Lower Salaries and No Options: Implementation of Minimzation Problems*

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This document derives versions of UPPS, EU and their derivatives that can be directly transformed into computer programs. This is done for several different minimization problems. For the problem with taxes there exists a separate document.

1 Base Case: The original problem

1.1 Expected Utility EU

From our basic assumptions (equations (2) and (3) in the paper) we obtain:

$$EU(\phi, n_S, n_O) = E\left[\frac{1}{1-\gamma}\left[TW + n_S \exp\{dT\} P_T + n_O \max\{P_T - K, 0\}\right]^{1-\gamma}\right],$$

*This note is a technical document to accompany our paper "Lower salaries and no options: the optimal structure of executive pay." It was originally intended as an internal document for the process of developing and writing the paper. We make it now publically available to provide the interested reader with additional details about our approach in the paper. We cannot guarantee that this document is self-contained. It should not be cited without the permission of the authors.

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$$\text{where } TW = (\phi + W_0)\exp\{r_f T\}$$

$$\begin{aligned} P_T &= P_0 \exp\left\{\left(r_f - d - \frac{\sigma^2}{2}\right)T + u\sqrt{T}\sigma\right\} \text{ with } u\tilde{\ }N(0,1) \\ &= PC\exp\{CV \cdot u\} \text{ with } PC = P_0\exp\left\{\left(r_f - d - \frac{\sigma^2}{2}\right)T\right\} \end{aligned}$$

$$\text{where } CV = \sigma\sqrt{T}$$

This implies

$$\begin{aligned} EU(\phi, n_S, n_O) = \frac{1}{1-\gamma}\frac{1}{\sqrt{2\pi}} & \\ & \left(\int_{-\infty}^{MD2}\left[TW + n_S\exp\{dT\}PC\exp\{uCV\}\right]^{1-\gamma}\exp\left\{-\frac{u^2}{2}\right\}du\right. \\ & + \int_{MD2}^{+\infty}\left[TW + (n_S\exp\{dT\} + n_O)\,PC\exp\{uCV\} - n_O K\right]^{1-\gamma} \\ & \left.\exp\left\{-\frac{u^2}{2}\right\}du\right) \end{aligned}$$

where $MD2$ is the u for which

$$P_T = K$$

$$\Leftrightarrow P_0\exp\left\{\left(r_f - \frac{\sigma^2}{2} - d\right)T + MD2\sqrt{T}\sigma\right\} = K$$

$$\Leftrightarrow \left(r_f - \frac{\sigma^2}{2} - d\right) T + MD2\sqrt{T}\sigma = \ln\left(\frac{K}{P_0}\right)$$

$$\Leftrightarrow MD2\sqrt{T}\sigma = \ln\left(\frac{K}{P_0}\right) - \left(r_f - \frac{\sigma^2}{2} - d\right) T$$

$$\Leftrightarrow MD2 = \frac{\ln\left(\frac{K}{P_0}\right) - (r_f - d)\,T}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$$

Note that (**??**) holds only for $\gamma \neq 1$. For $\gamma = 1$, we obtain:

$$\begin{aligned} EU(\phi, n_S, n_O) = {} & \frac{1}{\sqrt{2\pi}} \\ & \left(\int_{-\infty}^{MD2} \ln\{TW + n_S \exp\{dT\} PC \exp\{uCV\}\} \exp\left\{-\frac{u^2}{2}\right\} du \right. \\ & + \int_{MD2}^{+\infty} \ln\{TW + (n_S \exp\{dT\} + n_O)\, PC \exp\{uCV\} - n_O K\} \\ & \left. \exp\left\{-\frac{u^2}{2}\right\} du\right) \end{aligned}$$

1.2 UPPS

UPPS is defined as:

$$\begin{aligned} UPPS(\phi, n_S, n_O) &= \frac{d}{dP_0} \exp(-r_f T) E[U(W_T, e^*)] = \frac{d}{dP_0} \exp(-r_f T) E[V(W_T)] \\ & \text{where } W_T = TW + n_S \exp\{dT\} P_T + n_O \max\{P_T - K, 0\} \end{aligned}$$

