\left(\xi^y \xi^v\right) \cdot E\left[\left(dN_t\right)^2\right] - E\left(\xi^y\right) \cdot E\left(dN_t\right) \cdot E\left(\xi^v\right) \cdot E\left(dN_t\right), \text{ (by independence)} \end{aligned}$$

$$\begin{aligned} E\left(\xi^y \xi^v\right) &= E\left[E\left(\xi^y \xi^v | \xi^v\right)\right] \\ &= E\left[\xi^v E\left(\xi^y | \xi^v\right)\right] \\ &= E\left[\xi^v\left(\mu_y + \rho_J \xi^v\right)\right] \text{ (by (1.3))} \\ &= E\left[\mu_y \xi^v + \rho_J\left(\xi^v\right)^2\right] \\ &= \mu_y E\xi^v + \rho_J E\left[\left(\xi^v\right)^2\right] \\ &= \mu_y \mu_v + \rho_J 2\mu_v^2, \text{ (by (1.3))} \end{aligned}$$

$$E\left[\left(dN_t\right)^2\right] = \lambda_y dt. \text{ (by (1.3))}$$

In the SVCJ model, we have previously computed

$$E\xi^y = \mu_y + \rho_J \mu_v.$$

Thus, by (1.3),

$$E\left(\xi^y\right) \cdot E\left(dN_t\right) \cdot E\left(\xi^v\right) \cdot E\left(dN_t\right) = \left(\mu_y + \rho_J \mu_v\right) \cdot \lambda_y dt \cdot \mu_v \cdot \lambda_y dt = 0.$$

Consequently,

$$Cov\left(\xi^y dN_t^y, \xi^v dN_t^v\right) = \left(\mu_y \mu_v + 2\rho_J \mu_v^2\right) \lambda_y dt.$$

Finally,

$$\begin{aligned} Cov\left(dY_t, dV_t\right) &= \rho\sigma_v V_t dt + \left(\mu_y \mu_v + 2\rho_J \mu_v^2\right) \lambda_y dt \\ &= \left[\rho\sigma_v V_t + \lambda_y\left(\mu_y \mu_v + 2\rho_J \mu_v^2\right)\right] dt. \end{aligned}$$

- The SVIJ model

  Similar to the calculations in the SVCJ model, we have

$$Var\left(dY_t\right) = \left[V_t + \lambda_y E\left[\left(\xi^y\right)^2\right]\right] dt.$$

$$\begin{aligned} &\xi^y \sim N\left(\mu_y, \sigma_y^2\right), \\ &E\xi^y = \mu_y, \; Var\left(\xi^y\right) = \sigma_y^2, \; E\left[\left(\xi^y\right)^2\right] = \left(E\xi^y\right)^2 + Var\left(\xi^y\right) = \mu_y^2 + \sigma_y^2. \end{aligned}$$

  Thus,

$$Var\left(dY_t\right) = \left[V_t + \lambda_y\left(\mu_y^2 + \sigma_y^2\right)\right] dt.$$

  Similar to the calculations in the SVCJ model, we have

$$Var\left(dV_t\right) = \left(\sigma_v^2 V_t + 2\mu_v^2 \lambda_v\right) dt.$$

$$
\begin{aligned}
&Cov\left(dY_t, dV_t\right)\\
&= Cov\left(\mu dt + \sqrt{V_t}dW_t^y + \xi^y dN_t^y, \kappa\left(\theta - V_t\right)dt + \sigma_v\sqrt{V_t}\left(\rho dW_t^y + \sqrt{1-\rho^2}dW_t^v\right) + \xi^v dN_t^v\right)\\
&= Cov\left(\sqrt{V_t}dW_t^y + \xi^y dN_t^y, \sigma_v\sqrt{V_t}\left(\rho dW_t^y + \sqrt{1-\rho^2}dW_t^v\right) + \xi^v dN_t^v\right)\\
&= Cov\left(\sqrt{V_t}dW_t^y, \sigma_v\sqrt{V_t}\left(\rho dW_t^y + \sqrt{1-\rho^2}dW_t^v\right)\right) + Cov\left(\sqrt{V_t}dW_t^y, \xi^v dN_t^v\right)\\
&\quad + Cov\left(\xi^y dN_t^y, \sigma_v\sqrt{V_t}\left(\rho dW_t^y + \sqrt{1-\rho^2}dW_t^v\right)\right) + Cov\left(\xi^y dN_t^y, \xi^v dN_t^v\right).
\end{aligned}
$$

Similar to the calculations in the SV model, we have

$$
Cov\left(\sqrt{V_t}dW_t^y, \sigma_v\sqrt{V_t}\left(\rho dW_t^y + \sqrt{1-\rho^2}dW_t^v\right)\right) = \rho\sigma_v V_t dt.
$$

Since dW_t^y, ξ^y, dN_t^y, dW_t^v, ξ^v, dN_t^v are mutually independent, we have

$$
\begin{aligned}
&Cov\left(\sqrt{V_t}dW_t^y, \xi^v dN_t^v\right)\\
&= Cov\left(\xi^y dN_t^y, \sigma_v\sqrt{V_t}\left(\rho dW_t^y + \sqrt{1-\rho^2}dW_t^v\right)\right)\\
&= Cov\left(\xi^y dN_t^y, \xi^v dN_t^v\right)\\
&= 0.
\end{aligned}
$$

Finally,

$$
Cov\left(dY_t, dV_t\right) = \rho\sigma_v V_t dt.
$$

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