As $W_T(P_0)$ cannot be differentiated at $P_T = K$, we must split the integral before differentiation at $u = MD2$.

$$
\begin{aligned}
UPPS(\phi, n_S, n_O) = & \frac{d}{dP_0}\left(\exp(-r_f T)\right. \\
& \left.\int_{-\infty}^{MD2} \frac{(TW + n_S \exp\{dT\} P_T(u))^{1-\gamma}}{1-\gamma} f(u) du\right) \\
& + \frac{d}{dP_0}\left(\exp(-r_f T)\right. \\
& \left.\int_{MD2}^{\infty} \frac{(TW + (n_S \exp\{dT\} + n_O)\, P_T(u) - n_O K)^{1-\gamma}}{1-\gamma} f(u) du\right) \\
& = \int_{-\infty}^{MD2} (TW + n_S \exp\{dT\} P_T(u))^{-\gamma}\, n_S \exp\{dT\} \\
& \exp\left\{-dT - \frac{\sigma^2}{2} T + u\sqrt{T}\sigma\right\} f(u) du \\
& + \exp(-r_f T) \frac{(TW + n_S \exp\{dT\} P_T(MD2))^{1-\gamma}}{1-\gamma} \\
& f(MD2) \frac{dMD2(P_0)}{dP_0} \\
& + \int_{MD2}^{\infty} (TW + (n_S \exp\{dT\} + n_O)\, P_T(u) - n_O K)^{-\gamma} \\
& (n_S \exp\{dT\} + n_O) \exp\left\{-dT - \frac{\sigma^2}{2} T + u\sqrt{T}\sigma\right\} f(u) du \\
& - \exp(-r_f T) \frac{(TW + (n_S \exp\{dT\} + n_O)\, P_T(MD2) - n_O K)^{1-\gamma}}{1-\gamma} \\
& f(MD2) \frac{dMD2(P_0)}{dP_0}
\end{aligned}
$$

Note that the two derivatives of the integral boundaries cancel each other as $P_T(MD2) = K$:

$$
\begin{aligned}
UPPS(\phi, n_S, n_O) = & \int_{-\infty}^{MD2} \left(TW + n_S \exp\{dT\} P_T(u)\right)^{-\gamma} n_S \\
& \exp\{dT\} \exp\left\{-dT - \frac{\sigma^2}{2}T + u\sqrt{T}\sigma\right\} f(u)du \\
& + \int_{MD2}^{\infty} \left(TW + \left(n_S \exp\{dT\} + n_O\right) P_T(u) - n_O K\right)^{-\gamma} \\
& \left(n_S \exp\{dT\} + n_O\right) \exp\left\{-dT - \frac{\sigma^2}{2}T + u\sqrt{T}\sigma\right\} f(u)du
\end{aligned}
$$

Using the above defined symbols and $f(u) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{u^2}{2}\}$ we obtain:

$$
\begin{aligned}
UPPS(\phi, n_S, n_O) = LD & \left(\int_{-\infty}^{MD2} \left(TW + n_S \exp\{dT\} PC \exp\{uCV\}\right)^{-\gamma} \right. \\
& n_S \exp\{dT\} \exp\left\{uCV - \frac{u^2}{2}\right\} du \\
& + \int_{MD2}^{+\infty} \left(TW + \left(n_S \exp\{dT\} + n_O\right) PC \exp\{uCV\} - n_O K\right)^{-\gamma} \\
& \left. \left(n_S \exp\{dT\} + n_O\right) \exp\left\{uCV - \frac{u^2}{2}\right\} du \right)
\end{aligned}
$$

where $LD = \frac{1}{\sqrt{2\pi}} \exp\left\{-dT - \frac{\sigma^2}{2}T\right\}$

2 Gamma distributed stock prices

2.1 Incorporating different stock price dynamics

In order to generate a program that is flexible enough to switch to different stock price dynamics than the log-normal distribution, we rewrite the conditions EU and UPPS in the following way:

$$
f(u) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{u^2}{2}\right\}
$$

$$P_T(u) = P_0 \exp\left\{\left(r_f - d - \frac{\sigma^2}{2}\right)T + u\sqrt{T}\sigma\right\}$$

$$\frac{dP_T(u)}{dP_0} = \exp\left\{\left(r_f - d - \frac{\sigma^2}{2}\right)T + u\sqrt{T}\sigma\right\}$$

$$\begin{aligned}EU(\phi, n_S, n_O) = &\frac{1}{1-\gamma}\int_{-\infty}^{MD2}\left[TW + n_S\exp\{dT\}P_T(u)\right]^{1-\gamma} f(u)du \\ &+ \int_{MD2}^{+\infty}\left[TW + \left(n_S\exp\{dT\} + n_O\right)P_T(u) - n_O K\right]^{1-\gamma} f(u)du\end{aligned}$$

$$\begin{aligned}UPPS(\phi, n_S, n_O) = &\exp\{-r_f T\} \\ &\left(\int_{-\infty}^{MD2}\left(TW + n_S\exp\{dT\}P_T(u)\right)^{-\gamma} n_S\exp\{dT\}\frac{dP_T(u)}{dP0}f(u)du\right. \\ &+ \int_{MD2}^{\infty}\left(TW + \left(n_S\exp\{dT\} + n_O\right)P_T(u) - n_O K\right)^{-\gamma} \\ &\left.\left(n_S\exp\{dT\} + n_O\right)\frac{dP_T(u)}{dP0}f(u)du\right)\end{aligned}$$

2.2 Mean, variance and skewness of the log-normally distributed stock price

The stock price dynamic is given by

$$P_T(u) = P_0 \exp\left\{\left(r_f - d - \frac{\sigma^2}{2}\right)T\right\}u$$

where $\ln(u) + N(0, \sigma^2 T)$.Using the formulae for the mean, the variance and the coefficient of skewness for lognormally distributed random variates (p.102 in Eavans, Hastings & Peacock, 1993), we obtain:

$$E(P_T(u)) = P_0 \exp\left\{\left(r_f - d - \frac{\sigma^2}{2}\right)T\right\}\exp\left\{\frac{1}{2}\sigma^2 T\right\}$$
$$= P_0 \exp\left\{(r_f - d)\,T\right\}$$

$$Var\,(P_T(u)) = P_0^2 \exp\left\{2\left(r_f - d - \frac{\sigma^2}{2}\right)T\right\}\exp\left\{\sigma^2 T\right\}\left(\exp\left\{\sigma^2 T\right\} - 1\right)$$
$$= P_0^2 \exp\left\{2\,(r_f - d)\,T\right\}\left(\exp\left\{\sigma^2 T\right\} - 1\right)$$

$$\frac{E\,(P_T(u) - \mu_{P_T})^3}{\sigma_{P_T}^3} = \frac{P_0^3 \exp\left\{3\left(r_f - d - \frac{\sigma^2}{2}\right)T\right\}E\,(u - \mu_u)^3}{\left(P_0^2 \exp\left\{2\left(r_f - d - \frac{\sigma^2}{2}\right)T\right\}E\,(u - \mu_u)^2\right)^{3/2}}$$
$$= \frac{E\,(u - \mu_u)^3}{\sigma_u^3} = \sqrt{\exp\left\{\sigma^2 T\right\} - 1}\left(\exp\left\{\sigma^2 T\right\} + 2\right)$$

2.3 Gamma distributed stock prices

According to Hemmer, Kim and Verrecchia (2000), the stock price is given by$P_T(v) = P_v v$, where v is Gamma distributed with density function

$$f_v(x|b,c) = \left(\frac{x}{b}\right)^{c-1}\frac{\exp\{-x/b\}}{b\Gamma(c)}$$

Note that Hemmer et al. (2000) use $a = b$ and $k = c$ and assume that k is integer, so that $\Gamma(k) = (k-1)!$. The notation used is the notation in Eavans, Hastings & Peacock (1993).

This distribution has three parameters (P_v, b, c). We determine these parameters by equating the first three moments of the distribution with the corresponding moments of the exponential distribution.

The three moments of $P_T(v)$ are:

$$E(P_T(v)) = P_v bc$$

$$Var(P_T(v)) = P_v^2 b^2 c$$

$$\frac{E\left(P_T(v) - \mu_{P_T(v)}\right)^3}{\sigma^3_{P_T(v)}} = \frac{E\left(v - \mu_v\right)^3}{\sigma_v^3} = \frac{2}{\sqrt{c}}$$

As P_v and b only occur in the expectation and the variance and both times as product $P_v b$ only, these two parameters cannot be separated and the system of equations cannot be solved. The reason is that the Gamma distribution only has two parameters and multiplying it by a constant P_v not only changes the mean but also the variance. We therefore set $P_v = P_0 \exp\left\{\left(r_f - d\right) T\right\}$and only use the first two equations, the mean and the variance in order to identify the parameters b and c:

$$\begin{aligned} P_v bc &= P_0 \exp\left\{\left(r_f - d\right) T\right\} \\ P_v^2 b^2 c &= P_0^2 \exp\left\{2\left(r_f - d\right) T\right\}\left(\exp\left\{\sigma^2 T\right\} - 1\right) \end{aligned}$$

The solution is:

$$\begin{aligned} b &= \left(\exp\left\{\sigma^2 T\right\} - 1\right) \\ c &= \left(\exp\left\{\sigma^2 T\right\} - 1\right)^{-1} \end{aligned}$$

Hence, the necessary changes to the program are:

$$f(u) = \left(\frac{u}{b}\right)^{c-1} \frac{\exp\{-u/b\}}{b\Gamma(c)} = (uc)^{c-1} \frac{\exp\{-uc\}c}{\Gamma(c)}$$

$$P_T(u) = P_0 \exp\{(r_f - d)\,T\}\,u$$

$$\frac{dP_T(u)}{dP_0} = \exp\{(r_f - d)\,T\}\,u$$

In addition, the threshold $MD2$ must be adjusted. Recall that $MD2$ is defined by $P_T(MD2) = K$. Hence:

$$MD2 = \frac{K}{P_0 \exp\{(r_f - d)T\}}$$

Also, the integration starts at 0 (not at $-\infty$) and the value of the options to the firm must be calculated numerically as the Black-Scholes formula is not valid.

3 Problem with two types of options

In this section, we derive the expected utility and UPPS with stock and two types of options A and B. The two types of options differ in their strike price K_A and K_B , $K_B > K_A$, but have the same maturity T. Hence, the principal chooses four parameters $(\phi, n_S, n_O^A, n_O^B)$ in order to minimize the total costs $\pi = \phi + n_S P_0 + n_O^A BS_A + n_O^B BS_B$.

By setting $n_O^B = 0$ it is easy to derive the corresponding formula for a quadratic bonus scheme.

3.1 Expected Utility EU

Analogous to the derivation for the case with only one type of options, we get:

$$
\begin{aligned}
&EU(\phi, n_S, n_O^A, n_O^B) = \\
&\frac{1}{1-\gamma}\frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{MD2A} \left[TW + n_S \exp\{dT\} PC \exp\{uCV\}\right]^{1-\gamma} \exp\left\{-\frac{u^2}{2}\right\} du\right. \\
&+ \int_{MD2A}^{MD2B} \left[TW + \left(n_S \exp\{dT\} + n_O^A\right) PC \exp\{uCV\} - n_O^A K_A\right]^{1-\gamma} \exp\left\{-\frac{u^2}{2}\right\} du \\
&+ \int_{MD2B}^{+\infty} \left[TW + \left(n_S \exp\{dT\} + n_O^A + n_O^B\right) PC \exp\{uCV\} - n_O^A K_A - n_O^B K_B\right]^{1-\gamma} \\
&\left.\exp\left\{-\frac{u^2}{2}\right\} du\right)
\end{aligned}
$$

where

$$MD2A = \frac{\ln\left(\frac{K_A}{P_0}\right) - (r_f - d)\,T}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$$

and

$$MD2B = \frac{\ln\left(\frac{K_B}{P_0}\right) - (r_f - d)\,T}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}.$$

For $\gamma = 1$, we obtain:

$$
\begin{aligned}
&EU(\phi, n_S, n_O^A, n_O^B) = \\
&\frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{MD2A} \ln\left\{TW + n_S \exp\{dT\} PC \exp\{uCV\}\right\} \exp\left\{-\frac{u^2}{2}\right\} du\right. \\
&+ \int_{MD2A}^{MD2B} \ln\left\{TW + \left(n_S \exp\{dT\} + n_O^A\right) PC \exp\{uCV\} - n_O^A K_A\right\} \exp\left\{-\frac{u^2}{2}\right\} du\Bigg) \\
&+ \int_{MD2B}^{+\infty} \ln\left\{TW + \left(n_S \exp\{dT\} + n_O^A + n_O^B\right) PC \exp\{uCV\} - n_O^A K_A - n_O^B K_B\right\} \\
&\left.\exp\left\{-\frac{u^2}{2}\right\} du\right)
\end{aligned}
$$

3.2 UPPS

Analogous to the derivation for the case with only one type of options, we get:

$$
\begin{aligned}
&UPPS(\phi, n_S, n_O^A, n_O^B) = \\
&LD\left(\int_{-\infty}^{MD2} \left(TW + n_S \exp\{dT\} PC \exp\{uCV\}\right)^{-\gamma} n_S \exp\{dT\} \exp\left\{uCV - \frac{u^2}{2}\right\} du \right. \\
&+ \int_{MD2A}^{MD2B} \left(TW + \left(n_S \exp\{dT\} + n_O^A\right) PC \exp\{uCV\} - n_O^A K_A\right)^{-\gamma} \\
&\left(n_S \exp\{dT\} + n_O^A\right) \exp\left\{uCV - \frac{u^2}{2}\right\} du\Bigg) \\
&+ \int_{MD2B}^{+\infty} \left(TW + \left(n_S \exp\{dT\} + n_O^A + n_O^B\right) PC \exp\{uCV\} - n_O^A K_A - n_O^B K_B\right)^{-\gamma} \\
&\left(n_S \exp\{dT\} + n_O^A + n_O^B\right) \exp\left\{uCV - \frac{u^2}{2}\right\} du\Bigg)
\end{aligned}
$$

where $LD = \frac{1}{\sqrt{2\pi}} \exp\left\{-dT - \frac{\sigma^2}{2} T\right\}$

4 Estimation of the theoretical solution

This section considers the estimation of the parameters α_0 and α_1 of the theoretical solution (19) in the paper. The theoretical solution is (where $\delta > 0$ is a small number):

$$
\pi_T = \begin{cases} (\alpha_0 + \alpha_1 \ln P_T)^{1/\gamma} - W_0 \exp(r_f T) & \text{if} \quad P_T \geq \overline{P} \\ -W_0 \exp(r_f T) + \delta & \text{if} \quad P_T < \overline{P} \end{cases},
$$

where $\overline{P} = \exp\left(\left(\delta^\gamma - \alpha_0\right)/\alpha_1\right)$.

Hence we obtain for $W_T = W_0 \exp(r_f T) + \pi_T$:

$$
W_T = \begin{cases} (\alpha_0 + \alpha_1 \ln P_T)^{1/\gamma} & \text{if} \quad P_T \geq \overline{P} \\ \delta & \text{if} \quad P_T < \overline{P} \end{cases}.
$$

4.1 Expected Utility EU

For the expected utility, we get:

$$\begin{aligned}E(U(W_T)) &= E\left[\frac{1}{1-\gamma}\left\{\begin{array}{cl}(\alpha_0+\alpha_1\ln P_T)^{(1-\gamma)/\gamma} & \text{if} \quad P_T \geq \overline{P} \\ \delta^{1-\gamma} & \text{if} \quad P_T < \overline{P}\end{array}\right.\right] \\ &= \frac{1}{1-\gamma}\frac{1}{\sqrt{2\pi}}\left[\int_M^\infty (\alpha_0+\alpha_1(\ln PC + CVu))^{(1-\gamma)/\gamma}\exp\left(-\frac{u^2}{2}\right)du\right. \\ &+ \left.\int_{-\infty}^M \delta^{1-\gamma}\exp\left(-\frac{u^2}{2}\right)du\right],\end{aligned}$$

where the threshold M is given by

$$\begin{aligned}P_T(M) &= \exp\left(\left(\delta^\gamma-\alpha_0\right)/\alpha_1\right) \\ \Leftrightarrow PC\exp\left(CV\cdot M\right) &= \exp\left(\left(\delta^\gamma-\alpha_0\right)/\alpha_1\right) \\ \Leftrightarrow M &= \frac{\left(\delta^\gamma-\alpha_0\right)/\alpha_1-\ln(PC)}{CV}\end{aligned}$$

For $\gamma = 1$, we get:

$$\begin{aligned}E(U(W_T)) &= \frac{1}{\sqrt{2\pi}}\left[\int_M^\infty \ln\left(\alpha_0+\alpha_1(\ln PC + CVu)\right)^{1/\gamma}\exp\left(-\frac{u^2}{2}\right)du\right. \\ &+ \left.\int_{-\infty}^M \ln(\delta)\exp\left(-\frac{u^2}{2}\right)du\right],\end{aligned}$$

4.2 Expected costs for the firm

For the costs of the contract, we get:

$$\begin{aligned}E(W_T) &= E\left[\left\{\begin{array}{cl}(\alpha_0+\alpha_1\ln P_T)^{1/\gamma} & \text{if} \quad P_T \geq \overline{P} \\ \delta & \text{if} \quad P_T < \overline{P}\end{array}\right.\right] \\ &= \frac{1}{\sqrt{2\pi}}\left[\int_M^\infty (\alpha_0+\alpha_1(\ln PC + CVu))^{1/\gamma}\exp\left(-\frac{u^2}{2}\right)du\right. \\ &+ \left.\int_{-\infty}^M \delta\exp\left(-\frac{u^2}{2}\right)du\right]\end{aligned}$$

4.3 UPPS

$$\begin{aligned}
UPPS &= \frac{d}{dP_0}\exp(-r_f T)E(U(W_T)) \\
&= \frac{d}{dP_0}\exp(-r_f T)\left[\int_M^\infty \frac{(\alpha_0+\alpha_1(\ln PC + CVu))^{(1-\gamma)/\gamma}}{1-\gamma}f(u)du + \int_{-\infty}^M \frac{\delta^{1-\gamma}}{1-\gamma}f(u)du\right] \\
&= \exp(-r_f T)\int_M^\infty \frac{(\alpha_0+\alpha_1(\ln PC + CVu))^{(1-\gamma)/\gamma-1}}{\gamma} \\
\frac{\alpha_1}{PC}&\exp\left\{\left(r_f - d - \frac{\sigma^2}{2}\right)T\right\} f(u)du \\
&= \frac{\alpha_1 LD}{\gamma PC}\int_M^\infty (\alpha_0+\alpha_1(\ln PC + CVu))^{(1-2\gamma)/\gamma}\exp\left(-\frac{u^2}{2}\right)du
\end{aligned}$$

Note that the threshold M depends on the variable P_0 with respect to which we differentiate the integral. The two additional terms cancel each other (just like in the base case further above), because the integrand is continuous at the boundary.

4.4 Numerical problems and reformulation of EU and UPPS

The above derived formulae for expected utility and UPPS are numerically unstable, because the threshold M cannot be computed accurately. The reason is that δ^γ is small, because δ is measured in percent of the firm value P_0. δ^γ is especially small if γ becomes large. From this very small quantity (like $0.001^6 = 10^{-18}$) large numbers like α_0 and especially PC (which is of the order of the firm value) are deducted, so that δ^γ is effectively set to zero. Rescaling does not help, because both δ and PC would be rescaled. Therefore, we reformulate the integrals using a substitution that leaves only δ^γ as the lower boundary of the integral. We define:

$$
\begin{aligned}
u = g(x) &= \frac{(x-\alpha_0)/\alpha_1 - \ln(PC)}{CV} \\
x = g^{-1}(u) &= \alpha_0 + \alpha_1 (\ln PC + CVu) \\
g'(x) &= (\alpha_1 CV)^{-1} \\
g^{-1}(M) = \delta^\gamma,\ g^{-1}(\infty) &= \infty \text{ (if } \alpha_1 > 0),\ g^{-1}(-\infty) = -\infty
\end{aligned}
$$

Using the substitution rule for integrals, we obtain:

$$
\begin{aligned}
UPPS &= \frac{\alpha_1 LD}{\gamma PC} \int_{g^{-1}(M)}^{g^{-1}(\infty)} (\alpha_0 + \alpha_1 (\ln PC + CVg(x)))^{(1-2\gamma)/\gamma} \exp\left(-\frac{g(x)^2}{2}\right) g'(x)dx \\
&= \frac{LD}{\gamma PC \cdot CV} \int_{\delta^\gamma}^{\infty} x^{(1-2\gamma)/\gamma} \exp\left(-\frac{1}{2}\left(\frac{(x-\alpha_0)/\alpha_1 - \ln(PC)}{CV}\right)^2\right) dx
\end{aligned}
$$

For expected utility, we get:

$$
\begin{aligned}
E(U(W_T)) &= \frac{1}{1-\gamma}\frac{1}{\sqrt{2\pi}} \left[\int_{g^{-1}(M)}^{g^{-1}(\infty)} (\alpha_0 + \alpha_1 (\ln PC + CVg(x)))^{(1-\gamma)/\gamma} \exp\left(-\frac{g(x)^2}{2}\right) g'(x)dx \right. \\
&\quad \left. + \int_{g^{-1}(-\infty)}^{g^{-1}(M)} \delta^{1-\gamma} \exp\left(-\frac{g(x)^2}{2}\right) g'(x)dx \right] \\
&= \frac{1}{\sqrt{2\pi}\,(1-\gamma)\,\alpha_1 CV} \left[\int_{\delta^\gamma}^{\infty} x^{(1-\gamma)/\gamma} \exp\left(-\frac{1}{2}\left(\frac{(x-\alpha_0)/\alpha_1 - \ln(PC)}{CV}\right)^2\right) dx \right. \\
&\quad \left. + \int_{-\infty}^{\delta^\gamma} \delta^{1-\gamma} \exp\left(-\frac{1}{2}\left(\frac{(x-\alpha_0)/\alpha_1 - \ln(PC)}{CV}\right)^2\right) dx \right]
\end{aligned}
$$

For the expected costs to the firm, we get:

$$
\begin{aligned}
E(W_T) &= \frac{1}{\sqrt{2\pi}\alpha_1 CV} \left[\int_{\delta^\gamma}^{\infty} x^{1/\gamma} \exp\left(-\frac{1}{2}\left(\frac{(x-\alpha_0)/\alpha_1 - \ln(PC)}{CV}\right)^2\right) du \right. \\
&\quad \left. + \int_{\delta^\gamma}^{M} \delta \exp\left(-\frac{1}{2}\left(\frac{(x-\alpha_0)/\alpha_1 - \ln(PC)}{CV}\right)^2\right) du \right]
\end{aligned}
$$

4.5 Further numerical details

Depending on α_0, α_1, and PC the peak in the integrands can be very "sharp", so that they might not be found by the integration routines. This can be prevented by providing the approximate point where the integrand reaches its maximum. For our integrands, this is the point, where $g(x) = 0$, i.e. for $x \approx \alpha_1 \ln(PC) + \alpha_0$.

If the variable α_2 varies slightly, the other two variables α_0 and α_1 vary markedly while the objective function does not change much. As this causes numerical instability, we use a nested algorithm. The inner algorithm finds those α_0 and α_1 which satisfy the two constraints for a given α_2, while the outer algorithm optimizes the objective function over α_2 only.

For the inner optimization, the starting value turned out to be important. (-10,10) turned out to be very successful.